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Combinatorics and Reasoning

Representing, Justifying and Building Isomorphisms
This book is dedicated to the Kenilworth students who participated in the longitudinal study and from whom we continue to learn so much. We thank you for your continuing commitment, abundant trust, and generous sharing of how mathematical ideas and ways of reasoning are built.
Preface

Our research project on mathematical learning focuses on the accomplishments of a cohort group of learners from first grade through high school and beyond, concentrating on their work on a set of combinatorics tasks. We describe their impressive mathematical achievements over these years. We illustrate in detail the processes by which students learn to justify solutions to combinatorics problems that were challenging for their age and grade level. Based on transcribed video data and learners’ inscriptions, we provide a careful and detailed analysis of the process by which mathematical ideas are developed, discussed, modified, expanded, and justified.

Our work underscores the power of attending to basic ideas in building arguments; it shows the importance of providing opportunities for the co-construction of knowledge by groups of learners; and it demonstrates the value of careful construction of appropriate tasks. Moreover, it documents how reasoning that takes the form of proof evolves with young children and it discusses the conditions for supporting student reasoning.

We present in this book strong and compelling evidence that under appropriate conditions and with minimal intervention, learners can develop sophisticated ideas about proof and justification, generalization, isomorphism, and mathematical reasoning at an early age and can continue to refine and expand those ideas over time, developing increasingly sophisticated presentations and representations. We also describe an extension of this work with groups of undergraduate students, noting similarities and differences between the reasoning of the original cohort group of younger students and that of the college students.

We include a detailed discussion of all the mathematical tasks, which can be used in classrooms from elementary school to the graduate college level.
Acknowledgements

We are deeply grateful for the many colleagues who have made this book possible and would like to acknowledge their contributions.

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Introduction

Carolyn A. Maher, Arthur B. Powell, and Elizabeth B. Uptegrove

The theoretical foundation for the program of research on which this book is based comes from recognition that individual learning takes place within a community. The members of that community have access to and are influenced by the ideas of others. Individual learners are interconnected with other members of the community; engagement with others opens up possibilities for sharing and comparing representations of ideas and for revising existing schemes and building new ones. In the activity of problem solving, learners bring forth, communicate, and compare ideas. They explore whether the ways that others represent ideas correspond with their own representations, thereby extending their personal repertoires of tools for dealing with new ideas. In this way further learning takes place and understanding deepens (Davis & Maher, 1997; Maher, Martino, & Alston, 1993; Maher & Davis, 1990).

The data for this book come from a long-term program of research detailing the collective building of mathematical ideas, which we call the longitudinal study. In this book, we explore student work in one of the mathematics strands of the longitudinal study: counting and combinatorics. It investigates how students’ reasoning evolved from elementary and high school years to college.

The reasoning of learners is documented by their actions – that is, what they do, say, build, and write – as they work on strands of tasks. In studying how participants make sense of the complexity of problems, we trace the representations they share, the heuristics they invent and apply, and the modifications they make in building arguments and in offering justifications for solutions.

The authors of the constituent 17 chapters relate how an ordinary group of school children manifest over a 12-year period an extraordinary array of mathematical ideas that they discursively build and how – with time – their ideas modify and mature as they reason and justify their ideas. The book reports episodes from a long-term study of how mathematical ideas and ways of reasoning are built by students over time. The study has produced over 4,500 h of video, over several sites, involving far more
data than can be presented here. However, we have selected narratives that feature the voices of several children, as interpreted by a variety of researchers, to weave together a bigger story about how students can educate us about the multifaceted nature of mathematical development. In an important sense, the really big story is still being written as our work in preserving and further analyzing those 4,500 h of video through the Video Mosaic Collaborative continues to reveal new narratives. We invite readers to view the videos at http://www.video-mosaic.org/. Along with the narratives offered in the book chapters that follow, these videos enable readers to trace in detail the development of counting/combinatorics ideas and ways of reasoning of learners over more than a decade. Thus, while only some of the children’s voices appear in this book, we are indebted to all of them for sharing their developing mathematical ideas over time and in divergent contexts, which we continue to study and consider how these children’s extraordinary mathematical reasoning may inspire the fields of mathematics education, teacher education, and the learning sciences.

To structure a story that emerges from the chapters, the editors have divided this book into four parts. The two chapters of the first part, respectively, provide historical background of the research study from which the details of the later chapters emerge and describe the design of the study. The first chapter describes the study and the purpose of the research, how the study began, and the conditions under which the research was conducted. It also briefly describes the mathematical ideas and ways of reasoning that emerged from the study. (The details are presented in later chapters.) The second chapter presents the method of the study, its design, including selection of participants, data collection, and analysis, as well as the strand of tasks on which participants were invited to work. The chapter also discusses the importance of the task design for helping learners to develop ways of reasoning.

The second part of the book contains five chapters. These chapters chronicle the work of the study’s participants over a 7-year period from grades 2–8, tracing the development of their mathematical ideas, heuristics, and forms of reasoning. In particular, the reader will learn how the participating children represented their ideas; developed schemes and strategies; reasoned in specific ways; built inductive arguments; reasoned by cases and by recursion; and connected numbers in Pascal’s Triangle to results of previous problems. The authors of Chapter 3 discuss how young children use representations to express their mathematical ideas while building a solution to a particular counting problem (the shirts and jeans

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1 The Video Mosaic Collaborative is a research and development project sponsored by the National Science Foundation (award DRL-0822204) directed by C.A. Maher, G. Agnew, C.E. Hmelo-Silver, and M.F. Palius that is leveraging the Rutgers Community Repository to preserve the unique video collection amassed by The Robert B. Davis Institute for Learning at Rutgers University through two decades of research with over four millions dollars of grant funding from the NSF (awards MDR-9053597, REC-9814846, REC-0309062 and DRL-0723475). In addition to preserving the video collection, new tools are being developed for conducting design research and an empirical study that use the videos in the context of teacher education. The editors gratefully acknowledge this considerable support from the National Science Foundation and wish to clarify that all views expressed in this book are those of the authors are not necessarily those of the NSF.
problem, described fully in Appendix A, along with all combinatorics problems discussed herein). They show how children structure their representations in response to requests to justify their problem solution and build convincing arguments to early counting problems. The authors of Chapter 4 and 5 discuss students’ work on different versions of the towers problems (which involve determining how many towers can be built of various heights when selecting from cubes of various numbers of colors). They show the emergence of different forms of reasoning (cases, contradiction, recursion, and induction) and how, motivated by the need to find the sample space for a basic probability exploration, students revisit the inductive argument for building towers. Chapter 6 discusses how participants collaboratively build representations that help them use reasoning by cases and by recursion to develop justifications for their solutions to classes of pizza problems. (Pizza problems involve determining how many pizzas it is possible to make when selecting from various numbers of toppings and under various other constraints.) Completing this part of the book, Chapter 7 presents the results of an interview with 13-year-old Stephanie, who discusses the relationship between the towers problems and the binomial expansion, including how the towers answers can be found in Pascal’s Triangle.

The six chapters of the book’s third part closely examine the mathematical work of the research participants during their high school years. It shows how the students built important connections using sophisticated mathematical reasoning. In these chapters, the story revolves around the students’ proof making, use of representations, acquisition of standard notation, and forging of conceptual connections among isomorphic problems. Specifically, Chapter 8 shows that as they revisit their representations and arguments, students refine representations and clarify arguments. In Chapter 9, students working in groups on towers problems are seen to find and generalize formulas, using methods including controlling for variables, justification by cases, and induction. Chapter 10 shows how a tenth-grade student’s binary notation helped his group form connections among the pizza and towers problems, the binomial expansion, and Pascal’s Triangle.

Chapter 11 details how representations are a source for making connections in solutions to pizza and tower problems, resulting in the students mapping the structure of the solution of these problems to Pascal’s Triangle and how their increasingly sophisticated use of representations led to further development of mathematical reasoning and justification. Chapter 12 discusses how students moved from personal to standard notations in order to express in general form their understanding of solutions to the pizza and towers problems and to extend their understanding in creating an isomorphism from the numerical results in those problems to Pascal’s Triangle. The chapter also shows how the students’ understanding of extensions of the pizza and tower problems led to their understanding of the addition rule for Pascal’s Triangle. The final chapter of Part III, Chapter 13, reveals how as high school seniors, days before graduation, the students used their understanding of relationships between the pizza and tower problems and Pascal’s Triangle to solve a third isomorphic problem – the Taxicab Problem. (This problem involves finding the number of routes from the starting point – the taxicab stand – to various points on
a rectangular grid.) They recognized the isomorphism, used it to make conjectures about the new problem, saw the need to prove their conjectures, and provided a convincing argument. This chapter concludes by examining some of the extraordinary mathematical accomplishments of the cohort group of students.

The last part of the book, consisting of four chapters, takes stock and looks forward. Chapter 14 examines the epistemological growth of the students, viewed from their own perspectives. Students’ reflections on their learning over the years challenges common views about student engagement in learning, and gives insight into how students view their own sense making in doing mathematics.

Chapter 15 examines a different student population – college undergraduates – and their work with the set of combinatorics problems. The chapter shows that when adult college students are asked to justify ideas and make convincing arguments, an understanding of mathematical reasoning, proof, and generalization can emerge. In Chapter 16, Glass compares the strategies developed by children and older learners for solving the combinatorics problems and discusses the implications for adult learning.

In closing, Chapter 17 presents the epistemological and methodological contributions of the book. We argue that students must be actively and purposely engaged in their learning so as to take ownership and be proud of their accomplishments. Mathematics educators and teachers need to create opportunities for students to engage in ways similar to those described in this book. We have shown that in a program of carefully selected tasks, with minimal intervention by educators who pay careful attention to students’ arguments and justifications, students can perform mathematically at high levels. In addition to developing mathematical competency, students who participated in the study gained confidence and a sense of empowerment and were successful in their career choices. They learned to trust their own mathematical ability and they did not rely on outside authority for validation. This confidence, sense of empowerment and propensity to reason carefully has been carried over outside their mathematical work; these students found that the knowledge and ways of working that they gained through their participation in the longitudinal study continues to help them in many other areas of study and employment.
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Part I

Introduction, Background, and Methodology
Chapter 1
The Longitudinal Study

Carolyn A. Maher

1.1 Theoretical View

Where do new ideas come from? Our view is that building new ideas is a process; new ideas come from old ideas that are revisited, reviewed, extended, and connected (Davis, 1984; Maher & Davis, 1995). Building new ideas also involves the retrieval and modification of representations of existing ideas. The representations that a learner builds for a mathematical idea or procedure can take different forms – physical objects or actions on objects, words, and symbols, for example. As the learner’s experience increases, old representations become elaborated, extended, and linked to new ones (Maher, 2008; Davis & Maher, 1997).

The problem tasks that are posed to learners are critical to their learning (Francisco & Maher, 2005); they should be well defined, open-ended, and open to extension and generalization. The connections that the learner makes when analyzing and developing solutions to these problems provide further opportunity for growth in knowledge. Students are encouraged to revisit earlier problems because requirements to justify and generalize solutions can help students to see underlying mathematical structure. It is a widely accepted view that when learners understand the fundamental structure of a subject, the gap between “elementary” and “advanced” knowledge is reduced (Bruner, 1960). There is increasing evidence that learners, under certain conditions, can build meaningful, mathematical relationships and understand the structure of mathematical problems at an early age. For example, a study of Norwegian children indicated that even as young as Grade 3, learners are able to unearth the underlying structure of the mathematics of problem tasks (Torkildsen, 2006).

A central component of the learning process is encouraging students to communicate their ideas. Sfard (2001) suggests that students learn to think mathematically...
by participating in discourse about ideas – arguing, asking questions, and anticipating feedback. We have emphasized that justifying ideas in problem solving is an essential component of mathematical reasoning (Maher, 2002, 2005, 2008; Maher & Martino, 1996a; Martino & Maher, 1999). Learners, in communicating their ideas, share personal mental images – representations. When students make their representations public, they have an opportunity to talk further about them, compare them, and later revisit them. Similarities and differences in ideas naturally emerge. When learners try to convince others that their answers are correct, they can reorganize and reformulate their representations so as to make convincing arguments. In summary, students learn mathematics by engaging in the process of building their own personal representations, communicating them as ideas, and then providing support for those ideas by reorganizing and restructuring representations. Our view is that this process is a necessary prerequisite both for developing the idea of mathematical proof and for making suitable connections between problems of equivalent structure by building isomorphisms.

In this book, we discuss how a group of students developed new and increasing sophisticated mathematical ideas by revisiting, reviewing, extending, and connecting old ideas that they had begun developing in first grade. They developed and modified representations that became increasingly elaborated and extended. They participated in serious mathematical discourse. And ultimately they built a strong and durable understanding of the solutions to a set of mathematical tasks. Our longitudinal work is important because it reveals the processes that these learners used to build structural understanding of solutions to mathematical tasks.

1.2 Background of the Study

The longitudinal study began in 1987 in Kenilworth, New Jersey. This was during a time when behaviorism mainly governed mathematics instruction. It was a time before the reform movement in the United States emphasizing conceptual understanding had made its entry. The K-8 Harding Elementary School in the working-class community of Kenilworth, New Jersey, was typical of others at that time. Half-hour sessions were devoted to mathematics, and mathematics instruction was mainly rote. The rule was drill and practice for carrying out memorized procedures. For the most part, even the brightest students from the school did not excel when they moved on to high school mathematics classes, only doing average work. Most members of the community and most teachers had rather low expectations for student advancement.

But Fred Rica, principal of the Harding Elementary School, had higher expectations for the students in his school. Formerly an elementary grade classroom teacher in Kenilworth, Fred Rica knew his staff and students well. Like other concerned educators, he knew when the system was not serving its student population. He turned to Rutgers University for help with instruction, first in language and literacy and then in mathematics. It was shortly after this professional development work that Fred Rica and Carolyn Maher created a partnership between the Kenilworth Public
Schools and Rutgers University. It should be noted that the Rutgers–Kenilworth partnership, with its focus on students building meaning of mathematical ideas and working collaboratively with each other, began long before the National Council of Teachers of Mathematics published its reform standards.

Initially, the project began as a teacher development intervention in mathematics. The Rutgers University team of researchers and graduate students worked for 3 years to help teachers build an understanding of the mathematics they were expected to teach and to learn to be attentive to the developing understanding of their students. (See Davis & Maher, 1993; O’Brien, 1994, for a detailed study of the teacher development project.)

The project could not have survived the early years without the full support and active participation of the Kenilworth school administration. In particular, principal Rica actively participated in the teacher-training sessions, encouraged teachers to become involved, and made sure that students who were involved in the study were available to the researchers. Original financial support for the partnership came from the Kenilworth school district and through volunteer efforts of the Rutgers team. The Kenilworth school district continued to fund the study for several years as a component of its mathematics teacher development mission. The Rutgers research group received outside funding for the research from two National Science Foundation grants. The first grant awarded to Principal Investigators Robert B. Davis and Carolyn A. Maher was when the students were in Grade 4; the second grant awarded to Principal Investigator Carolyn A. Maher was when students were in high school.

1.2.1 Teacher Development Component

It is not surprising that the teachers at the Harding Elementary School were not prepared to teach mathematics with understanding. What is surprising was the expectation of principal Fred Rica that the teachers were capable, with some professional development and classroom support, of understanding the mathematics they were expected to teach. In fact, this view was remarkable for its time.

The teacher development team was made up of mathematics education doctoral students who had considerable experience in schools; its first members were Alice S. Alston and Judith H. Landis. The team worked closely with Fred Rica and his teachers to establish a program of activities that involved not only videotaped teacher workshops and classroom sessions, but also study of those workshops and sessions. The Rutgers team worked directly with students and with their teachers, first observing classroom sessions and later collaborating with the teachers in the design and implementation of lessons. Alice Alston also worked in the classrooms alongside the teachers.

Principal Rica obtained school funding to support teachers’ summer work to revise the existing curriculum. Two years of summer professional development assisted by John O’Brien and Alice Alston resulted in a movement from a “drill and kill” approach to one in which students’ building of mathematical understanding
was central. Curriculum revisions included the use of more engaging and thoughtful lessons for the students and the introduction of manipulatives that allowed students to build models of their solutions.

Some Kenilworth teachers who participated in the teacher development programs also became involved in classroom action research. As teachers were introduced to new resources and tools, they developed new units and piloted them during the school year. Through course work opportunities at Rutgers, some teachers studied the mathematical learning of their own students (Landis & Maher, 1989; Landis, 1990; Maher 1988; O’Brien, 1994).

1.2.2 Intervention Design

The Rutgers team was interested in what mathematical concepts students could learn with minimal intervention from teachers. Classrooms were organized so that children might work together and collaborate on problem tasks. Children were encouraged to use each other as resources in their investigations, to construct models of solutions with available tools, and to revisit tasks and discuss their strategies and solutions. An important observation during the first 3 years was that students produced arguments that took on a variety of forms of reasoning to support their solutions to the problems. By Grade 4, it became increasingly clear to researchers that students’ reasoning, in a natural way, took the form of proof. Children began their investigations by searching for patterns, organizing solutions, searching for completeness, deriving strategies for keeping track and checking, and then reorganizing justifications into arguments that were proof-like in structure. Using each other as resources, children freely shared ideas, questioned each other, argued about the reasonableness of ideas, and became comfortable in sharing and communicating with each other.

What encouraged both the school staff and the university collaborators was the enthusiastic feedback from students. The children enjoyed talking about their ideas; they engaged with each other with energy and enthusiasm, becoming increasingly more comfortable making their ideas public. Their way of working underscored a demand for sense making, which then evolved as a cultural norm.

This book explores student work for one of the mathematics strands of the longitudinal study: counting and combinatorics. It investigates how students’ reasoning evolved over the course of the longitudinal study that continued from elementary and high school years to college.

1.3 Longitudinal Study: Grades 1–3

In order to study the effectiveness of the intervention, the Rutgers team decided to follow a class of students throughout their elementary grades as they worked on mathematical investigations that were not part of the school curriculum. The study began with a class of 18 first-grade students from the Harding School. These children, randomly assigned to one of three first grades, became the initial focus
group; they were together for Grades 1–3 as part of the school design. Throughout
the study, students engaged in strands of thoughtful mathematics activities designed
by the researchers. Although the mathematical investigations were not part of the
curriculum, the concepts that were introduced would later become part of the regular
school mathematics curriculum.

1.4 Longitudinal Study, Grades 4–8

After Grade 3, the students were distributed among different classrooms, accord-
ing to school policy. However, the principal worked with Rutgers researchers to
facilitate maintenance of a focus group of 12 students for research purposes. When
families moved and new families entered the district, the composition of the focus
group changed, but an attempt was made to maintain a group of comparable back-
ground and interest. Although some students stayed with the study from the start
(and are still in touch today), some students moved from the district and new stu-
dents joined. During middle school, the school arranged for the cohort group to
continue working with researchers during school hours, 4–6 times a year in two
90-min sessions and one 45-min session each time.

1.5 Longitudinal Study: High School Years

In 1996 the high school in Kenilworth was closed, as the school district became
part of a regional system. The community joined forces to protest the merger and
succeeded after 1 year. Hence, the first year of high school (ninth grade) proved
disruptive for the students, although some math problem-solving sessions were
conducted with small groups of students during that year in local homes, usu-
ally on Saturdays. After Kenilworth de-regionalized and the students returned to
Kenilworth for the remaining 3 years of high school, groups of students resumed
participation in the longitudinal study in informal, after-school sessions that were
held during the year, usually on Friday. While students no longer met with
researchers during regular class hours, 14 students (some from the original group
of first graders and others who had joined the study at various times during middle
school and high school) made time in their schedules to meet after school about
4–6 times a year for problem-solving sessions that lasted 1–2 h or longer. This
group included ten students who had been with the study since Grade 1, two stu-
dents who had joined the study in Grade 6, and two who joined in high school
(Grade 11).

1.6 Longitudinal Study: Beyond High School

All students in the focus group applied to Rutgers University, and all were
accepted – a remarkable achievement for the district. However, not all students
attended Rutgers; they attended a variety of universities, public and private; besides
Rutgers, these included Cornell University, Kean University, St. John’s University, and the University of Pennsylvania. Majors included accounting, American studies, animal science, computer science, criminal justice, economics, engineering, English, and mathematics. All are now either employed or in graduate school.

Some of the students have continued to meet occasionally with researchers during and after college. They do not generally work on problems (although sometimes old problems are revisited), but they talk about how being in the study has affected them, their academic careers, and their future plans.

In the next chapter, we detail how the study was conducted and we discuss selected problems that formed the cornerstone of the student investigations over the years.
2.1 Introduction

In this chapter, we discuss how data were collected and analyzed, and we briefly describe some results, which will be more fully explored in later chapters. We summarize student work on fundamental problems and note how this work led to exceptional growth in the students’ mathematical understanding.

Researchers (professors at the Rutgers University Graduate School of Education and their students) conducted all problem-solving sessions with the students; the sessions were always videotaped with one or more cameras. Researchers observed, described, and coded the videotape data, and they kept written and electronic files of the emerging theoretic, analytic, and interpretative ideas about the students’ mathematical behaviors. Researchers paid careful attention to children’s use of inscriptions, the connections they made between and among codes, and their emerging and extended ideas and ways of reasoning. Critical events in children’s reasoning were flagged and transcribed and transcripts were coded according to the research questions. The connected series of events that formed a trace led to the emergence of a narrative (Maher & Martino, 1996a; Powell, Francisco, & Maher, 2003).

The videotapes, researcher notes, and student notes did not capture every interaction or every case of student learning. Some students sat silently during discussions; but they had quietly absorbed a problem or quietly developed a solution that came to light some time later in a different situation. Therefore, although we can make inferences about what is observed, we cannot assume that a student who is quiet does not understand.

By videotaping children as they worked together on mathematical tasks over long periods of time, we were able to trace the origin and development of their mathematical ideas. We observed what children said to one another and showed to one another. We used videotapes and transcripts to study the meanings that children gave
to mathematical situations and to note the different representations they made public. A detailed analysis of data made it possible to trace the origin and evolution of children’s arguments. Our data indicate how children expressed their ideas through spoken and written language, through the physical models they built, through the drawings and diagrams they made, and through the mathematical notations they invented.

2.2 Theoretical Perspectives

Guiding our work is the view that children come to mathematical investigations with theories they can modify and refine. We observe them do so in settings that combine personal exploration and suitable social interaction. The theories we consider can include criteria to decide (1) what, at some given moment, needs to be investigated, (2) how to conduct such an investigation, (3) what key features need to be explored in detail, (4) when useful progress has been made, and, given such progress, (5) if further investigation might be needed. We have found that theories of this kind often empower striking and effective ways for children to work conceptually with mathematical ideas, often using concrete objects as specific anchors for their thinking.

2.3 Selected Problems

Mathematics arose from the need to count, measure, and calculate, but the discipline evolved to include abstraction, logical reasoning, and the search for and analysis of patterns. Good mathematical problems are therefore those which give rise to the need for abstraction, systematization, and pattern recognition. A focus of the study was therefore to select problems that would give rise to these needs.

Another focus of the longitudinal study was on doing problems that were not part of the regular curriculum, because it was important for the students to come to the problems fresh, without pre-taught algorithms. A major strand of the longitudinal study therefore consisted of problems in combinatorics, because in working on these problems, students can find the need to organize their work systematically, look for patterns, and generalize their findings; also, counting problems were at the time outside the regular elementary school curriculum and therefore unfamiliar to students. In addition, these problems lend themselves to the use of multiple personal representations that can be shared. Freudenthal (1991) cites the study of combinatorics as “a most important matter for reinvention” (p. 53), specifically because combinatorics can be learned through paradigmatic examples and because problems in combinatorics give rise to the need for convincing proof, including mathematical induction.

Another purpose of the longitudinal study was to provide an environment in which certain socio-mathematical norms could be established to elicit in children