INVERSE PROBLEMS

MATHEMATICAL AND ANALYTICAL TECHNIQUES
WITH APPLICATIONS TO ENGINEERING
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WITH APPLICATIONS TO ENGINEERING

Alan Jeffrey, Consulting Editor

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INVERSE PROBLEMS

MATHEMATICAL AND ANALYTICAL TECHNIQUES WITH APPLICATIONS TO ENGINEERING

ALEXANDER G. RAMM
To Luba and Olga
CONTENTS

Foreword xv
Preface xvii
1. Introduction 1

1.1 Why are inverse problems interesting and practically important? 1

1.2 Examples of inverse problems 2

1.2.1 Inverse problems of potential theory 2
1.2.2 Inverse spectral problems 2
1.2.3 Inverse scattering problems in quantum physics; finding the potential from the impedance function 2
1.2.4 Inverse problems of interest in geophysics 3
1.2.5 Inverse problems for the heat and wave equations 3
1.2.6 Inverse obstacle scattering 4
1.2.7 Finding small subsurface inhomogeneities from the measurements of the scattered field on the surface 5
1.2.8 Inverse problem of radiomeasurements 5
1.2.9 Impedance tomography (inverse conductivity) problem 5
1.2.10 Tomography and other integral geometry problems 5
1.2.11 Inverse problems with “incomplete data” 6
1.2.12 The Pompeiu problem, Schiffer’s conjecture, and inverse problem of plasma theory 7
1.2.13 Multidimensional inverse potential scattering 8
1.2.14 Ground-penetrating radar 8
1.2.15 A geometrical inverse problem 9
1.2.16 Inverse source problems 10
1.2.17 Identification problems for integral-differential equations 12
1.2.18 Inverse problem for an abstract evolution equation 12
1.2.19 Inverse gravimetry problem 12
1.2.20 Phase retrieval problem (PRP) 12
1.2.21 Non-overdetermined inverse problems 12
1.2.22 Image processing, deconvolution 13
1.2.23 Inverse problem of electrodynamics, recovery of layered medium from the surface scattering data 13
1.2.24 Finding ODE from a trajectory 13

1.3 Ill-posed problems 14

1.4 Examples of Ill-posed problems 15
  1.4.1 Stable numerical differentiation of noisy data 15
  1.4.2 Stable summation of the Fourier series and integrals with randomly perturbed coefficients 15
  1.4.3 Solving ill-conditioned linear algebraic systems 15
  1.4.4 Fredholm and Volterra integral equations of the first kind 16
  1.4.5 Deconvolution problems 16
  1.4.6 Minimization problems 16
  1.4.7 The Cauchy problem for Laplace’s equation 16
  1.4.8 The backwards heat equation 17

2. Methods of solving ill-posed problems 19
  2.1 Variational regularization 19
    2.1.1 Pseudoinverse. Singular values decomposition 19
    2.1.2 Variational (Phillips-Tikhonov) regularization 20
    2.1.3 Discrepancy principle 22
    2.1.4 Nonlinear ill-posed problems 23
    2.1.5 Regularization of nonlinear, possibly unbounded, operator 24
    2.1.6 Regularization based on spectral theory 25
    2.1.7 On the notion of ill-posedness for nonlinear equations 26
    2.1.8 Discrepancy principle for nonlinear ill-posed problems with monotone operators 26
    2.1.9 Regularizers for Ill-posed problems must depend on the noise level 29
  2.2 Quasisolutions, quasinversion, and Backus-Gilbert method 30
    2.2.1 Quasisolutions for continuous operator 30
    2.2.2 Quasisolution for unbounded operators 31
    2.2.3 Quasiinversion 32
    2.2.4 A Backus-Gilbert-type method: Recovery of signals from discrete and noisy data 32
  2.3 Iterative methods 40
  2.4 Dynamical system method (DSM) 41
    2.4.1 The idea of the DSM 41
    2.4.2 DSM for well-posed problems 42
    2.4.3 Linear ill-posed problems 45
    2.4.4 Nonlinear ill-posed problems with monotone operators 49
    2.4.5 Nonlinear ill-posed problems with non-monotone operators 57
    2.4.6 Nonlinear ill-posed problems: avoiding inverting of operators in the Newton-type continuous schemes 59
    2.4.7 Iterative schemes 62
2.4.8 A spectral assumption 64
2.4.9 Nonlinear integral inequality 65
2.4.10 Riccati equation 70

2.5 Examples of solutions of ill-posed problems 71
2.5.1 Stable numerical differentiation: when is it possible? 71
2.5.2 Stable summation of the Fourier series and integrals with perturbed coefficients 85
2.5.3 Stable solution of some Volterra equations of the first kind 87
2.5.4 Deconvolution problems 87
2.5.5 Ill-conditioned linear algebraic systems 88

2.6 Projection methods for ill-posed problems 89

3. One-dimensional inverse scattering and spectral problems 91

3.1 Introduction 92
3.1.1 What is new in this chapter? 92
3.1.2 Auxiliary results 92
3.1.3 Statement of the inverse scattering and inverse spectral problems 97
3.1.4 Property C for ODE 98
3.1.5 A brief description of the basic results 99

3.2 Property C for ODE 104
3.2.1 Property C_+ 104
3.2.2 Properties C_ϕ and C_θ 105

3.3 Inverse problem with I-function as the data 108
3.3.1 Uniqueness theorem 108
3.3.2 Characterization of the I-functions 110
3.3.3 Inversion procedures 112
3.3.4 Properties of I(k) 112

3.4 Inverse spectral problem 122
3.4.1 Auxiliary results 122
3.4.2 Uniqueness theorem 124
3.4.3 Reconstruction procedure 126
3.4.4 Invertibility of the reconstruction steps 128
3.4.5 Characterization of the class of spectral functions of the Sturm-Liouville operators 130
3.4.6 Relation to the inverse scattering problem 130

3.5 Inverse scattering on half-line 132
3.5.1 Auxiliary material 132
3.5.2 Statement of the inverse scattering problem on the half-line. Uniqueness theorem 137
3.5.3 Reconstruction procedure 139
3.5.4 Invertibility of the steps of the reconstruction procedure 143
3.5.5 Characterization of the scattering data 145
3.5.6 A new Marchenko-type equation 147
3.5.7 Inequalities for the transformation operators and applications 148

3.6 Inverse scattering problem with fixed-energy phase shifts as the data 156
3.6.1 Introduction 156
3.6.2 Existence and uniqueness of the transformation operators independent of angular momentum 157
5. Stability of the solutions to 3D Inverse scattering problems with fixed-energy data 255

5.1 Introduction 255
  5.1.1 The direct potential scattering problem 256
  5.1.2 Review of the known results 256

5.2 Inverse potential scattering problem with fixed-energy data 264
  5.2.1 Uniqueness theorem 264
  5.2.2 Reconstruction formula for exact data 264
  5.2.3 Stability estimate for inversion of the exact data 267
  5.2.4 Stability estimate for inversion of noisy data 270
  5.2.5 Stability estimate for the scattering solutions 273
  5.2.6 Spherically symmetric potentials 274

5.3 Inverse geophysical scattering with fixed-frequency data 275

5.4 Proofs of some estimates 277
  5.4.1 Proof of (5.1.18) 277
  5.4.2 Proof of (5.1.20) and (5.1.21) 278
  5.4.3 Proof of (5.2.17) 283
  5.4.4 Proof of (5.4.49) 285
  5.4.5 Proof of (5.4.51) 286
  5.4.6 Proof of (5.2.13) 287
  5.4.7 Proof of (5.2.23) 289
  5.4.8 Proof of (5.1.30) 292

5.5 Construction of the Dirichlet-to-Neumann map from the scattering data and vice versa 293

5.6 Property C 298

5.7 Necessary and sufficient condition for scatterers to be spherically symmetric 300

5.8 The Born inversion 307

5.9 Uniqueness theorems for inverse spectral problems 312

6. Non-uniqueness and uniqueness results 317

6.1 Examples of nonuniqueness for an inverse problem of geophysics 317
  6.1.1 Statement of the problem 317
  6.1.2 Example of nonuniqueness of the solution to IP 318

6.2 A uniqueness theorem for inverse boundary value problems for parabolic equations 319

6.3 Property C and an inverse problem for a hyperbolic equation 321
  6.3.1 Introduction 321
  6.3.2 Statement of the result. Proofs 321

6.4 Continuation of the data 330

7. Inverse problems of potential theory and other inverse source problems 333

7.1 Inverse problem of potential theory 333

7.2 Antenna synthesis problems 336

7.3 Inverse source problem for hyperbolic equations 337
xii  Contents

8. Non-overdetermined inverse problems 339
   8.1 Introduction 339
   8.2 Assumptions 340
   8.3 The problem and the result 340
   8.4 Finding $\varphi_j(s)$ from $\varphi_j^2(s)$ 342
   8.5 Appendix 347

9. Low-frequency inversion 349
   9.1 Derivation of the basic equation. Uniqueness results 349
   9.2 Analytical solution of the basic equation 353
   9.3 Characterization of the low-frequency data 355
   9.4 Problems of numerical implementation 355
   9.5 Half-spaces with different properties 356
   9.6 Inversion of the data given on a sphere 357
   9.7 Inversion of the data given on a cylinder 358
   9.8 Two-dimensional inverse problems 359
   9.9 One-dimensional inversion 362
   9.10 Inversion of the backscattering data and a problem of integral geometry 363
   9.11 Inversion of the well-to-well data 364
   9.12 Induction logging problems 366
   9.13 Examples of non-uniqueness of the solution to an inverse problem of geophysics 369
   9.14 Scattering in absorptive medium 371
   9.15 A geometrical inverse problem 371
   9.16 An inverse problem for a biharmonic equation 373
   9.17 Inverse scattering when the background is variable 375
   9.18 Remarks concerning the basic equation 377

10. Wave scattering by small bodies of arbitrary shapes 379
   10.1 Wave scattering by small bodies 379
      10.1.1 Introduction 379
      10.1.2 Scalar wave scattering by a single body 380
      10.1.3 Electromagnetic wave scattering by a single body 383
      10.1.4 Many-body wave scattering 385
   10.2 Equations for the self-consistent field in media consisting of many small particles 388
      10.2.1 Introduction 388
      10.2.2 Acoustic fields in random media 390
      10.2.3 Electromagnetic waves in random media 394
10.3 Finding small subsurface inhomogeneities from scattering data 395
   10.3.1 Introduction 396
   10.3.2 Basic equations 397
   10.3.3 Justification of the proposed method 398

10.4 Inverse problem of radiomeasurements 401

11. The Pompeiu problem 405
   11.1 The Pompeiu problem 405
      11.1.1 Introduction 405
      11.1.2 Proofs 407
   11.2 Necessary and sufficient condition for a domain, which fails to have
      Pompeiu property, to be a ball 414
      11.2.1 Introduction 414
      11.2.2 Proof 416

Bibliographical Notes 421
References 425
Index 441
The importance of mathematics in the study of problems arising from the real world, and the increasing success with which it has been used to model situations ranging from the purely deterministic to the stochastic, is well established. The purpose of the set of volumes to which the present one belongs is to make available authoritative, up to date, and self-contained accounts of some of the most important and useful of these analytical approaches and techniques. Each volume provides a detailed introduction to a specific subject area of current importance that is summarized below, and then goes beyond this by reviewing recent contributions, and so serving as a valuable reference source.

The progress in applicable mathematics has been brought about by the extension and development of many important analytical approaches and techniques, in areas both old and new, frequently aided by the use of computers without which the solution of realistic problems would otherwise have been impossible.

A case in point is the analytical technique of singular perturbation theory which has a long history. In recent years it has been used in many different ways, and its importance has been enhanced by it having been used in various fields to derive sequences of asymptotic approximations, each with a higher order of accuracy than its predecessor. These approximations have, in turn, provided a better understanding of the subject and stimulated the development of new methods for the numerical solution of the higher order approximations. A typical example of this type is to be found in the general study of nonlinear wave propagation phenomena as typified by the study of water waves.
Elsewhere, as with the identification and emergence of the study of inverse problems, new analytical approaches have stimulated the development of numerical techniques for the solution of this major class of practical problems. Such work divides naturally into two parts, the first being the identification and formulation of inverse problems, the theory of ill-posed problems and the class of one-dimensional inverse problems, and the second being the study and theory of multidimensional inverse problems.

On occasions the development of analytical results and their implementation by computer have proceeded in parallel, as with the development of the fast boundary element methods necessary for the numerical solution of partial differential equations in several dimensions. This work has been stimulated by the study of boundary integral equations, which in turn has involved the study of boundary elements, collocation methods, Galerkin methods, iterative methods and others, and then on to their implementation in the case of the Helmholtz equation, the Lamé equations, the Stokes equations, and various other equations of physical significance.

A major development in the theory of partial differential equations has been the use of group theoretic methods when seeking solutions, and in the introduction of the comparatively new method of differential constraints. In addition to the useful contributions made by such studies to the understanding of the properties of solutions, and to the identification and construction of new analytical solutions for well established equations, the approach has also been of value when seeking numerical solutions. This is mainly because of the way in many special cases, as with similarity solutions, a group theoretic approach can enable the number of dimensions occurring in a physical problem to be reduced, thereby resulting in a significant simplification when seeking a numerical solution in several dimensions. Special analytical solutions found in this way are also of value when testing the accuracy and efficiency of new numerical schemes.

A different area in which significant analytical advances have been achieved is in the field of stochastic differential equations. These equations are finding an increasing number of applications in physical problems involving random phenomena, and others that are only now beginning to emerge, as is happening with the current use of stochastic models in the financial world. The methods used in the study of stochastic differential equations differ somewhat from those employed in the applications mentioned so far, since they depend for their success on the Ito calculus, martingale theory and the Doob-Meyer decomposition theorem, the details of which are developed as necessary in the volume on stochastic differential equations.

There are, of course, other topics in addition to those mentioned above that are of considerable practical importance, and which have experienced significant developments in recent years, but accounts of these must wait until later.

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This book can be used for courses at various levels in ill-posed problems and inverse problems. The bibliography of the subject is enormous. It is not possible to compile a complete bibliography and no attempt was made to do this. The bibliography contains some books where the reader will find additional references. The author has used extensively his earlier published papers, and referenced these, as well as the papers of other authors that were used or mentioned.

Let us outline some of the novel features in this book.

In Chapter 1 the statement of various inverse problems is given.

In Chapter 2 the presentation of the theory of ill-posed problems is shorter and sometimes simpler than that published earlier, and quite a few new results are included. Regularization for ill-posed operator equations with unbounded nonlinear operators is studied. A novel version of the discrepancy principle is formulated for nonlinear operator equations. Convergence rate estimates are given for Backus-Gilbert-type methods. The DSM (Dynamical systems method) in ill-posed problems is presented in detail. The presentation is based on the author's papers and the joint papers of the author and his students. These results appear for the first time in book form. Papers [R216], [R217], [R218], [R220], [ARS3], [AR1] have been used in this Chapter.

In Chapter 3 the presentation of one-dimensional inverse problems is based mostly on the author's papers, especially on [R221]. It contains many novel results, which are described at the beginning of the Chapter. The presentation of the classical results, for example, Gel’fand-Levitan's theory, and Marchenko’s theory, contains many novel points. The presentation of M. G. Krein’s inversion theory with complete proofs is
given for the first time. The Newton-Sabatier inversion theory, which has been in the
literature for more than 40 years, and was presented in two monographs [CS], [N], is
analyzed and shown to be fundamentally wrong in the sense that its foundations are
wrong (cf. [R206]). This Chapter is based on the papers [R221], [R199], [R197],
[R196], [R195], [R192], [R185]. One of the first papers on inverse spectral prob-
lems was Ambartsumian's paper (1929) [Am], where it was proved that one spectrum
determines the one-dimensional Neumann Schrödinger's operator uniquely. This re-
sult is an exceptional one: in general one spectrum does not determine the potential
uniquely (see Section 3.7 and [PT]). Only 63 years later a multidimensional analog of
Ambartsumian's result was obtained ([RSt1]). The main technical tool in this Chapter
and in Chapter 5 is Property C, that is, completeness of the set of products of solu-
tions to homogeneous differential equations. For partial differential equations this tool
has been introduced in [R87] and developed in many papers and in the monograph
[R139]. For ordinary differential equations completeness of the products of solutions
to homogeneous ordinary equations has been used in different forms in [B], [L1]. In
our book Property C for ODE is presented in the form introduced and developed by
the author in [R196].

In Chapter 4 the presentation of inverse obstacle scattering problems contains many
novel points. The requirements on the smoothness of the boundary are minimal,
stability estimates for the inversion procedure corresponding to fixed-frequency data
are given, the high-frequency inversion formulas are discussed and the error of the
inversion from noisy data is estimated. Analysis of the currently used numerical methods
is given. This Chapter is based on [R83], [R155], [R162], [R164], [R167], [R171],
[RSa].

In Chapter 5 a presentation of the solution of the 3D inverse potential scattering
problem with fixed-energy noisy data is given. This Chapter is based on the series
of the author's papers, especially on the paper [R203]. The basic concept used in the
analysis of the inverse scattering problem in Chapter 5 is the concept of Property C,
i.e., completeness of the set of products of solutions to homogeneous partial differential
equations. This concept was introduced by the author ([R87]) and applied to many
inverse problems (see [R139] and references therein). An important part of the theory
consists of obtaining stability estimates for the potential, reconstructed from fixed-
energy noisy data (and from exact data). Error estimates for the Born inversion are
given under suitable assumptions. It is shown that the Born inversion may fail while
the Born approximation works well. In other words, the Born approximation may
be applicable for solving the direct scattering problem, while the Born inversion, that
is, inversion based on the Born approximation, may fail. The Born inversion is still
popular in applications, therefore these error estimates will hopefully be useful for
practitioners.

The author's inversion method for fixed-energy scattering data, is compared with
that based on the usage of the Dirichlet-to-Neumann map. The author shows why
the difficulties in numerical implementation of his method are less formidable than the
difficulties in implementing the inversion method based on the Dirichlet-to-Neumann
map.
A necessary and sufficient condition for a scatterer to be spherically symmetric is given ([R.128]).

In Chapter 6 an example of non-uniqueness of the solution to a 3D problem of geophysics is given. It illustrates the crucial role of the uniqueness theorems in a study of inverse problems. One may try to solve numerically such a problem, by a parameter-fitting, which is very popular among practitioners. But if the uniqueness result is not established, the numerical results may be meaningless. Some uniqueness theorems for inverse boundary value problem and for an inverse problem for hyperbolic equations are established in this Chapter.

In Chapter 7, inverse problems of potential theory and antenna synthesis are briefly discussed. The presentation of the theory on this topic is not complete: there are books and many papers on antenna synthesis (e.g., [MJ], [ZK], [AVST], [R.21], [R.26]) including nonlinear problems of antenna synthesis ([R.23], [R.27]).

Chapter 8 contains a discussion of non-overdetermined problems. These are, roughly speaking, the inverse problems in which the unknown function depends on the same number of variables as the data function. Examples of such problems are given. Most of these problems are open: even uniqueness theorems are not available. Such a problem, namely, recovery of an unknown coefficient in a Schrödinger equation in a bounded domain from the knowledge of the values of the spectral function \( \rho(s, s, \lambda) \) on the boundary is discussed under the assumption that all the eigenvalues are simple, that is, the corresponding eigenspaces are one-dimensional. The presentation follows [R.198].

In Chapter 9 the theory of the inversion of low-frequency data is presented. This theory is based on the series of author’s papers, starting with [R.68], [R.77], and uses the presentation in [R.83] and [R.139]. Almost all of the results in this Chapter are from the above papers and books.

Chapter 10 is a summary of the author’s results regarding the theory of wave scattering by small bodies of arbitrary shapes. These results have been obtained in a series of the author’s papers and are summarized in [R.65], [R.50]. The solution of inverse radiomeasurements problem ([R.33], [R.65]) is based on these results. Also, these results are used in the solution of the problem of finding small subsurface inhomogeneities from the scattering data, measured on the surface. The solution to this problem can be used in modeling ultrasound mammography, in finding small holes in metallic objects, and in many other applied problems.

In Chapter 11 the classical Pompeiu problem is presented following the papers [R.177], [R.186].

The author thanks several publishers of his papers, mentioned above, for the permission to use these papers in the book.

There are many questions that the author did not discuss in this book: inverse scattering for periodic potentials and other periodic objects, such as gratings, periodic objects, (see, e.g., [L] for one-dimensional scattering problems for periodic potentials), the Carleman estimates and their applications to inverse problems ([Bu2], [H], [LRS]), the inverse problems for elasticity and Maxwell’s equations ([RK], [Ya]), the methods based on controllability results ([Bel]), problems of tomography and integral geometry ([RKa], [R.139]), etc. Numerous parameter-fitting schemes for solving various
engineering problems are not discussed. There are many papers published, which use parameter-fitting for solving inverse problems. However, in most cases there are no error estimates for parameter-fitting schemes for solving inverse problems, and one cannot guarantee any accuracy of the inversion result. In [GRS] the concept of stability index is introduced and applied to a parameter-fitting scheme for solving a one-dimensional inverse scattering problem in quantum physics. This concept allows one to get some idea about the error estimate in a parameter-fitting scheme.

The applications of inverse scattering to integration of nonlinear evolution equations are not discussed as there are many books on this topic (see e.g., [M], [FT] and references therein).
1. INTRODUCTION

1.1 WHY ARE INVERSE PROBLEMS INTERESTING AND PRACTICALLY IMPORTANT?

Inverse problems are the problems that consist of finding an unknown property of an object, or a medium, from the observation of a response of this object, or medium, to a probing signal. Thus, the theory of inverse problems yields a theoretical basis for remote sensing and non-destructive evaluation. For example, if an acoustic plane wave is scattered by an obstacle, and one observes the scattered field far from the obstacle, or in some exterior region, then the inverse problem is to find the shape and material properties of the obstacle. Such problems are important in identification of flying objects (airplanes missiles, etc.), objects immersed in water (submarines, paces of fish, etc.), and in many other situations.

In geophysics one sends an acoustic wave from the surface of the earth and collects the scattered field on the surface for various positions of the source of the field for a fixed frequency, or for several frequencies. The inverse problem is to find the subsurface inhomogeneities. In technology one measures the eigenfrequencies of a piece of a material, and the inverse problem is to find a defect in this material, for example, a hole in a metal. In geophysics the inhomogeneity can be an oil deposit, a cave, a mine. In medicine it may be a tumor, or some abnormality in a human body.

If one is able to find inhomogeneities in a medium by processing the scattered field on the surface, then one does not have to drill a hole in a medium. This, in turn, avoids expensive and destructive evaluation. The practical advantages of remote sensing are what makes the inverse problems important.
1.2 EXAMPLES OF INVERSE PROBLEMS

1.2.1 Inverse problems of potential theory

Suppose a body $D \subset \mathbb{R}^3$ with a density $\rho(x)$, $x \in \mathbb{R}^3$, generates gravitational potential

$$u(x) = \int_D \frac{\rho(y)}{4\pi|x-y|} \, dy.$$ 

Is it possible to find $\rho$, given the potential $u(x)$ for $x \in B'_R := \{ x : |x| \geq R \}$, far away from $D$? A point mass $m$ and a uniformly distributed mass $m$ in a ball of radius $a$ produce the same potential $u(x) = \frac{m}{|x|}$ for $|x| \geq R > a$. Thus, it is not possible to find $\rho(y)$ uniquely from the knowledge of $u$ in $B'_R$. However, if one knows a priori that $\rho(x) = 1$ in $D$, then it is possible to find $D$ from the knowledge of $u(x)$ in $B'_R$, provided that $D$ is, for example, star-shaped, that is, every ray issued from some interior point $O \in D$, intersects the boundary $S = \partial D$ of $D$ at only one point.

1.2.2 Inverse spectral problems

Let $\ell = -\frac{d^2}{dx^2} + q(x)$ be the Sturm-Liouville operator defined by the Dirichlet boundary conditions as self-adjoint operator in $L^2[0, 1]$, and $0 < \lambda_1 < \lambda_2 \leq \cdots$ be its eigenvalues. To what extent does the set of these eigenvalues determine $q(x)$? Roughly speaking, one spectrum, that is, the set $\{ \lambda_j \ | j = 1, 2, \ldots \}$, determines “half of $q(x)$,” in the sense that if $q(x)$ is known on $[\frac{1}{2}, 1]$, then one spectrum determines uniquely $q(x)$ on $[0, \frac{1}{2}]$. A classical result due to Borg [B] and Marchenko [M] says that two spectra uniquely determine the operator $\ell$, i.e., the potential $q$ and the boundary conditions at $x = 0$ and $x = 1$ of the type $u'(1) + h_1u(1) = 0$ and $u'(0) + h_0u(0) = 0$, where $h_0$ and $h_1$ are constants, and one assumes that the two spectra correspond to the same $h_0$ and two distinct $h_1$. The author (see [R,196]) asked the following question: if $q(x)$ is known on the segment $[b, 1]$, $0 < b < 1$, then what part of the spectrum one needs to know in order to uniquely recover $q(x)$ on $[0, b]$? It is assumed that $q$ is real valued: $q = \bar{q}$, and $q \in L(0, 1)$.

Let $\rho(\lambda)$ be the spectral measure of the self-adjoint operator $l$. This notion is defined in Chapter 3. The inverse spectral problem is: given $\rho(\lambda)$, find $q(x)$, and the boundary conditions, characterizing $\ell$.

Similar problems can be formulated in the multidimensional cases, when a bounded domain $D$ plays the role of the segment $[0, 1]$, the role of the spectral data is played by the eigenvalues and the values on the boundary $S$ of $D$ of the normal derivatives of the normalized eigenfunctions, $\frac{\partial \phi_j}{\partial N}\big|_S$, $\forall j$. One may choose other spectral data.

1.2.3 Inverse scattering problems in quantum physics; finding the potential from the impedance function

Consider the Dirichlet operator $\ell = -\frac{d^2}{dx^2} + q(x)$, $q \in L_{1,1} := \{ q : q = \bar{q}, \int_0^\infty x|q(x)| \, dx < \infty \}$ in $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ := [0, \infty)$. Denote by $f(x, k)$ the Jost solution, by $f(k) := f(0, k)$ the Jost function, by $I(k) := \frac{f(0, k)}{I'(k)}$ the $I$-function (impedance function), and by $\mathcal{J} := \{ S(k), k_j, s_j, 1 \leq j \leq J \}$ the scattering data (see Chapter 3).
The inverse problem of quantum scattering on the half-axis consists of finding $q(x)$, given $S$. It was studied in [M].

The inverse problem of finding $q(x)$, given $I(k) \forall k > 0$, is of interest in many applications. The $I$-function has the physical meaning of the impedance function, it is the ratio $\frac{H}{E}$ in the problem of electromagnetic wave falling perpendicularly onto the earth, when the dielectric permittivity and conductivity of the earth depend on the vertical coordinate only. One can prove that $I(k)$ coincides with the Weyl function $m(k)$ (Chapter 3). It turns out that $I(k)$ known $\forall k > 0$ determines uniquely $q(x)$, and one can explicitly calculate $S$ and $\rho(\lambda)$, given $I(k)$ ([R196]).

1.2.4 Inverse problems of interest in geophysics

There are many inverse problems of interest in geophysics. A typical one consists of finding an unknown inhomogeneity in the velocity profile (refraction coefficient) from the scattered acoustic field measured on the surface of the earth and generated by a point source, situated on the surface of the earth at varying positions.

Its mathematical formulation (in a simplified form) is:

$$[\Delta + k^2 n_0(x) + k^2 \nu(x)]u(x, k) = -\delta(x - y) \quad \text{in} \quad \mathbb{R}^3, \quad x \in \mathbb{R}^3,$$

where $k = \text{const} > 0$, $n_0(x)$ is the known background refraction coefficient, $D := \text{supp} \nu(x) \subset \mathbb{R}^3_+ := \{x : x_3 < 0\}$, where $\text{supp} \nu(x)$ is the support of $\nu$, $\nu \in L^2(D)$, and $\nu$ is an inhomogeneity in the refraction coefficient (or in the velocity profile), $u$ is the acoustic pressure, $u$ satisfies the radiation condition at infinity (or the limiting absorption principle). An inverse problem of geophysics consists of finding the function $\nu$ from the knowledge of the scattered field on the surface of the Earth, that is from $u(x, y, k)$ known for all $x, y \in P := \{x : x_3 = 0\}$ at a fixed $k > 0$, or for all $k \in (0, k_0)$, where $k_0 > 0$ is a small number (the case of low-frequency surface data) (cf [LRS], [Ro], [R83], [R139]).

Another problem is to find the conductivity of the medium from the measurements of the electromagnetic waves, scattered by a source that moves in a borehole along the vertical line. ([R83], [R139])

1.2.5 Inverse problems for the heat and wave equations

A typical inverse problem for the heat equation

$$u_t = u_{xx} - q(x)u, \quad 0 \leq x \leq 1, \quad t > 0, \quad u(x, 0) = u_0(x), \quad u(0, t) = 0, \quad u(1, t) = a(t),$$

is to find $q(x)$ from the flux measurements: $u_x(1, t) = b(t)$. The extra data (measured data), $b(t) \forall t > 0$, allow one to find $q(x)$. Another inverse problem is to find the unknown conductivity $\sigma(x)$ from boundary measurements. For example, let $u_t = (\sigma(x)u_x)_x, \quad u|_{x=0} = u_0(x), \quad u(0, t) = 0, \quad u(1, t) = a(t), \quad u_x(1, t) = b(t)$. Can one find $\sigma(x)$, given $u_0(x), a(t)$ and $b(t) \forall t > 0$?
Consider the inverse conductivity problem: let $\nabla (\sigma (x) \nabla u) = 0$ in $D \subset \mathbb{R}^3$, $u = f$, $\sigma u_N = g$ on $S := \partial D$, where $N$ is the outer unit normal to $S$, the extra data is the flux $g$ at the boundary. Suppose that the set $\{f, g\}_{f \in H^1_2(S)}$ is known. Can one determine $\sigma (x)$ uniquely? Here $D$ is a bounded domain with a sufficiently smooth boundary $S$, and $H^1(S)$ is the Sobolev space. In applications in medicine, $f$ is the electrostatic potential, which can be applied to a human chest, and $g$ is the flux of the electrostatic field, which can be measured. If one can determine $\sigma (x)$ from these measurements then some diagnostic information is obtained ([R157], [R139], [R131], [R103], [Gro], [LRS], [Ro], [Is1]).

There are many inverse problem for the wave equation. One of them is to find the velocity $c(x)$ in the equation $u_{tt} c^2(x) - \Delta u = \delta(x - y) \delta(t)$, $u = u_t = 0$ at $t = 0$, $u = 0$ on $S$, given the extra data $u_N$ on $S$ for a fixed $y$ and all $t > 0$, or for $y$, varying on $S$, and $t \in [0, T]$, where $T > 0$ is some number. ([LRS], [Ro], [RRa], [RSj]).

1.2.6 Inverse obstacle scattering

Let $D \subset \mathbb{R}^3$ be bounded domain with a Lipschitz boundary $S$, $D' := \mathbb{R}^3 \setminus D$ be the exterior domain, $S^2$ be the unit sphere in $\mathbb{R}^3$. The scattering problem consists of finding the scattering solution, i.e., the solution to the problem

\[(\nabla^2 + k^2)u = 0 \text{ in } D', \]
\[u = u_0 + v, \quad u_0 := e^{ikr}, \quad \alpha \in S^2,\]
\[u|_{S} = 0, \quad v = A(\alpha', \alpha, k) e^{ikr} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \alpha' := \frac{x}{r}.\]

The coefficient $A$ is called the scattering amplitude.

Existence and uniqueness of the solution to (1.2.2)–(1.2.3) are proved in [RSA] without any assumption on the smoothness of the boundary. If the Neumann boundary condition

\[u_N|_S = 0\]

is used in place of (1.2.3), then the existence and uniqueness of the solution to (1.2.2)–(1.2.3N) are proved in [RSA] under the assumption of compactness of the embedding $H^1(D_R) \to L^2(D_R)$, where $D_R := B_R \setminus D$, $B_R := \{x : |x| \leq R\}$, $R > 0$ is such that $B_R \supset D$, and $H^1(D_R)$ is the Sobolev space. See also [GoR]. The inverse obstacle scattering problem consists of finding $S$ and the boundary condition on $S$, given $A(\alpha', \alpha, k)$ in the following cases:

1. either at a fixed $\alpha = \alpha_0$ for all $\alpha' \in S^2$ and all $k > 0$, or,
2. at a fixed $k = k_0 > 0$ for all $\alpha'$ and $\alpha$ running through open subsets of $S^2$, or,
3. for fixed $\alpha = \alpha_0$ and $k = k_0$ and all $\alpha' \in S^2$.

Uniqueness of the solution of the first inverse problem is proved by M. Schiffer (1964), (see [R83]) of the second by A. G. Ramm (1986) (see [R83]), and the third
problem is still open. See also [R154], [R155], [R162], [R159], [R164], [R167], [R171], [CK].

One may consider a penetrable layered obstacle, and ask if the scattering amplitude at a fixed \( k = k_0 > 0 \) allows one to determine the boundaries of all the layers uniquely, and the constant velocity profiles in each of the layers. See [RPY] for an answer to this question.

1.2.7 Finding small subsurface inhomogeneities from the measurements of the scattered field on the surface

Suppose there are few small, in comparison with the wave-length, holes in the metallic body. A source of acoustic waves is on the surface of the body, and the scattered field is measured on the surface of the body for various positions of the acoustic source, at a fixed frequency.

The inverse problem is to find the number of the small holes, their locations, and their volume. A similar problem is important in medicine, where the small bodies are the cancer cells to be found in the healthy tissue of a human’s body. In the ultrasound mammography modeling, one deals with the tissue of a woman’s breast ([R193], [GR1]).

1.2.8 Inverse problem of radiomasurement

Suppose a complicated electromagnetic field distribution \((E, H)\) exists in the aperture of a mirror antenna. For many practical reasons one wants to know this distribution. Let \((E', H')\) be the field scattered by a small probe placed at a point \(x\) in the aperture of the antenna. Given the shape and electromagnetic constants \(\varepsilon, \mu, \text{ and } \sigma\), of the probe, the inverse problem of radiomasurement consists of finding \((E(x), H(x))\) from the knowledge of \(E' H'\). (See [R65] for a solution to this problem).

1.2.9 Impedance tomography (inverse conductivity) problem

This problem was briefly mentioned in Section 1.2.5.

1.2.10 Tomography and other integral geometry problems

Define \(\hat{f}(\alpha, p) = \int_{l_{ap}} f(x) ds\), where \(l_{ap} := \{x : \alpha \cdot x = p\}\), \(x \in \mathbb{R}^n\), \(n \geq 2\), \(\alpha \in S^{n-1}\), \(S^{n-1}\) is the unit sphere in \(\mathbb{R}^n\), \(p \in \mathbb{R}\). The function \(\hat{f}(\alpha, p)\) is called the Radon transform of \(f\). The function \(f\) can be assumed piecewise-continuous and absolutely integrable over every plane \(l_{ap}\), so that the Radon transform would be well defined in the classical sense. But in fact, one can define the Radon transform for much larger sets of functions and on distributions [R170], [RKa], [Hel].

Given \(\hat{f}(\alpha, p)\), one can uniquely recover \(f(x)\) provided, for example, that \(f(x) \in L^1(\mathbb{R}^n)\), or \(f(x) \in L^1(\mathbb{R}^n, \frac{1}{1+|x|})\), where \(L^1(\mathbb{R}, w)\) is the weighted space with the norm \(\int_{\mathbb{R}} |f(x)| w(x) dx := \|f\|_{L^1(\mathbb{R}, w)}\). Practically interesting questions are:

(a) How are singularities of \(f\) and \(\hat{f}\) related?
(b) Given the noisy measurements of \(f\) at a grid, how does one find the discontinuities of \(f\)?
A grid is a set of points \( x_j := (j_1 h_1, j_2 h_2, j_n h_n) \), where \( h_i > 0, 1 \leq i \leq n, j_m = 0, \pm 1, \pm 2, \pm 3, \ldots, 1 \leq m \leq n \). The noisy measurement are \( u_j = f(x_j) + n_j \), where \( n_j \) are identically distributed, independent random variables with zero mean value and a finite variance \( \sigma^2 < \infty \). See [R176], [RKa] for a detailed investigation of the above problem. An open problem is: what are the minimal assumptions on the growth of \( f(x) \) at infinity that guarantee the injectivity of the Radon transform? There is an example of a smooth function \( f \neq 0 \), such that \( \int_{u_p} |f| ds < \infty \) for all \( \alpha, p \), and \( \hat{f}(\alpha, p) \equiv 0 \), [R.Ka].

In many applications one integrates \( f \) not over the planes \( l_{kp} \), but over some other family of manifolds. The problem of integral geometry is to recover \( f \) from the knowledge of its integral over a family of manifolds.

For example, if the family of manifolds is a family of spheres of various radii \( r > 0 \) and centers \( s \) running over some surface \( S \), then \( Mf := m(s, r) := \int_{|s| = r} f(x) dx \) are the spherical means of \( f \), \( s \in S \), and the problem is to recover \( f \) from the knowledge of \( m(s, z) \forall s \in S \) and \( \forall z > 0 \).

Conditions on \( S \) that guarantee the injectivity of the operator \( M \) are given in [R211], where some inversion formulas are also derived.

### 1.2.11 Inverse problems with “incomplete data”

Suppose that not all the scattering data in Section 1.2.3 are given, for example, \( r(k) \forall k > 0 \) is given, but \( k_j, s_j \) and \( f \) are unknown.

In general, one cannot recover a \( q \in L_{1,1} \) from these “incomplete” data. However, if one knows a priori that \( q(x) \) has compact support, or \( |q(x)| \leq ce^{-c|x|}, \), \( \delta > 1 \), then the data \( r(k) \forall k > 0 \) alone determine \( q(x) \) uniquely. Such type of inverse problems we call inverse problems with “incomplete” data. The “incompleteness” of the data is remedied by the additional a priori assumption about \( q(x) \), so, in fact, the data are complete in the sense that \( q(x) \) is determined uniquely by these data.

Another example of an inverse problem with “incomplete” data, is recovery of \( q(x) \in L_{1,1}(\mathbb{R}) := \{ q : \int_{-\infty}^{\infty} (1 + |x|)|q| dx, \ q = \tilde{q}, \ q(x) = 0 \ for \ x < 0, \ \text{from the knowledge of the reflection coefficient} \ r(k) \forall k > 0 \), in the full-axis (full-line) scattering problem:

\[
\begin{align*}
   u'' + k^2 u - q(x)u &= 0, \quad -\infty < x < \infty, \\
   u &= e^{ikx} + r(k)e^{-ikx} + o(1), \quad x \to -\infty, \\
   u &= t(k)e^{ikx} + o(1), \quad x \to +\infty.
\end{align*}
\]

The coefficients \( r(k) \) and \( t(k) \) are reflection and transmission coefficients, respectively.

A general \( q \in L_{1,1}(\mathbb{R}) \) cannot be uniquely recovered from the knowledge of \( r(k) \) alone: one needs to know additionally the bound states and norming constants to recover \( q \) uniquely. However if one knows a priori that \( q(x) = 0 \) for \( x < x_0 \), for example, for \( x < 0 \), then \( q(x) \) is uniquely determined by \( r(k) \forall k > 0 \) alone.
1.2.12 The Pompeiu problem, Schiffer’s conjecture, and inverse problem of plasma theory

Let \( f \not\equiv 0 \in L^1_{\text{loc}}(\mathbb{R}^n) \). Assume
\[
\int_D f(gx + y)dx = 0 \quad \forall g \in SO(n) \forall y \in \mathbb{R}^n, \quad n \geq 2, \tag{1.2.4}
\]
where \( SO(n) \) is the group of rotations, and \( D \subset \mathbb{R}^n \) is a bounded domain. The problem (going back to Pompeiu (1929)) is to prove that (1.2.4) implies that \( D \) is a ball. Originally Pompeiu claimed that (1.2.4) implies that \( f = 0 \), but this claim is wrong. References related to this problem are given in [R186], [R177], [Z]. One can prove that (1.2.4) holds iff (if and only if) \( \tilde{\chi}(k\alpha) = 0 \) for all \( \alpha \in S^{n-1} \) and some \( k > 0 \), where \( \tilde{f}(\xi) := \int_D f(x)e^{i\xi \cdot x}dx \), and \( \chi := \begin{cases} 1 & \text{in } D, \\ 0 & \text{in } D \end{cases} \). If \( \tilde{f}(k\alpha) = 0 \) \( \forall \alpha \in S^{n-1} \) and some \( k > 0 \), then the overdetermined problem
\[
\left( \nabla^2 + k^2 \right)u = 1 \text{ in } D, \quad u|_S = u_N|_S = 0, \tag{1.2.5}
\]
has a solution.

The Schiffer’s conjecture is: if \( D \) is a bounded connected domain homeomorphic to a ball, and
\[
\left( \nabla^2 + k^2 \right)u = 0 \text{ in } D, \quad u|_S = 0, \quad u_N|_S = 1, \quad k^2 = \text{const} > 0, \tag{1.2.6}
\]
then \( D \) is a ball.

The Pompeiu problem in the form (1.2.4) is equivalent to the following conjecture: if
\[
\left( \nabla^2 + k^2 \right)u = 0 \text{ in } D, \quad u|_S = 1, \quad u_N|_S = 0, \quad k^2 = \text{const} > 0, \tag{1.2.7}
\]
and \( D \) is homeomorphic to a ball, then \( D \) is a ball.

An inverse problem of plasma theory consists of the following.

Let
\[
\nabla^2 u + f(u) = 0 \text{ in } D, \quad u|_S = 0, \tag{1.2.8}
\]
where \( u \) is a non trivial solution to (1.2.8), (i.e., if \( f(0) = 0 \) then \( u \not\equiv 0 \)), and let the extra data (measured data) be the value \( u_N|_S := h(s), \forall s \in S \). Assume that \( f(u) \) is an entire function of \( u \). The inverse problem, of interest in plasma theory, is: given \( h(s) \) \( \forall s \in S \), can one recover \( f(u) \) uniquely. Even for \( f(u) = \epsilon_0 + \epsilon_1 u, \quad \epsilon_j = \text{const}, \quad j = 0, 1 \), the problem is open (cf [Vog]).
1.2.13 Multidimensional inverse potential scattering

Let

$$\left[\nabla^2 + k^2 - q(x)\right]u = 0 \text{ in } \mathbb{R}^n, \quad n > 2,$$

(1.2.9)

$$u = e^{ik\alpha \cdot x} + A(\alpha', \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right),$$

(1.2.10)

where \(\alpha\) is given, and \(k = \text{cons} t > 0.\) The coefficient \(A(\alpha', \alpha, k)\) is called the scattering amplitude, and the solution to (1.2.9)–(1.2.10) is called the scattering solution. The direct scattering problem is: given \(q, k > 0\) and \(\alpha \in S^{n-1},\) find \(u(x, \alpha, k),\) and, in particular, \(A(\alpha, \alpha, k).\) This problem has been studied in great detail (see, e.g., [CFKS], [[R121], Appendix]) under various assumptions on \(q(x).\) We assume that \(q \in Q = \{q = \bar{q}, \ q(x) = 0 \text{ for } |x| > a, \ q(x) \in L^2(B_a)\}\) and often we assume additionally that \(q \in Q := Q_a \cap L^\infty(\mathbb{R}^n).\)

The inverse scattering problem (ISP) consists of finding \(q(x),\) given \(A(\alpha, \alpha, k).\) Consider several cases:

1. \(A\) is given for all \(\alpha', \alpha \in S^2\) and all \(k > 0,\)
2. \(A\) is given for all \(\alpha', \alpha\) and a fixed \(k = k_0 > 0,\)
3. \(A\) is given for a fixed \(\alpha = \alpha_0\) all \(\alpha' \in S^2\) and all \(k > 0,\)
4. \(A (\alpha, \alpha, k)\) is given for all \(\alpha \in S^2\) and all \(k > 0\) (back scattering data).

In case (1) uniqueness of the solution to ISP has been established long ago, and follows easily from the asymptotics of \(A\) as \(k \to \infty.\) An inversion formula based on high-energy asymptotics of \(A\) is known (Born inversion), (cf [Sai]). In case (2) the uniqueness of the solution to ISP is proved by Ramm [R109], [R100], (see also [R105], [R112], [R114], [R115], [R120], [R125], [R130], [R133], [R140], [R142], [R143], [RSt2], [R203]), an inversion formula for the exact data is derived in [R109], [R143], an inversion formula for the noisy data is derived by in [R143], and stability estimates for the inversion formulas for the exact and noisy data are derived by in [R143], [R203]. In case (3) and (4) uniqueness of the solution to ISP is an open problem, but in the case (4) uniqueness holds if one assumes a priori that \(q\) is sufficiently small. A generic uniqueness result is given in [St1]. See also [StU].

1.2.14 Ground-penetrating radar

Let the source of electromagnetic waves be located above the ground, and the scattered field be observed on the ground. From these data one wants to get information about the properties of the ground. Mathematical modeling of this problem is based on the Maxwell equations

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}, \quad \nabla \times H = \varepsilon \frac{\partial E}{\partial t} + \sigma E + j,$$

(1.2.11)

where \(\mu = \text{cons} t > 0, \varepsilon = \varepsilon(z), \sigma = \sigma(z),\) \(z = x_3\) is the vertical coordinate, \(z > 0\) is the region of the ground, \(j = f(t) \delta(x) \delta(z - z_0)\varepsilon \gamma\) is the source, \(z_0 < 0,\) \(f(t)\)
penetrating radar. The ground-penetrating radar inverse problem is: given 

\( E \)

Here

\[ 1.2.15 \text{ A geometrical inverse problem} \]

\[ \text{Let} \]

\[ w \]

\[ \text{Let} \]

\[ \text{where} \]

\[ \text{describes the shape of the pulse of the current} \]

\[ \gamma = x_2 \text{ axis at the height} \mid z_0 \mid \text{ above the ground. Assume} \]

\[ \epsilon = \epsilon_0, \mu = \mu_0 \text{ for} \]

\[ z < 0 \text{ (in the air),} \]

\[ \epsilon = \epsilon_1 = \text{const,} \mu = \mu_0 \text{ for} \]

\[ z > L, \]

\[ f(t) = 0 \text{ for} t < 0 \text{ and} \]

\[ t > T, \epsilon_0 \text{ and} \mu_0 \text{ are dielectric and magnetic constants,} \epsilon = \epsilon(z), \mu = \mu(z) \]

\[ \text{for} \]

\[ 0 < z < L. \]

\[ \text{Differentiate the second equation} (1.2.1) \]

\[ \text{with respect to} \]

\[ t, \text{and get} \]

\[ -\nabla \times \nabla \times E = \epsilon \mu \frac{\partial E_t}{\partial t} + \sigma \mu E_t + \mu j_t. \]

\[ \text{Let} \]

\[ E = \tilde{E}(x, z, t) e^{i\gamma}, u(z, k, \lambda) := \tilde{\delta}(z, k, \lambda)/(ik\mu h(k)), \]

\[ \tilde{E} := \int_0^\infty dt \int_{-\infty}^\infty dx \tilde{E}(x, z, t)e^{ikt}, \]

\[ \tilde{f}(k) := \int_0^\infty f(t)e^{ikt} dt. \]

\[ \text{Then} \]

\[ u'' - \lambda^2 u + k^2 A^2(z) u + ik B(z) u = -\delta(z-z_0), u(\pm \infty, k, \lambda) = 0, \]

\[ (1.2.12) \]

\[ \text{where} \]

\[ u' = \frac{du}{dz}, \]

\[ A^2(z) := \epsilon(z)\mu, \]

\[ B(z) := \sigma(z)\mu, \]

\[ B(z) = 0 \text{ for} z \notin [0, 1] \]

\[ \text{ground-penetrating radar. The ground-penetrating radar inverse problem is: given} \]

\[ \tilde{\delta}(x, 0, t) \]

\[ \forall x \in \mathbb{R}, \forall t > 0, \text{find} \epsilon(z) \text{ and} \sigma(z). \]

\[ \text{One may use the source} \]

\[ j = f(t)\delta(r-r_0)\delta(z-z_0)e^{i\phi}, \]

\[ \text{which is a current along a loop of wire,} e^{i\phi} \]

\[ \text{is the unit vector in cylindrical coordinates. In this case, one looks for} \]

\[ E \]

\[ \text{of the form:} \]

\[ E = \tilde{\delta}(r, z)e^{i\phi}, \]

\[ \text{and from} (1.2.11) \text{one gets:} \]

\[ A^2(z)\tilde{E}_{tt} + B(z)\tilde{E}_t - \tilde{E}_{zz} - \tilde{E}_z - \frac{1}{r} \tilde{E}_r + \frac{\tilde{\delta}}{r^2} = -\mu f, \delta(t-r_0) \delta(z-z_0), \]

\[ (1.2.13) \]

\[ \tilde{E} := \int_0^\infty \tilde{\delta}(z, z, t)e^{ikt} dt, \]

\[ \text{then} \]

\[ \tilde{\delta}'' + k^2 A^2(z) \tilde{\delta} + ik B(z) \tilde{\delta} + \frac{1}{r} \tilde{\delta}_r - \frac{\tilde{\delta}}{r^2} = -ik \mu h(k) \delta(r-r_0) \delta(z-z_0) \]

\[ \tilde{\delta}' := \frac{d \tilde{\delta}}{dz}, \]

\[ h(k) := \int_0^\infty f(t)e^{ikt} dt. \]

\[ (1.2.14) \]

\[ \text{Let} \]

\[ w := \int_0^\infty \tilde{\delta}(r, z, k) J_1(\lambda r)r dr, \]

\[ \text{where} \]

\[ J_1(r) \]

\[ \text{is the Bessel function. Set} \]

\[ u := \frac{w}{ik(\tilde{\delta}e^{i\phi})}. \]

\[ \text{Then} \]

\[ u \]

\[ \text{solves} (1.2.12), \]

\[ \text{and the inverse problem is the same as above} \]

\[ ([R.185]). \]

\[ 1.2.15 \text{ A geometrical inverse problem} \]

\[ \text{Let} \]

\[ \Delta u = 0 \text{ in} \]

\[ D \subset \mathbb{R}^n, \]

\[ n \geq 2, \]

\[ u|_{S_0} = u_0, \]

\[ u_N|_{S_0} = u_1, \]

\[ |u_0| + |u_1| \neq 0, \]

\[ u|_{S_1} = 0 \text{ or} \]

\[ u_N|_{S_1} = 0. \]

\[ \text{Here} D \]

\[ \text{is domain homeomorphic to an annulus,} S_0 \]

\[ \text{is its inner boundary,} S_1 \]

\[ \text{is its outer boundary. The geometrical inverse problem is: given} u_0, u_1, \text{and} S_0, \text{find} S_1. \]
One can interpret the data as the Cauchy data on \( S_0 \) for an electrostatic potential \( u \), and then \( S_1 \) is the surface on which the potential is vanishing if \( u|_{S_1} = 0 \), or the charge distribution is vanishing if \( u N|_{S_1} = 0 \) ([R139]).

### 1.2.16 Inverse source problems

#### (1) Inverse source problems in acoustics

Let \((\nabla^2 + k^2)u = f\) in \(\mathbb{R}^3\), \(f = 0\) for \(|x| \geq a\), \(u\) satisfies the radiation condition \(\lim_{r \to \infty} r \left( \frac{\partial u}{\partial r} - iku \right) = 0\) uniformly in \(\alpha' := \frac{x}{r}, r = |x|\). Define the radiation pattern \(A\) by the formula: \(u = Ae^{ikr} + o\left(\frac{1}{r}\right)\) as \(r \to \infty\), \(\frac{x}{r} = \alpha'\). Then

\[
A(\alpha', k) := -\frac{1}{4\pi} \int_{B_a} f(y)e^{-ik\alpha' \cdot y} dy, \quad B_a := \{x : |x| \leq a\}. \tag{1.2.15}
\]

The inverse source problems are:

(i) Given \(A(\alpha', k)\) \(\forall \alpha' \in S^2\) and \(\forall k > 0\), find \(f(x)\).

(ii) Given \(A(\alpha', k)\) \(\forall \alpha' \in S^2\) and a fixed \(k = k_0 > 0\), find \(f(x)\).

Clearly, by (1.2.15) problem (i) has at most one solution, but an a priori given function \(A(\alpha', k)\) may be not of the form (1.2.15): the right-hand side of (1.2.15) is an entire function of exponential type of the vector \(\xi := k\alpha\).

Problem (ii), in general, may have many solutions, since \(A(\alpha', k_0)\) may vanish for all \(\alpha' \in S^2\) at some \(k_0 > 0\).

#### (2) Inverse source problem in electrodynamics

Consider Maxwell’s equations (1.2.11) in \(\mathbb{R}^3\), and assume \(j = 0\) for \(|x| \geq a\), The radiation condition for \((E, H)\) is:

\[
E = \frac{e^{ikr}}{r} A(\alpha', k) + o\left(\frac{1}{r}\right), \quad r := |x| \longrightarrow \infty, \quad \alpha' := \frac{x}{r}, \quad k = \omega \sqrt{\varepsilon \mu},
\]

where \(E\) and \(H\) in (1.2.11) are assumed monochromatic with \(e^{-i\omega t}\) time dependence, \(\sigma(x) = 0\), and \(H = \sqrt{\frac{\varepsilon_0}{\mu_0}} [\alpha', E] + o\left(\frac{1}{r}\right)\), where \(\varepsilon_0, \mu_0\) are the constant values of \(\varepsilon\) and \(\mu\) near infinity.

The inverse source problem is: given \(A(\alpha', k)\), find \(j\). Again, one should specify for what \(\alpha'\) and \(k\) the function \(A(\alpha', k)\) is known. One can derive the relation between \(A\) and \(j\). Namely, assuming \(\varepsilon\) and \(\mu\) constants, and \(j\) smooth and compactly supported, one starts with the equations

\[
\nabla \times E = i \omega \mu H, \quad \nabla \times H = -i \omega \varepsilon E + j(x),
\]

then gets

\[
\nabla \times \nabla \times E = k^2 E + \omega \mu j, \quad k^2 = \omega^2 \varepsilon \mu,
\]
Thus

\[ \left( -\Delta - k^2 \right) E = i\omega \mu j - \nabla \nabla \cdot E, \quad \nabla \cdot E = (i\omega \epsilon)^{-1} \nabla \cdot j, \]

so

\[ E = i\omega \mu \int_g (x, y) j \, dy - (i\omega \epsilon)^{-1} \int g \nabla_x (\nabla_y \cdot j) \, dy, \quad g := \frac{e^{ik|x-y|}}{4\pi |x-y|}, \quad \int := \int_{\mathbb{R}^3}. \]

Let \( \gamma := \frac{e^{ikr}}{r} \), and \( J := \frac{1}{4\pi} \int e^{-ik'z} j \, dy \). Then

\[ E = \gamma i\omega \mu j - \frac{1}{i\omega \epsilon} \nabla_x \nabla_y \cdot \gamma J = \gamma i\omega \mu (J - \alpha', \alpha' \cdot J) + o\left( \frac{1}{r} \right). \]

Thus \( E = -\gamma i\omega \mu [\alpha'[\alpha', J]] \), where \([a, b]\) is the vector product. So

\[ A(\alpha', k) = -ik\sqrt{\frac{\mu}{\epsilon}} [\alpha', [\alpha', J]], \quad J := \frac{1}{4\pi} \int e^{-ik\alpha' \cdot y} j(y) \, dy. \tag{1.2.17} \]

and \( J = \alpha'(\alpha', J) + \sqrt{\frac{\epsilon}{\mu ik}} A. \)

It is now clear, that even if \( A \) is known for all \( k > 0 \) and all \( \alpha' \in S^2 \), vector \( J \) is not uniquely determined, but only its component orthogonal to \( \alpha' \) is determined. Therefore, the solution to the inverse source problem in electrodynamics is not unique and may not exist, in general. The antenna synthesis problems are inverse source-type problems of electrodynamics. For example, if \( j \) is the current along a linear antenna (which is a wire along \( x_3 := z \) axis, \( -\ell < z \leq \ell \), \( 2\ell \) is the length of the antenna, \( j = j(z) \delta(x) \delta(y) e_3 \)), then

\[ J = e_3 f := \frac{e_3}{4\pi} \int_{-\ell}^{\ell} e^{-ikz\cos \theta} j(z) \, dz, \quad \cos \theta = e_3 \cdot \alpha' \text{ and} \]

\[ A = -ik\sqrt{\frac{\mu}{\epsilon}} [\alpha', [\alpha', e_3]] f, \tag{1.2.18} \]

so \( f = f(\theta) = \frac{1}{4\pi} \int_{-\ell}^{\ell} e^{-ikz\cos \theta} j(z) \, dz \) is determined uniquely by the data \( A \). Finding \( j(z) \), which produces the desired diagram \( f(\theta) \), is the problem of linear antenna synthesis. There is a large body of literature on this subject.

Let \( u_{tt} - \Delta u = f(x, t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3, \quad u|_{t=0} = u_0(x), \quad u|_{t=0} = u_1(x), \quad u|_{t=T} = v_0(x), \quad u|_{t=T} = 0 \) for \( t \in [0, T] \).

The inverse source problem is: given \( u_0, u_1, v_0, v_1 \) \( \forall x \in \mathbb{R}^3 \) \( \forall t \in \mathbb{R} \), find \( f(x, t), \quad x \in \mathbb{R}^3, \quad t \in [0, T] \).

The questions mentioned in this subsection were discussed in many papers and books ([AVST], [MJ], [ZK], [R11], [R21], [R26], [R27], [R28], [R73], [Is2]).
Clearly, the solution to this problem is not unique: $f$ and $\gamma$ results for PRP.

family $U$ densely defined, linear, operators on a Banach space $X$ located in the region $\phi(x) = f(x)$, given $|F(\xi)|$. Clearly, the solution to this problem is not unique: $f(x)e^{i\phi}$, $\phi \in \mathbb{R}$, produces the same $|F(\xi)|$. Under suitable assumptions on $f(x)$ and $|F(\xi)|$, one can get uniqueness results for PRP.

See [KST], [R.139].

1.2.21 Non-overdetermined inverse problems

Formally we call an inverse problem non-overdetermined if the unknown function, which is to be found, depends on the same number of variables as the data. For example, problems in cases (1) and (2) in Section 1.2.13 are overdetermined, while in cases (3) and (4) they are not overdetermined. In multidimensional inverse scattering problems uniqueness of the solution is an open problem for most of the non-overdetermined