NUMERICAL METHODS IN FINANCE

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NUMERICAL METHODS IN FINANCE

Edited by
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GERAD and HEC Montréal

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Foreword

GERAD celebrates this year its 25th anniversary. The Center was created in 1980 by a small group of professors and researchers of HEC Montréal, McGill University and of the École Polytechnique de Montréal. GERAD's activities achieved sufficient scope to justify its conversion in June 1988 into a Joint Research Centre of HEC Montréal, the École Polytechnique de Montréal and McGill University. In 1996, the Université du Québec à Montréal joined these three institutions. GERAD has fifty members (professors), more than twenty research associates and post doctoral students and more than two hundreds master and Ph.D. students.

GERAD is a multi-university center and a vital forum for the development of operations research. Its mission is defined around the following four complementarily objectives:

- The original and expert contribution to all research fields in GERAD's area of expertise;
- The dissemination of research results in the best scientific outlets as well as in the society in general;
- The training of graduate students and post doctoral researchers;
- The contribution to the economic community by solving important problems and providing transferable tools.

GERAD's research thrusts and fields of expertise are as follows:

- Development of mathematical analysis tools and techniques to solve the complex problems that arise in management sciences and engineering;
- Development of algorithms to resolve such problems efficiently;
- Application of these techniques and tools to problems posed in related disciplines, such as statistics, financial engineering, game theory and artificial intelligence;
- Application of advanced tools to optimization and planning of large technical and economic systems, such as energy systems, transportation/communication networks, and production systems;
- Integration of scientific findings into software, expert systems and decision-support systems that can be used by industry.

One of the marking events of the celebrations of the 25th anniversary of GERAD is the publication of ten volumes covering most of the Center's research areas of expertise. The list follows: **Essays and Surveys in Global Optimization**, edited by C. Audet, P. Hansen and G. Savard; **Graph Theory and Combinatorial Optimization**, edited by D. Avis, A. Hertz and O. Marcotte; **Numerical Methods in Finance**, edited by H. Ben-Ameur and M. Breton; **Analysis, Control and Optimization of Complex Dynamic Systems**, edited by E.K. Boukas and R. Malhamé; **Column Generation**, edited by G. Desaulniers, J. Desrosiers and M.M. Solomon; **Statistical Modeling and Analysis for Complex Data Problems**, edited by P. Duchesne and B. Rémillard; **Performance Evaluation and Planning Methods for the Next Generation Internet**, edited by A. Girard, B. Sansò and F. Vázquez-Abad; **Dynamic Games: Theory and Applications**, edited by A. Haurie and G. Zaccour; **Logistics Systems: Design and Optimization**, edited by A. Langevin and D. Riopel; **Energy and Environment**, edited by R. Loulou, J.-P. Waaub and G. Zaccour.

I would like to express my gratitude to the Editors of the ten volumes, to the authors who accepted with great enthusiasm to submit their work and to the reviewers for their benevolent work and timely response. I would also like to thank Mrs. Nicole Paradis, Francine Benoît and Louise Letendre and Mr. André Montpetit for their excellent editing work.

The GERAD group has earned its reputation as a worldwide leader in its field. This is certainly due to the enthusiasm and motivation of GERAD's researchers and students, but also to the funding and the infrastructures available. I would like to seize the opportunity to thank the organizations that, from the beginning, believed in the potential and the value of GERAD and have supported it over the years. These are HEC Montréal, École Polytechnique de Montréal, McGill University, Université du Québec à Montréal and, of course, the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Fonds québécois de la recherche sur la nature et les technologies (FQRNT).

Georges Zaccour
Director of GERAD

Avant-propos

Le Groupe d'études et de recherche en analyse des décisions (GERAD) fête cette année son vingt-cinquième anniversaire. Fondé en 1980 par une poignée de professeurs et chercheurs de HEC Montréal engagés dans des recherches en équipe avec des collègues de l'Université McGill et de l'École Polytechnique de Montréal, le Centre comporte maintenant une cinquantaine de membres, plus d'une vingtaine de professionnels de recherche et stagiaires post-doctoraux et plus de 200 étudiants des cycles supérieurs. Les activités du GERAD ont pris suffisamment d'ampleur pour justifier en juin 1988 sa transformation en un Centre de recherche conjoint de HEC Montréal, de l'École Polytechnique de Montréal et de l'Université McGill. En 1996, l'Université du Québec à Montréal s'est jointe à ces institutions pour parrainer le GERAD.

Le GERAD est un regroupement de chercheurs autour de la discipline de la recherche opérationnelle. Sa mission s'articule autour des objectifs complémentaires suivants :

- la contribution originale et experte dans tous les axes de recherche de ses champs de compétence ;
- la diffusion des résultats dans les plus grandes revues du domaine ainsi qu'auprès des différents publics qui forment l'environnement du Centre ;
- la formation d'étudiants des cycles supérieurs et de stagiaires post-doctoraux ;
- la contribution à la communauté économique à travers la résolution de problèmes et le développement de coffres d'outils transférables.

Les principaux axes de recherche du GERAD, en allant du plus théorique au plus appliqué, sont les suivants :

- le développement d'outils et de techniques d'analyse mathématiques de la recherche opérationnelle pour la résolution de problèmes complexes qui se posent dans les sciences de la gestion et du génie ;
- la confection d'algorithmes permettant la résolution efficace de ces problèmes ;
- l'application de ces outils à des problèmes posés dans des disciplines connexes à la recherche opérationnelle telles que la statistique, l'ingénierie financière, la théorie des jeux et l'intelligence artificielle ;
- l'application de ces outils à l'optimisation et à la planification de grands systèmes technico-économiques comme les systèmes énergétiques, les réseaux de télécommunication et de transport, la logistique et la distributique dans les industries manufacturières et de service ;

- l'intégration des résultats scientifiques dans des logiciels, des systèmes experts et dans des systèmes d'aide à la décision transférables à l'industrie.

Le fait marquant des célébrations du 25^e du GERAD est la publication de dix volumes couvrant les champs d'expertise du Centre. La liste suit : **Essays and Surveys in Global Optimization**, édité par C. Audet, P. Hansen et G. Savard ; **Graph Theory and Combinatorial Optimization**, édité par D. Avis, A. Hertz et O. Marcotte ; **Numerical Methods in Finance**, édité par H. Ben-Ameur et M. Breton ; **Analysis, Control and Optimization of Complex Dynamic Systems**, édité par E.K. Boukas et R. Malhamé ; **Column Generation**, édité par G. Desaulniers, J. Desrosiers et M.M. Solomon ; **Statistical Modeling and Analysis for Complex Data Problems**, édité par P. Duchesne et B. Rémillard ; **Performance Evaluation and Planning Methods for the Next Generation Internet**, édité par A. Girard, B. Sansò et F. Vázquez-Abad ; **Dynamic Games : Theory and Applications**, édité par A. Haurie et G. Zaccour ; **Logistics Systems : Design and Optimization**, édité par A. Langevin et D. Riopel ; **Energy and Environment**, édité par R. Loulou, J.-P. Waaub et G. Zaccour.

Je voudrais remercier très sincèrement les éditeurs de ces volumes, les nombreux auteurs qui ont très volontiers répondu à l'invitation des éditeurs à soumettre leurs travaux, et les évaluateurs pour leur bénévolat et ponctualité. Je voudrais aussi remercier Mmes Nicole Paradis, Francine Benoît et Louise Letendre ainsi que M. André Montpetit pour leur travail expert d'édition.

La place de premier plan qu'occupe le GERAD sur l'échiquier mondial est certes due à la passion qui anime ses chercheurs et ses étudiants, mais aussi au financement et à l'infrastructure disponibles. Je voudrais profiter de cette occasion pour remercier les organisations qui ont cru dès le départ au potentiel et la valeur du GERAD et nous ont soutenus durant ces années. Il s'agit de HEC Montréal, l'École Polytechnique de Montréal, l'Université McGill, l'Université du Québec à Montréal et, bien sûr, le Conseil de recherche en sciences naturelles et en génie du Canada (CRSNG) et le Fonds québécois de la recherche sur la nature et les technologies (FQRNT).

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Preface

This volume collects twelve chapters dealing with a wide range of topics in numerical finance. It is divided in three parts. The first part contains surveys and tutorial contributions, reviewing the current state of the art in diverse mathematical methods and models and their applications in finance. The second part examines asset pricing, proposing numerical methods and specific applications. Finally, the third part deals with asset portfolios, presenting methods for efficiency testing, performance evaluation and optimal selection, with empirical experiments.

Part I

In Chapter 1, P. François presents a survey of the major models of the structural approach for the valuation of corporate debt in a continuous-time arbitrage-free economy. Numerous models are presented, including endogenous capital structure, discrete coupon payments, flow-based state variables, interest rate risk, strategic debt service and advanced default rules. Finally, the author presents an assessment of the performance of these structural models in capturing the empirical patterns of the term structure of credit spread.

Chapter 2 prepared by D. Dufresne is a concise account of the connection between Bessel processes and the integral of geometric Brownian motion. The main motivation for the study of this integral is the pricing of Asian options. The author reviews the definition and properties of Bessel processes. The expressions for the density function of the integral and of the Laplace transform for Asian option prices are given. Some new derivations and alternative proofs for these results are also presented.

Chapter 3, by J.-P. Aubin, D. Pujal and P. Saint-Pierre, presents the main results of the viability/capturability approach for the valuation and hedging of contingent claims with transaction costs in the tychastic control framework (or dynamic game against nature). A viability/capturability algorithm is proposed, and it is shown that this provides both the value of the contingent claim and the hedging portfolio. An outline of the viability/capturability strategy establishing these results is subsequently provided.

In Chapter 4, P. Bernhard presents an overview of the robust control approach to option pricing and hedging, based on an interval model for security prices. This approach does not assume a probabilistic knowledge of market prices behaviour. The theory developed allows for continuous or discrete trading for hedging options, while taking transaction costs into account. A numerical algorithm implementing the theory and efficiently computing option prices is also provided.

Part II

Chapter 5, by J. de Frutos, presents a finite element method for pricing two-factor bonds with conversion, call and put embedded options. The method decouples the state and temporal discretizations, thus allowing the use of efficient numerical procedures for each one of the decoupled problems. Numerical experiments are presented, showing stability and accuracy.

In Chapter 6, M. Bellalah proposes a finite difference method for the valuation of index options, where the index price volatility has two components, one of which is specific and the other is related to the interest rate volatility. An extension of the Alternating Direction Implicit numerical scheme is proposed. Numerical illustrations are provided, showing the impact of interest rate volatility on early exercise of American options.

Chapter 7 by T. Berrada studies American options with uncertain maturities. The author shows how to use the exercise premium decomposition to value such options by a backward integral equation. Two application examples are presented: real options to invest in projects and employee stock options. Numerical illustrations in both cases show the effect of stochastic maturity on the optimal exercise boundary.

In Chapter 8, E. Clark uses an American option framework to study the expropriation decision by a host country, in order to estimate the expropriation risk in foreign direct investment projects. The model is used to illustrate the impact of incomplete information about the expropriation costs on the valuation of foreign direct investment projects.

Part III

Chapter 9, prepared by M.-C. Beaulieu, J.-M. Dufour and L. Khalaf, propose exact inference procedures for asset pricing models. The statistical approach presented allows for possibly asymmetric, heavy tailed distributions, based on Monte-Carlo test techniques. The methods proposed are applied to a mean-variance efficiency problem using portfolio returns of the NYSE and show significant goodness-of-fit improvement over standard distribution frameworks.

In Chapter 10, M. Ayadi and L. Kryzanowski use a general asset pricing framework to evaluate the performance of actively managed fixed-income mutual fund portfolios. Their approach is independent of asset-pricing models and distributional assumptions. Applying this to Canadian fixed-income mutual funds, they find that the measured unconditional performance of fund managers is negative.

In Chapter 11, P. Boyle and B. Ding propose a linear approximation for the third moment of a portfolio in a mean-absolute deviation-skewness approach for portfolio optimization. Their model can be used to obtain a high skewness and a relatively lower variance, while keeping the expected return fixed, with respect to a base portfolio. The model is then used to analyse the potential for put options to increase the skewness of portfolios. Numerical experiments use historical data from the Toronto Stock Exchange.

Chapter 12, by N. Gülpınar and B. Rustem, presents a continuous min-max approach for single-period portfolio selection in a mean-variance context. The optimization is performed assuming a range of expected returns and various covariance scenarios. The optimal investment strategy is robust in the sense that it has the best lower bound performance. Computational experiments using historical prices of FTSE stocks are provided, and illustrate the robustness of the min-max strategy.

Acknowledgements

The Editors would like to express their gratitude to the authors for their contributions and timely responses to their comments and suggestions. They also wish to thank Francine Benoît, André Montpetit and Nicole Paradis for their expert editing of the volume.

HATEM BEN-AMEUR
MICHÈLE BRETON

Chapter 1

CORPORATE DEBT VALUATION: THE STRUCTURAL APPROACH

Pascal François

Abstract This chapter surveys the contingent claims literature on the valuation of corporate debt. Model summaries are presented in a continuous-time arbitrage-free economy. After a review of the basic model, I extend the approach to models with an endogenous capital structure, discrete coupon payments, flow-based state variables, interest rate risk, strategic debt service, and more advanced default rules. Finally, I assess the empirical performance of structural models in light of the latest tests available.

1. Introduction

The purpose of this chapter is to review the structural models for valuing corporate straight debt. Beyond the scope of this survey are the reduced-form models of credit risk[1] as well as the structural models for vulnerable securities and for risky bonds with option-like provisions.[2] Earlier reviews of this literature may be found in Cooper and Martin (1996); Bielecki and Rutkowski (2002) and Lando (2004). This survey covers several topics that were previously hardly surveyed (in particular Sections 5, 7, 8, and 9). Model summaries are presented in a continuous-time arbitrage-free economy. Adaptations to the binomial setting may be found in Garbade (2001).

In Section 2, I present the basic model (valuation of finite-maturity corporate debt with a continuous coupon and an exogenous default

[1] See for instance Jarrow and Turnbull (1995); Jarrow et al. (1997); Duffie and Singleton (1999) or Madan and Unal (2000).

[2] See, e.g., Klein (1996); Rich (1996), and Cao and Wei (2001) for vulnerable options, Ho and Singer (1984) for bonds with a sinking-fund provision, Ingersoll (1977) and Brennan and Schwartz (1980) for convertibles, and Acharya and Carpenter (2002) for callables.

threshold). Then I extend the approach to models with an endogenous capital structure (Section 3), discrete coupon payments (Section 4), flow-based state variables (Section 5), interest rate risk (Section 6), strategic debt service (Section 7), and more advanced default rules (Section 8). In Section 9, I discuss the empirical efficacy of structural models measured by their ability to reproduce observed patterns of term structure of credit spreads. I conclude in Section 10.

2. The basic model

2.1 Contingent claims pricing assumptions

Throughout I consider a firm with equity and debt outstanding. This version of the basic model was initially derived by Merton (1974) in the set-up defined by Black and Scholes (1973). It relies on the following assumptions

1 The assets of the firm are continuously traded in an arbitrage-free and complete market. Uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ stands for the historical probability measure. From Harrison and Pliska (1981) we have that there exists a unique probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which asset prices discounted at the risk-free rate are martingales.

2 The term structure of interest rates is flat. The constant r denotes the instantaneous risk-free rate (this assumption is relaxed in Section 6).

3 Once debt is issued, the capital structure of the firm remains unchanged (this assumption is relaxed in Section 3.3).

4 The value of the firm assets $V(t)$ is independent of the firm capital structure and, under $\mathbb{Q}$, it is driven by the geometric Brownian motion

$$\frac{dV(t)}{V(t)} = (r - \delta)\, dt + \sigma\, dz_t$$

where δ and σ are two constants and $(z_t)_{t\geq 0}$ is a standard Brownian motion. This equation states that the instantaneous return on the firm assets is r and that a proportion δ of assets is continuously paid out to claimholders. Firm business risk is captured by $(z_t)_{t\geq 0}$, and the risk-neutral firm profitability is Gaussian with mean r and standard deviation σ. Other possible state variables are examined in Section 5. Other dynamics for $V(t)$ are possible,[3] but the pricing technique remains the same.

[3]Mason and Bhattacharya (1981) postulate a pure jump process for the value of assets. Zhou (2001a) investigates the jump-diffusion case.

Absent market frictions such as taxes, bankruptcy costs or informational asymmetry costs, assumption 4 is consistent with the Modigliani–Miller paradigm. In this framework, the value of the firm assets is identical to the total value of the firm and Merton (1977) shows that capital structure irrelevance still holds in the presence of costless default risk. This setup can however be extended to situations where optimal debt level matters. In that case, the total value of the firm is $V(t)$ net of the present value of market frictions.

The debt contract is a bond with nominal M and maturity T (possibly infinite) paying a continuous coupon c. Let $D(t, V)$ denote the value of the bond. According to the structural approach of credit risk, $D(t, V)$ is a claim contingent to the value of the firm assets. In the absence of arbitrage, it verifies

$$rD\,dt = c\,dt + E_{\mathbb{Q}}(dD)$$

where $E_{\mathbb{Q}}(\cdot)$ denotes the expectation operator under $\mathbb{Q}$. Using Itô's lemma, we obtain the following PDE for D

$$rD = c + (r - \delta)VD_V + \tfrac{1}{2}\sigma^2 V^2 D_{VV} + D_t \qquad (1.1)$$

where D_x stands for the partial derivative of D with respect to x.

To account for the presence of default risk in corporate debt contracts, two types of boundary conditions are typically attached to the former PDE. The first condition ensures that in case of no default, the debtholder receives the contractual payments. Let T_d denote the random default date. The no-default condition associated to the debt contract defined above may be written as

$$D(T, V) = M \cdot 1_{T_d > T},$$

where 1_ω stands for the indicator function of the event ω.

The second condition characterizes default. This event is fully described by its timing and its magnitude. In the structural approach, the timing of default is modeled as the *first hitting time* of the state variable to a given level. Let $V_d(t)$ denote the default threshold. The default date T_d may be written as

$$T_d = \inf\{t \geq 0 : V(t) = V_d(t)\}.$$

The magnitude of default represents the loss in debt value following the default event. Formally, we have that

$$D(T_d, V_d) = \Psi(V_d)$$

where $\Psi(\cdot)$ is the function relating the remaining debt value to the firm asset value at the time of default.

2.2 Default magnitude

The function $\Psi(\cdot)$ depends on three key factors:

1. The nature of the claim held by debtholders after default. If default leads to immediate liquidation, the remaining assets of the firm are sold and debtholders share the proceeds. In that case debt value may be considered as a fraction of $V_d(t)$, where the proportional loss reflects the discount caused by fire asset sales and/or by the inefficient piecewise reallocation of assets.[4] If default leads to the firm reorganization, the debtholders obtain a new claim whose value may be defined as a fraction of the initially promised nominal M (aka the recovery rate) or as a fraction of the equivalent risk-free bond with same nominal and maturity. Altman and Kishore (1996) provide extensive evidence on recovery rates.
2. The total costs associated to the event of default. One can distinguish direct costs (induced by the procedure resolving financial distress) from indirect costs (induced by foregone investment opportunities). Again, if default is assumed to lead to immediate liquidation, it is convenient to express these costs as a fraction of the remaining assets.[5]
3. In case default is resolved through the legal bankruptcy procedure, the *absolute priority rule* (APR) states that debtholders have highest priority to recover their claims. In practice however, equityholders may bypass debtholders and perceive some of the proceeds of the firm liquidation. Franks and Torous (1989) and Eberhart et al. (1990) provide evidence of very frequent (but relatively small) deviations from the APR in the US bankruptcy procedure.

To account for all these factors, we denote by α the total proportional costs of default and by γ the proportional deviation from the APR (calculated from the value of remaining assets *net of default costs*).

---

[4] Liquidation costs may be calculated as the firm's going concern value minus its liquidation value, divided by its going concern value. Using this definition, Alderson and Betker (1995) and Gilson (1997) report liquidation costs equal to 36.5% and 45.5% for the median firm in their samples.

[5] Empirical studies by Warner (1977); Weiss (1990), and Betker (1997) report costs of financial distress between 3% and 7.5% of firm value one year before default. Bris et al. (2004) find that bankruptcy costs are very heterogeneous and sensitive to measurement method.

2.3 Exogenous default threshold

Firm asset value follows a geometric Brownian motion and can therefore be written as

$$V(t) = V \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right)t + \sigma z_t\right].$$

The default threshold under consideration is exogenous with exponential shape $V_d(t) = V_d \exp(\lambda t)$ and terminal point $V_d(T) = M$. Default occurs the first time before T we have

$$z_t = \frac{1}{\sigma} \ln \frac{V_d}{V} - \left(\frac{r - \delta - \lambda}{\sigma} - \frac{\sigma}{2}\right)t,$$

or otherwise if

$$z_T = \frac{1}{\sigma} \ln \frac{M}{V} - \left(\frac{r - \delta}{\sigma} - \frac{\sigma}{2}\right)T.$$

Knowing the distribution of $(z_t)_{t \geq 0}$, one obtains the following result.

PROPOSITION 1.1 *Consider a corporate bond with maturity T, nominal M and continuous coupon c. The issuer is a firm whose asset value follow a geometric Brownian motion with volatility σ. The default threshold starts at V_d, grows exponentially at rate λ and jumps at level M upon maturity. In case of default, a fraction α of remaining assets is lost as third party costs and an additional fraction γ accrues to equityholders. Initial bond value is given by*

$$\begin{aligned}
D = {} & M e^{-rT}\left[\Phi(d_1) - \left(\frac{V_d}{V}\right)^{2R/\sigma^2 - 1} \Phi(d_2)\right] + \chi V e^{-\delta T}\big(\Phi(d_3) - \Phi(d_4)\big) \\
& + \chi V e^{-\delta T}\left(\frac{V_d}{V}\right)^{2R/\sigma^2 + 1} \big(\Phi(d_5) - \Phi(d_6)\big) \\
& + \chi V \left[\left(\frac{V_d}{V}\right)^{(R+\sigma^2/2+\rho)/\sigma^2} \Phi(d_7) + \left(\frac{V_d}{V}\right)^{(R+\sigma^2/2-\rho)/\sigma^2} \Phi(d_8)\right] \\
& + \frac{c}{r}\left[1 - \left(\frac{V_d}{V}\right)^{(R-\sigma^2/2+\rho)/\sigma^2} \Phi(d_7) - \left(\frac{V_d}{V}\right)^{(R-\sigma^2/2-\rho)/\sigma^2} \Phi(d_8)\right] \\
& - \frac{c}{r} e^{-rT}\left[\Phi(d_9) - \left(\frac{V_d}{V}\right)^{2R/\sigma^2 - 1} \Phi(d_{10})\right],
\end{aligned}$$

where δ is the firm payout rate, r is the constant risk-free rate and

$$R = r - \delta - \lambda$$

$$\rho = \sqrt{\left(r - \delta - \lambda - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}$$

$$\chi = (1-\gamma)(1-\alpha)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{V}{M} + \left(r - \delta - \frac{\sigma^2}{2}\right)T\right) \qquad d_6 = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{V_d}{V} + \left(R + \frac{\sigma^2}{2}\right)T\right)$$

$$d_2 = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{V_d^2}{MV} + \left(R - \frac{\sigma^2}{2}\right)T\right) \qquad d_7 = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{V_d}{V} + \rho T\right)$$

$$d_3 = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{V}{V_d} + \left(R + \frac{\sigma^2}{2}\right)T\right) \qquad d_8 = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{V_d}{V} - \rho T\right)$$

$$d_4 = d_1 + \sigma\sqrt{T} \qquad d_9 = d_3 - \sigma\sqrt{T}$$

$$d_5 = d_2 + \sigma\sqrt{T} \qquad d_{10} = d_6 - \sigma\sqrt{T}$$

and $\Phi(\cdot)$ is the cumulative normal distribution function.

Proposition 1.1 embeds as special cases the pricing formulae by Black and Cox (1976) (when $c = 0$ and $\chi = 1$, that is a discount bond with no costs of financial distress nor deviations from the APR), by Leland and Toft (1996) (when the exogenous default threshold is a constant ($\lambda = 0$)), and by Merton (1974) (the exogenous default threshold is zero).

From the Feynman–Kac representation theorem, Proposition 1.1 (and subsequent results) may either be obtained by solving PDE (1.1) with appropriate boundary conditions, or by applying the martingale property of discounted prices under the risk-neutral probability measure. Ericsson and Reneby (1998) emphasize the modularity of the latter methodology. The time-t value $x(t)$ of any claim on V promising a single payoff at date T can be written as

$$x(t) = E_{\mathbb{Q}}\left(x(T)\exp\left(-\int_t^T r(u)\,du\right)\right).$$

Corporate debt can then be decomposed into such claims that are valued as building blocks of the whole contract.

3. Debt pricing and capital structure

Since the structural approach links the value of corporate securities to an economic fundamental related to firm value, it has by construction a balance-sheet view of the firm and is therefore well suited to connect the issue of pricing risky debt to the capital structure decision. This connection provides a natural way to endogenize the decision to default: The optimal amount of debt is chosen in order to maximize the value of the firm, and, based on this amount, shareholders select the default threshold that maximizes equity value.

3.1 Infinite maturity debt

The default threshold V_d can be endogenized as shareholders' choice to maximize equity value. If debt is a perpetuity, the PDE for D can be written as

$$rD = c + (r - \delta)VD_V + \tfrac{1}{2}\sigma^2V^2D_{VV} \tag{1.2}$$

with boundary conditions:

1 When $V = V_d$, the firm is immediately liquidated[6] and creditors take possession of the residual assets net of costs of default and deviations from the APR

$$D(V_d) = (1 - \gamma)(1 - \alpha)V_d,$$

2 As $V \to \infty$, debt value converges to that of the risk-free perpetuity

$$\lim_{V \to +\infty} D(V) = \frac{c}{r}.$$

The PDE (1.2) with the above conditions admits the following closed-form solution

$$D(V) = \frac{c}{r} + \left[(1 - \gamma)(1 - \alpha)V_d - \frac{c}{r}\right]\left(\frac{V_d}{V}\right)^{\xi}$$

with

$$\xi = \frac{r - \delta - \sigma^2/2}{\sigma^2} + \sqrt{\left(\frac{r - \delta - \sigma^2/2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

Equity value, denoted by $S(V)$, is now determined as the residual claim value on the firm, i.e.,

$$S(V) = v(V) - D(V)$$

where $v(V)$ denotes the firm value.

Leland (1994) proposes to rely on the static trade-off capital structure theory to determine firm value. In this framework, v equals the value of

[6] In practice, resolution of financial distress may take on several forms other than liquidation. In Section 8, we study other types of default rules.

the firm's assets (V) plus the tax advantage of debt ($\mathrm{TB}(V)$) minus the present value of bankruptcy costs ($\mathrm{BC}(V)$). Both $\mathrm{TB}(V)$ and $\mathrm{BC}(V)$ obey the same PDE (1.2) and their corresponding boundary conditions are respectively:

$$\mathrm{TB}(V_d) = 0 \qquad \lim_{V\to\infty} \mathrm{TB}(V) = \tau\frac{c}{r},$$
$$\mathrm{BC}(V_d) = \alpha V_d \qquad \lim_{V\to\infty} \mathrm{BC}(V) = 0,$$

where τ stands for the corporate tax rate.

Solving for $\mathrm{TB}(V)$ and $\mathrm{BC}(V)$ yields firm value and equity value is given by

$$S(V) = V - (1-\tau)\frac{c}{r} + \left[(1-\tau)\frac{c}{r} - \big(1-\gamma(1-\alpha)\big)V_d\right]\left(\frac{V_d}{V}\right)^{\xi}.$$

Shareholders' optimal default rule is then obtained using the following smooth pasting condition:

$$\left.\frac{\partial S}{\partial V}\right|_{V=V_d} = \gamma(1-\alpha),$$

which yields

$$V_d = \frac{\xi}{(\xi+1)}\frac{(1-\tau)c}{[1-\gamma(1-\alpha)]r}.$$

The endogenous default threshold is interpreted as the value of the option to wait for defaulting $\big(\xi/(\xi+1)\big)$ times the opportunity cost of servicing the debt.

3.2 Finite maturity debt with stationary capital structure

Leland and Toft (1996) examine a firm with a debt service that is invariant through time, which allows for a constant default threshold. The firm constant debt level is M. For each period, M/T units of bonds are issued with maturity T while a fraction M/T of former bonds is reimbursed. This roll over strategy maintains the debt service at a constant level $C + M/T$ where C denotes the sum of all coupons.

The value of a *single* bond issue with nominal m and continuous coupon c is given by (for clarity of exposition, we set $\gamma = 0$):

$$d(V, V_d, T) = \int_0^T e^{-rs}c\big(1-F(s)\big)\,ds + e^{-rT}m\big(1-F(T)\big)$$
$$+ \int_0^T e^{-rs}(1-\alpha)V_d f(s)\,ds,$$

where $f(t)$ and $F(t)$ stand for the density and the cumulative distribution function of the default date T_d respectively.

From Proposition 1.1, we get

$$d(V, V_d, T) = \frac{c}{r} + \left((1-\alpha)V_d - \frac{c}{r}\right)\left(\frac{V_d}{V}\right)^{(r-\delta-\sigma^2/2+\rho)/\sigma^2} \Phi(d_7) + \left((1-\alpha)V_d - \frac{c}{r}\right)\left(\frac{V_d}{V}\right)^{(r-\delta-\sigma^2/2-\rho)/\sigma^2} \Phi(d_8) + \left(m - \frac{c}{r}\right)e^{-rT}\left[\Phi(d_9) - \left(\frac{V_d}{V}\right)^{2(r-\delta)/\sigma^2-1} \Phi(d_{10})\right].$$

Total debt is the sum of all bond issues with nominal $M = mT$ and coupon $C = cT$. Its value $D(V, V_d, T)$ is given by

$$D(V, V_d, T) = \int_0^T d(V, V_d, t)\, dt,$$

and Leland and Toft (1996) obtain

$$D(V, V_d, T) = \frac{C}{r} + \left(M - \frac{C}{r}\right)\left(\frac{1-e^{-rT}}{rT} - I(T)\right) + \left((1-\alpha)V_d - \frac{C}{r}\right)J(T)$$

with

$$I(T) = \frac{1}{rT}\left[\left(\frac{V_d}{V}\right)^{(r-\delta-\sigma^2/2+\rho)/\sigma^2} \Phi(d_7) + \left(\frac{V_d}{V}\right)^{(r-\delta-\sigma^2/2-\rho)/\sigma^2} \Phi(d_8)\right] - \frac{e^{-rT}}{rT}\left[\Phi(-d_9) + \left(\frac{V_d}{V}\right)^{2(r-\delta)/\sigma^2-1} \Phi(d_{10})\right],$$

and

$$J(T) = \frac{\sigma}{\rho\sqrt{T}}\left(\frac{V_d}{V}\right)^{(r-\delta-\sigma^2/2+\rho)/\sigma^2} \Phi(d_7)d_7 - \frac{\sigma}{\rho\sqrt{T}}\left(\frac{V_d}{V}\right)^{(r-\delta-\sigma^2/2-\rho)/\sigma^2} \Phi(d_8)d_8.$$

Equity value, $S(V, V_d, T)$, is again obtained as the difference between firm value and total debt value. Since capital structure is stationary, the tax advantage of debt as well as the present value of bankruptcy costs are computed over an infinite horizon, that is they both obey PDE (1.2). Which yields

$$S(V,V_d,T) = V + \tau\frac{C}{r}\left[1-\left(\frac{V_d}{V}\right)^{\xi}\right] - \alpha V_d\left(\frac{V_d}{V}\right)^{\xi} - D(V,V_d,T).$$

The smooth pasting condition on $S(V,V_d,T)$ yields the endogenous default threshold

$$V_d = \frac{C(A/rT - B)/r - AM/rT - \tau C\xi/r}{1+\alpha\xi-(1-\alpha)B}$$

with

$$\begin{aligned}A = &\left[\frac{2(r-\delta)}{\sigma^2}-1\right]e^{-rT}\Phi\left(\frac{(r-\delta)}{\sigma}\sqrt{T}-\frac{\sigma}{2}\sqrt{T}\right)-2\frac{\rho}{\sigma^2}\Phi\left(\frac{\rho}{\sigma}\sqrt{T}\right)\\ &-\frac{2}{\sigma\sqrt{T}}\phi\left(\frac{\rho}{\sigma}\sqrt{T}\right)+\frac{2e^{-rT}}{\sigma\sqrt{T}}\phi\left(\frac{(r-\delta)}{\sigma}\sqrt{T}-\frac{\sigma}{2}\sqrt{T}\right)\\ &+\frac{\rho}{\sigma^2}-\frac{2(r-\delta)}{\sigma^2}+1,\\ B = &-\left(\frac{2\rho}{\sigma^2}+\frac{2}{\rho T}\right)\Phi\left(\frac{\rho}{\sigma}\sqrt{T}\right)-\frac{2}{\sigma\sqrt{T}}\phi\left(\frac{\rho}{\sigma}\sqrt{T}\right)+\frac{\rho}{\sigma^2}\\ &-\frac{2(r-\delta)}{\sigma^2}+1+\frac{1}{\rho T},\end{aligned}$$

where $\phi(\cdot)$ denotes the normal density function.

3.3 Dynamic capital structure

In models presented in Sections 3.1 and 3.2, the optimal capital structure is determined at initial date and the level of debt is not changed subsequently. In practice, firms have the flexibility to adjust their level of debt to current economic conditions. In the Fischer et al. (1989) model, the value of firm assets V is assumed to follow a geometric Brownian motion and, for a fixed face value of debt M, so does the value-to-debt ratio $y = V/M$. Debt value D and equity value S obey a PDE similar to (1.2) adjusted for a simple tax regime where τ_c is the corporate tax rate and τ_p is the tax rate on income revenues, that is

$$\begin{aligned}r(1-\tau_p)D &= \widehat{\mu}yD_y + \tfrac{1}{2}\sigma^2y^2D_{yy} + (1-\tau_p)iM\\ r(1-\tau_p)S &= \widehat{\mu}yS_y + \tfrac{1}{2}\sigma^2y^2S_{yy} - (1-\tau_c)iM,\end{aligned}$$

where $\widehat{\mu}$ stands for the risk-adjusted expected return on the firm's assets (yet to be characterized).

The firm may recapitalize and issue additional debt when its value-to-debt ratio reaches an upper bound $\overline{y}$. Recapitalization induces a proportional cost k, hence firm value must verify

$$v(\overline{y}, M) = v\left(y_0, \frac{\overline{y}}{y_0}M\right) - k\frac{\overline{y}}{y_0}M,$$

where y_0 stands for the initial value-to-debt ratio. Similarly, the firm may reduce its level of debt when its value-to-debt ratio reaches an lower bound $\underline{y}$. However, this debt reduction is possible provided the firm is not already in bankruptcy. Denoting by α the proportional bankruptcy costs, the value of the firm at the lower recapitalization level $v\left(\underline{y}, M\right)$ is given by

$$\begin{cases} \max\left[v\left(y_0, \dfrac{\underline{y}}{y_0}M\right) - k\dfrac{\underline{y}}{y_0}M - \alpha M, 0\right], \\ \qquad\qquad\qquad\qquad \text{if } v\left(y_0, \dfrac{\underline{y}}{y_0}M\right) - k\dfrac{\underline{y}}{y_0}M < M, \\ v\left(y_0, \dfrac{\underline{y}}{y_0}M\right) - k\dfrac{\underline{y}}{y_0}M, \quad \text{otherwise.} \end{cases}$$

In the absence of arbitrage, firm value just after recapitalization equals the value of assets plus recapitalization costs, hence

$$v(y, M) = yM + kM.$$

In particular, at the recapitalization bounds, this yields

$$\begin{aligned} v\left(y_0, \frac{\overline{y}}{y_0}M\right) &= \overline{y}M + k\frac{\overline{y}}{y_0}M \\ v\left(y_0, \frac{\underline{y}}{y_0}M\right) &= \underline{y}M + k\frac{\underline{y}}{y_0}M. \end{aligned}$$

Combining with the expressions for $v(\overline{y}, M)$ and $v(\underline{y}, M)$, we get

$$\begin{aligned} v(\overline{y}, M) &= \overline{y}M \\ v(\underline{y}, M) &= \begin{cases} \max[(\underline{y} - \alpha)M, 0], & \text{if } \underline{y} < 1 \\ \underline{y}M, & \text{otherwise.} \end{cases} \end{aligned}$$

Debt value is retrieved as the difference between firm value and equity value. Assuming debt is issued and callable at par, this yields

$$\begin{aligned} D(\overline{y}, M) &= M \\ D(\underline{y}, M) &= \begin{cases} \max[(\underline{y} - \alpha)M, 0], & \text{if } \underline{y} < 1, \\ M, & \text{otherwise,} \end{cases} \end{aligned}$$

and these expressions are used as boundary conditions to solve the PDE for debt value. Fischer et al. (1989) obtain

$$D(y,M) = D_1 y^{m_1} + D_2 y^{m_2} + \frac{iM}{r},$$

where

$$\begin{aligned}
D_1 &= \frac{M}{\Delta}\left\{\left(1-\frac{i}{r}\right)\underline{y}^{m_2} - \left[(\underline{y}-\alpha)^+ - \frac{i}{r}\right]\overline{y}^{m_2}\right\}\\
D_2 &= \frac{M}{\Delta}\left\{\left[(\underline{y}-\alpha)^+ - \frac{i}{r}\right]\overline{y}^{m_1} - \left(1-\frac{i}{r}\right)\underline{y}^{m_1}\right\}\\
\Delta &= \overline{y}^{m_1}\underline{y}^{m_2} - \overline{y}^{m_2}\underline{y}^{m_1}\\
m_1 &= \frac{1}{2} - \frac{\widehat{\mu}}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\widehat{\mu}}{\sigma^2}\right)^2 + \frac{2r(1-\tau_p)}{\sigma^2}}\\
m_2 &= \frac{1}{2} - \frac{\widehat{\mu}}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\widehat{\mu}}{\sigma^2}\right)^2 + \frac{2r(1-\tau_p)}{\sigma^2}}.
\end{aligned}$$

To characterize the optimal recapitalization policy, Fischer et al. (1989) define the advantage of leverage as

$$\delta = r(1-\tau_p) - \widehat{\mu}.$$

The equilibrium is found by maximizing firm value net of recapitalization costs, that is

$$\max_{\overline{y},\underline{y},M,i} v(y_0, M, \overline{y}, \underline{y}) - kM$$

subject to

$$\begin{aligned}
v(y_0, M, \overline{y}, \underline{y}) &= y_0 M + kM\\
\left.\frac{\partial S(y, M, \overline{y}, \underline{y})}{\partial y}\right|_{y=\underline{y}} &\geq 0\\
S(y_0, M, \overline{y}, \underline{y}) &= M
\end{aligned}$$

The first condition is a no-arbitrage condition, the second one is the smooth-pasting condition preserving the limited liability property of equity, and the third one states that debt is initially issued at par. Solving this program yields the initial optimal leverage (M), the optimal recapitalization policy ($\overline{y}$ and $\underline{y}$) as well as the risk-adjusted expected return on the firm's assets $\widehat{\mu}$ and the coupon rate i.

The basic model is extended in several directions. Leland (1998) examines the case of finite-maturity debt in a framework similar to that of

Leland and Toft (1996) with a possibility to call the debt at some upper boundary for asset value (downside restructuring is not addressed). Goldstein et al. (2001) and Dangl and Zechner (2004) also value corporate debt within a dynamic capital structure model. Because they use a different underlying state variable, we shall review their approach in Section 5. Ju et al. (2003) build a model of dynamic recapitalization within the static trade-off capital structure framework (i.e., the optimal amount of debt results from trading off the tax advantage with expected bankruptcy costs) at the cost of assuming an exogenous exponential default boundary.

4. Discrete coupon payments

In practice, coupons are paid annually or semi-annually and the continuous coupon assumption may not be appropriate. Geske (1977) extends the basic model to the case of a discrete coupon-bearing debt. Debt service is a sequence of coupon payments $\{C_{t_i}, i = 1, \ldots, n\}$ to be paid at date t_i (with $t_n = T$). At date t_{n-1}, debt is zero-coupon and may be priced with Merton's (1974) formula:

$$D(t_{n-1}) = C_{t_n} e^{-r(t_n - t_{n-1})}\Phi(d_1) + V(t_{n-1})e^{-\delta(t_n - t_{n-1})}[1 - \Phi(d_4)].$$

At date t_{n-2}, there are two debt payments remaining. If $V(t_{n-1}) > V_d(t_{n-1})$, debtholders receive $C_{t_{n-1}} + D(t_{n-1})$. Otherwise, they get the residual value of assets $V_b(t_{n-1})$ (for simplicity, we set $\alpha = 0$). Which yields

$$\begin{aligned} D(t_{n-2}) = C_{t_n} e^{-r(t_n - t_{n-2})}\Phi_2(h_{n-1}, h_n, \theta) + C_{t_{n-1}} e^{-r(t_{n-1} - t_{n-2})}\Phi(h_{n-1}) \\ + [1 - \Phi_2(h_{n-1} + \sigma\sqrt{t_n - t_{n-1}}, h_n + \sigma\sqrt{t_n - t_{n-2}}, \theta)] \\ \times V(t_{n-2})e^{-\delta(t_n - t_{n-2})} \end{aligned}$$

with

$$\begin{aligned} h_n &= \frac{1}{\sigma\sqrt{T - t_{n-2}}}\left[\ln\frac{V(t_{n-2})}{C_{t_n}} + \left(r - \delta - \frac{\sigma^2}{2}\right)(T - t_{n-2})\right] \\ h_{n-1} &= \frac{1}{\sigma\sqrt{T - t_{n-1}}}\left[\ln\frac{V(t_{n-2})}{V_d(t_{n-1})} + \left(r - \delta - \frac{\sigma^2}{2}\right)(T - t_{n-1})\right] \\ \theta &= \sqrt{\frac{t_{n-1} - t_{n-2}}{T - t_{n-2}}} \end{aligned}$$

and $\Phi_2(\cdot)$ stands for the bivariate cumulative normal distribution function.