Preface

Vladimir Drinfeld’s many profound contributions to mathematics reflect breadth and great originality. The ten research articles in this volume, covering a diversity of topics predominantly in algebra and number theory, reflect Drinfeld’s vision in significant areas of mathematics, and are dedicated to him on the occasion of his 50th birthday.

The paper by Goncharov and Fock is devoted to the study of cluster varieties and their quantizations. This subject has its origins in the work of Fomin and Zelevinsky on cluster algebras and total positivity on the one hand, and, on the other hand, on various attempts to understand Kashiwara’s theory of crystals and quantizations of moduli spaces of curves.

Starting with a split semisimple real Lie group $G$ with trivial center, Goncharov and Fock define a family of varieties with additional structures called cluster $\mathcal{X}$-varieties. These varieties have a natural Poisson structure. The authors define a Poisson map from a cluster variety to the group $G$ equipped with the standard Poisson–Lie structure as defined by V. Drinfeld. The map is birational and thus provides $G$ with canonical rational coordinates. Further, Goncharov and Fock show how to construct complicated cluster $\mathcal{X}$-varieties from more elementary ones using an amalgamation procedure. This is used, in particular, to produce canonical (Darboux) coordinates for the Poisson structure on a Zariski open subset of the group $G$.

Some of the cluster varieties are very closely related to the double Bruhat cells studied by A. Berenshtein, S. Fomin, and A. Zelevinisky. On the other hand, the results of the paper play a key role in describing the cluster structure of the moduli spaces of local systems on surfaces, as studied by Goncharov and Fock in an earlier work.

The important role of Drinfeld’s ideas—indeed, one of the central themes of his research—is evident in the paper by Frenkel and Gaitsgory, which is devoted to the (local) geometric Langlands correspondence from the point of view of $D$-modules and the representation theory of affine Kac–Moody algebras.

Let $\mathfrak{g}$ be a simple complex Lie algebra and $G$ a connected algebraic group with Lie algebra $\mathfrak{g}$. The affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the universal central extension of the formal loop algebra $\mathfrak{g}(\!(t)\!)$. Representations of $\widehat{\mathfrak{g}}$ have a parameter, an invariant bilinear form on $\mathfrak{g}$, which is called the level. Representations corresponding to the bilinear form that is equal to minus one-half of the Killing form are called
representations of critical level. Such representations can be realized in spaces of global sections of twisted $D$-modules on the quotient of the loop group $G((t))$ by its “compact” subgroup $K$ equal to $G[[t]]$, or to the Iwahori subgroup $I$.

This work by Frenkel and Gaitsgory is the first in a series of papers devoted to the study of the categories of representations of the affine Kac–Moody algebra $\hat{g}$ of the critical level and $D$-modules on $G((t))/K$ from the point of view of a geometric version of the local Langlands correspondence. The local Langlands correspondence sets up a relation between two different types of data. Roughly speaking, the first data consist of the equivalence classes of homomorphisms from the Galois group of a local non-archimedean field $\mathbb{K}$ to $G(\mathbb{C})^\vee$, the Langlands dual group of $G$. The second data consist of the isomorphism classes of irreducible smooth representations, denoted by $\pi$, of the locally compact group $G(\mathbb{K})$.

A naive analogue of this correspondence in the geometric situation seeks to assign to a $G(\mathbb{C})^\vee$-local system on the formal punctured disc a representation of the formal loop group $G((t))$. However, the authors show that in contrast to the classical setting, this representation of $G((t))$ should be defined not on a vector space, but on a category, as explained in the paper.

In the contribution by Ihara, and in the closely related appendix by Tsfasman, the authors study the $\zeta$-function $\zeta_{\mathbb{K}}(s)$ of a global field $\mathbb{K}$. Specifically, they are interested in the so-called Euler–Kronecker constant $\gamma_{\mathbb{K}}$, a real number attached to the power series expansion of the $\zeta$-function at the point $s = 1$. In the special case of the field $\mathbb{K} = \mathbb{Q}$ of rational numbers, this constant reduces to the Euler constant

$$\gamma_{\mathbb{Q}} = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right).$$

The constant $\gamma_{\mathbb{K}}$ plays an important role in analytic number theory. On the other hand, for $\mathbb{K} = \mathbb{F}_q(X)$, the field of rational functions on a complete algebraic curve $X$ over a finite field, the corresponding Euler–Kronecker constant is closely related to the number of $\mathbb{F}_q$-rational points of $X$.

Ihara addresses the question of how negative the constant $\gamma_{\mathbb{K}}$ may be, depending on the field $\mathbb{K}$. In the number field case, this happens when $\mathbb{K}$ has many primes with small norm. In the function field case, there are known towers of curves with many $\mathbb{F}_q$-rational points; the author studies the behavior of $\gamma_{\mathbb{K}}$ using the generalized Riemann hypothesis. In this way, he obtains very interesting explicit estimates of $\gamma_{\mathbb{K}}$. For instance, in the case $\mathbb{K} = \mathbb{F}_q(X)$ Ihara establishes an upper bound

$$\gamma_{\mathbb{K}} \leq 2 \log((g - 1) \log q) + \log q,$$

where $g$ denotes the genus of the curve $X$. He also obtains similar estimates for the lower bound.

Hrushovski and Kazhdan in their paper lay the foundations of integration theory over, not necessarily locally compact, valued fields of residue characteristic zero. A valued field is a field $\mathbb{K}$ equipped with a “ring of integers” $\mathfrak{o} \subseteq \mathbb{K}$, satisfying the property that $\mathbb{K} = \mathfrak{o} \cup (\mathfrak{o} \setminus \{0\})^{-1}$. In particular, the authors obtain new and base-field independent foundations for integration over local fields of large residue characteristic, extending results of Denef, Loeser, and Cluckers.
The work of Hrushovski and Kazhdan is on the border of logic and algebraic geometry. Their methods involve an analysis of definable sets. Specifically, they obtain a precise description of the Grothendieck semigroup of definable sets in terms of related groups over the residue field and value group. This yields new invariants of all definable bijections, as well as invariants of measure-preserving bijections. Their results are intended to be applied to the construction of Hecke algebras associated with reductive groups over a not necessarily locally compact valued field. In the case of a two-dimensional local field, the corresponding Hecke algebra is expected to be closely related to the double affine Hecke algebra introduced by Cherednik.

Kisin’s paper is devoted to $p$-adic algebraic geometry and number theory. This subject is rapidly developing at this point in time. Following the ideas of Berger and Breuil, Kisin gives a new classification of crystalline representations. The objects involved may be viewed as local, characteristic 0 analogues of the “shtukas” introduced by Drinfeld. Kisin also gives a classification of $p$-divisible groups and finite flat group schemes, conjectured by Breuil. Furthermore, he shows that a crystalline representation with Hodge–Tate weights 0, 1 arises from a $p$-divisible group—a result conjectured by Fontaine.

Let $k$ be a perfect field of characteristic $p > 0$, $W = W(k)$ its ring of Witt vectors, $K_0 = W(k)[\frac{1}{p}]$, and $K : K_0$ a finite totally ramified extension. Breuil proposed a new classification of $p$-divisible groups and finite flat group schemes over the ring of integers $O_K$ of $K$. For $p$-divisible groups and $p > 2$, this classification was established in an earlier paper by Kisin, who also used a variant of Breuil’s theory to describe flat deformation rings, and thereby establish a modularity lifting theorem for Barsotti–Tate Galois representations.

In the present paper, the author generalizes Breuil’s theory to describe crystalline representations of higher weight or, equivalently, their associated weakly admissible modules.

Krichever’s paper analyzes deep and important relations between the theory of integrable systems and the Riemann–Schottky problem. The Riemann–Schottky problem on the characterization of the Jacobians of curves among abelian varieties is more than 120 years old. Quite a few geometrical characterizations of the Jacobians have been found. None of them, however, provides an explicit system of equations for the image of the Jacobian locus in the projective space under the level-2 theta imbedding.

The link of this problem to integrable systems was first discovered in the 1980s. Specifically, T. Shiota established the first effective solution of the Riemann–Schottky problem, known as Novikov’s conjecture. The conjecture says the following: An indecomposable principally polarized abelian variety $(X, \vartheta)$ is the Jacobian of a curve of a genus $g$ if and only if there exist $g$-dimensional vectors $U \neq 0, V, W$ such that the function

$$u(x, y, t) = -2\varphi_x^2 \ln \theta(Ux + V y + W t + Z)$$

is a solution of the Kadomtsev–Petviashvili (KP) equation

$$3u_{yy} = (4u_t + 6uu_x - u_{xxx})_x.$$
(Here $\theta(Z) = \theta(Z|B)$ is the Riemann theta function.)

In the present paper, Krichever proves that an indecomposable principally polarized abelian variety $X$ is the Jacobian of a curve if and only if there exist vectors $U \neq 0, V$ such that the roots $x_i(y)$ of the theta functional equation $\theta(Ux + Vy + Z) = 0$ satisfy the equations of motion of the formal infinite-dimensional Calogero–Moser system.

The main goal of Laumon’s paper is to identify the fibers of the affine Springer resolution for the group $\text{GL}_n$ with coverings of compactified Jacobians of projective singular curves. This work is part of a more general project of obtaining a geometric version of the “Fundamental Lemma” that appears in Langlands’ works on automorphic forms.

Let $F$ be a local non-archimedean field of equal characteristic, let $\mathcal{O}_F$ be its ring of integers, and let $k$ be the residue field. Let $E$ be a finite-dimensional $F$-vector space. The author considers the affine Grassmannian formed by $\mathcal{O}_F$-lattices $M$ in $E$.

Given a regular semisimple and topologically nilpotent endomorphism $\gamma$ of $E$, one defines the affine Springer fiber, $X_\gamma$, as the closed reduced subscheme of the affine Grassmannian formed by the $\gamma$-stable lattices $M \subset E$. Kazhdan and Lusztig have shown that $X_\gamma$ is a scheme, locally of finite type over $k$. Moreover, this scheme comes equipped with a natural free action of an abelian algebraic group $\Lambda_\gamma$ such that the quotient $Z_\gamma = X_\gamma/\Lambda_\gamma$ is a projective $k$-scheme.

In his paper, the author attaches to $\gamma$ a projective algebraic curve $C_\gamma$ over $k$ with a single singular point such that the completed local ring at this point is isomorphic to $\mathcal{O}_F[\gamma] \subset F[\gamma] \subset \text{Aut}_F(E)$. Furthermore, the author relates the varieties $X_\gamma$ and $Z_\gamma$ with the compactified Jacobian of the curve $C_\gamma$. This allows him to reprove some irreducibility results about compactified Jacobians due to Altman and Kleiman. In addition, the techniques developed in the paper provide an approach to an important “purity conjecture” concerning the cohomology of certain affine Springer fibers, due to Goresky, Kottwitz, and MacPherson.

The goal of the work of Manin presented in this volume is to study properties of the iterated integrals of modular forms in the upper half-plane. This setting generalizes simultaneously the theory of modular symbols and that of multiple zeta values. Multiple zeta values are the numbers given by the $k$-multiple Dirichlet series

$$\zeta(m_1, \ldots, m_k) = \sum_{0 < n_1 < \cdots < n_k} \frac{1}{n_1^{m_1} \cdots n_k^{m_k}} \quad (0.1)$$

or, equivalently, by the $m$-multiple iterated integrals $m = m_1 + \cdots + m_k$,

$$\zeta(m_1, \ldots, m_k) = \int_0^1 \frac{dz_1}{z_1} \int_0^{z_1} \frac{dz_2}{z_2} \int_0^{z_2} \cdots \int_0^{z_{m_k-1}} \frac{dz_{m_k}}{1 - z_{m_k}} \cdots \quad (0.2)$$

Multiple zeta values are interesting because they and their generalizations appear in many different contexts involving mixed Tate motives, deformation quantization (Kontsevich), knot invariants, etc.

Multiple zeta values satisfy certain combinatorial relations, called double-shuffle relations. The relations in question can be succinctly written in terms of formal
generating series for (regularized) iterated integrals (0.2). Such integrals appeared more than 15 years ago in the celebrated work by Drinfeld on what is nowadays known as the Drinfeld associator. However, the question about interdependence of (double-) shuffle and associator relations does not seem to be settled at the moment.

In the paper, the author defines 1-forms of modular and cusp modular type and studies iterated integrals and the total Mellin transform for families of such forms. The functional equation for the total Mellin transform is deduced. This result extends the classical functional equation for $L$ series. The author also introduces an iterated modular symbol as a certain noncommutative 1-cohomology class of the relevant subgroup of the modular group. The paper establishes some analogues of the classical identity (0.1) = (0.2) but different from it in two essential respects. First, iterated integrals are only linear combinations of certain multiple Dirichlet series. Second, the identities obtained in the paper involve integrals which are not of the usual type, \[
\sum_{0 < n_1 < \ldots < n_k} \frac{a_{1,n_1} \cdots a_{n,n_k}}{n_1^{m_1} \cdots n_k^{m_k}},
\]
in fact, their coefficients depend on pairwise differences $n_j - n_i$.

In the paper by Eskin and Okounkov, the authors prove that natural generating functions for enumeration of branched coverings of the pillowcase orbifold are level-2 quasimodular forms. This gives us a way to compute the volumes of the strata of the moduli space of quadratic differentials.

Consider a complex torus $T^2 = \mathbb{C}/L$, where $L \subset \mathbb{C}$ is a lattice. Its quotient

$$P = T^2/\pm 1$$

by the automorphism $z \mapsto -z$ is a sphere with four $(\mathbb{Z}/2)$-orbifold points, which is sometimes called the pillowcase orbifold. The map $T^2 \to P$ is essentially the Weierstraß $\wp$-function. The quadratic differential $(dz)^2$ on $T^2$ descends to a quadratic differential on $P$. Viewed as a quadratic differential on the Riemann sphere, $(dz)^2$ has simple poles at corner points.

Let $\mu$ be a partition and $\nu$ a partition of an even number into odd parts. The authors are interested in enumeration of degree $2d$ maps

$$\pi : C \to P$$

with the following ramification data. Viewed as a map to the sphere, $\pi$ has profile $(\nu, 2^{d-|\nu|/2})$ over $0 \in P$ and profile $(2^d)$ over the other three corners of $P$. Additionally, $\pi$ has the profile $(\mu_i, 1^{2d-\mu_i})$ over some $\ell(\mu)$ given points of $P$ and unramified elsewhere. Here $\ell(\mu)$ is the number of parts in $\mu$.

The paper by Schechtman may be viewed as a continuation of the work by Gorbunov, Malikov, and Schechtman on the chiral de Rham complex. Specifically, the paper in the volume introduces a certain chiral analogue of the third Chern–Simons class of a vector bundle.
respectively, the Chern–Simons classes
\[ c_i^{CS}(E) \in H^i(X, \Omega^i_X) \],
where
\[ \Omega^{i,2i-1}_X = \sigma_{i \leq 2i-1} \Omega^{i}_X \rightarrow \cdots \rightarrow \Omega^{2i-1,cl}_X \],
and where \( \Omega^{i,cl}_X \subset \Omega^i_X \) stands for the subsheaf of closed forms. These classes are related via the canonical morphism
\[ H^i(X, \Omega^{i,2i-1}_X) \rightarrow H^{2i}(X, \Omega^{2i,cl}_X) \],
which sends the class \( c_i^{CS}(E) \) to \( c_i^{DR}(E) \). One also defines the corresponding “Chern character” by setting \( \text{ch}_1 = c_1, \text{ch}_2 = c_1^2 - \frac{c_2}{2}, \) etc.

The present paper addresses the problem of giving explicit de Rham representatives for the classes \( \text{ch}_i^{CS}(TX) \) for \( i = 1, 2, 3 \), where \( TX \) denotes the tangent sheaf on \( X \). Writing a de Rham representative for \( \text{ch}_i^{CS}(TX) \) involves a choice of flat connection on \( \text{det} TX \). Similarly, it is shown in the paper that the data required for writing de Rham representatives for the classes \( \text{ch}_i^{CS}(TX), i = 2, 3 \) involve three maps
\[ \gamma : \mathcal{O}_X \otimes_k T_X \rightarrow \Omega^1_X, \quad \langle, \rangle : S^2 T_X \rightarrow \mathcal{O}_X, \quad \text{and} \quad c : \Lambda^2 T_X \rightarrow \Omega^1_X. \]
These maps must satisfy certain identities.

It turns out that exactly the same data of three maps \( \gamma, \langle, \rangle, \) and \( c \) appears in the theory of vertex algebras. Specifically, Gorbunov, Malikov, and Schechtman have given a mathematical definition of a special class of vertex algebras, called sheaves of chiral differential operators. The main result of the present paper by Schechtman says that the sheaves of chiral differential operators on a manifold \( X \) form a gerbe over the complex \( \Omega^{(1,2)}_X \), and the characteristic class of that gerbe is equal to \( \text{ch}_2^{CS}(TX) \). This provides “la raison d’être” for the appearance of the class \( \text{ch}_2^{CS}(TX) \).

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June 2006
Volodya1 Drinfeld was born in 1954 in Kharkov, Ukraine. He graduated from Moscow State University in 1974 at the age of 20, and defended his Ph.D. thesis in 1978. His vision of mathematics was, to a great extent, influenced by Yu. I. Manin, his advisor, and by the Algebraic Geometry Seminar (“Manin’s Seminar”) that functioned with regularity at Moscow State University for about two decades.

Because of his Jewish origin and the absence of a Moscow “propiska,”2 the Soviet system made it extremely difficult for the talented Drinfeld, in spite of his obvious mathematical achievements, to get any reasonable job in mathematics in Moscow.

Therefore, after receiving his Ph.D., Drinfeld went to Ufa, a town in the Ural Mountains, where he taught mathematics at a local university. Later, he moved back to his native city, Kharkov, where he lived with his family until after the collapse of the Soviet Union.

It was in Kharkov where Drinfeld learned that he was to be awarded the Fields Medal, which he received at the Kyoto International Congress of Mathematicians (ICM) in 1990. In 1998, Drinfeld left Kharkov. Not long after migrating to the United States, he became a Distinguished Service Professor at the University of Chicago.

Almost immediately upon his arrival in Chicago, Drinfeld and A. Beilinson jointly organized the Geometric Langlands Seminar. Following, perhaps, in the tradition of the famous Gelfand Seminar in Moscow, the Geometric Langlands Seminar now runs regularly on Mondays from 4:30PM until both the speaker and the participants are completely exhausted.

In the course of his mathematical career, Drinfeld has worked on many different subjects, but his most fundamental contributions are in two fields. The first pertains to Drinfeld’s fascination with quantum groups, which were discovered by the Leningrad school—L. D. Faddeev’s students and collaborators. Drinfeld’s outstanding contributions revitalized the entire subject. With his celebrated talk at the Berkeley ICM

1 Volodya is the Russian diminutive for “Vladimir.”
2 People in the Soviet Union had their addresses written in their passports, the so-called “propiska.” A person was not allowed to get a job in any town different from the one indicated in the propiska. Changing one’s propiska was close to impossible.
in 1986, Drinfeld effectively “played a decisive role in the crystallization of this new domain.” To be sure, we do not discount the articles of M. Jimbo and others who drew attention of this field to so many mathematicians today.

The second major contribution of Drinfeld is in the area known as the “Langlands program.” Although the program itself was launched by Langlands in the late 1960s and early 1970s, it was Drinfeld who contributed crucial geometric insights. Drinfeld himself proved the Langlands conjecture in the special case of the group GL₂ over function fields; this, together with his achievements in quantum groups, earned him the Fields Medal. Drinfeld’s ideas have been extended to the GLₙ case by L. Lafforgue (2001), and a geometric refinement of this result was proved by D. Gaitsgory shortly afterwards. The general case of the Langlands conjecture still remains wide open.

Almost as important as Drinfeld’s own works were the remarks and ideas that he generously shared, either during private discussions or in his letters to other mathematicians. For instance, in a (widely circulated) letter to V. Schechtman, Drinfeld outlined his vision of deformation theory, emphasizes the role of DG-algebras, Maurer–Cartan equations, and stacks. All of these have later found their place in M. Kontsevich’s approach to deformation theory via $A_\infty$-algebras.

As another example, one may cite a classic work of Deligne and Lusztig that to a large extent owes its existence to a remark made by Drinfeld to T. Springer in a private conversation. In that remark, Drinfeld sketched a geometric construction of representations of the groups $\text{SL}_2(\mathbb{F}_q)$ in terms of what is nowadays known as Deligne–Lusztig varieties.

In the same spirit, I remember how Volodya once asked me, while walking in the corridor of Moscow State University sometime around 1987, whether or not the convolution of two spherical perverse sheaves on the loop grassmannian was again a perverse sheaf. The following day, I told him that this was indeed true and could be deduced from Lusztig’s results on Hecke algebras. In this way, thanks to his question, Drinfeld effectively created the theory of geometric Satake isomorphism.

I would like to finish with a couple of examples that show, I believe, that many of Drinfeld’s insights are still awaiting “discovery.” One such example is related to symplectic reflection algebras, a notion introduced by P. Etingof and myself in 2002. After having worked on the subject for several years, we discovered (in January 2005) that the definition of symplectic reflection algebras was essentially contained in two lines of Drinfeld’s paper “Degenerate Affine Hecke Algebras and Yangians,” written 15 years earlier! Although the paper itself is very well known, it seems nobody has read those two lines of Drinfeld’s very densely written text carefully enough.

The second example is equally amazing. I was preparing for a course on representation theory, which I teach regularly in Chicago. Volodya mentioned to me that he had some old notes with exercises on representation theory, written for his students in Kharkov back in the 1980s. As usual, Volodya’s notes were very systematic; they contained both the exercises and the solutions. Somewhere in the middle of the notes, I found a digression on “$q$-analogues” that contained computations equivalent, essentially, to the important geometric construction of the quantum group discovered by Beilinson, Lusztig, and MacPherson 10 years later!
We wish Volodya Drinfeld many more years of good health and inspirational mathematics which have contributed so much to so many of us from all over the mathematical world.

Victor Ginzburg  
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June 2006
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Algebraic Geometry
and Number Theory
Pillowcases and quasimodular forms

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To Vladimir Drinfeld on his 50th birthday.

Summary. We prove that natural generating functions for enumeration of branched coverings of the pillowcase orbifold are level 2 quasimodular forms. This gives a way to compute the volumes of the strata of the moduli space of quadratic differentials.

Subject Classifications: Primary 14N10, 14N30. Secondary 11F23, 14N35.

1 Introduction

1.1 Pillowcase covers and quadratic differentials

Consider a complex torus $\mathbb{T}^2 = \mathbb{C}/L$, where $L \subset \mathbb{C}$ is a lattice. Its quotient

$$\mathbb{P} = \mathbb{T}^2 / \pm$$

by the automorphism $z \mapsto -z$ is a sphere with four $(\mathbb{Z}/2)$-orbifold points which is sometimes called the pillowcase orbifold. The map $\mathbb{T}^2 \to \mathbb{P}$ is essentially the Weierstraß $\wp$-function. The quadratic differential $(dz)^2$ on $\mathbb{T}^2$ descends to a quadratic differential on $\mathbb{P}$. Viewed as a quadratic differential on the Riemann sphere, $(dz)^2$ has simple poles at corner points.

Let $\mu$ be a partition and $\nu$ a partition of an even number into odd parts. We are interested in enumeration of degree $2d$ maps
\( \pi : C \to \mathcal{P} \)

(1)

with the following ramification data. Viewed as a map to the sphere, \( \pi \) has profile \((\nu, 2^d - |\nu|/2)\) over \(0 \in \mathcal{P}\) and profile \((2^d)\) over the other three corners of \( \mathcal{P} \). Additionally, \( \pi \) has profile \((\mu_i, 1^{2^d - \ell(\mu_i)})\) over some \( \ell(\mu) \) given points of \( \mathcal{P} \) and is unramified elsewhere. Here \( \ell(\mu) \) is the number of parts in \( \mu \). This ramification data determines the genus of \( C \) by

\[
\chi(C) = \ell(\mu) + \ell(\nu) - |\mu| - |\nu|/2.
\]

In principle, one could allow more general ramifications over 0 and the nonorbifold points, but this more general problem is readily reduced to the one above.\(^1\)

Pulling back \((dz)^2\) via \( \pi \) gives a quadratic differential on \( C \) with zeros of multiplicities \( \{\nu_i - 2\} \) and \( \{2\mu_i - 2\} \). The periods of this differential, by construction, lie in a translate of a certain lattice. The enumeration of covers \( \pi \) is thus related to lattice point enumeration in the natural strata of the moduli space of quadratic differentials. In particular, the \( d \to \infty \) asymptotics gives the volumes of these strata. These volumes are of considerable interest in ergodic theory, in particular in connection with billiards in rational polygons; see [6, 18]. Their computation was the main motivation for the present work.

A different way to compute the volume of the principal stratum was found by M. Mirzakhani [19].

1.2 Generating functions

1.2.1

Two covers \( \pi_i : C_i \to \mathcal{P}, i = 1, 2 \), are identified if there is an isomorphism \( f : C_1 \to C_2 \) such that \( \pi_1 = f \circ \pi_2 \). In particular, associated to every cover \( \pi \) is a finite group \( \text{Aut}(\pi) \). This group is trivial for most connected covers; see, e.g., [7, Section 3.1]. We form the generating function

\[
Z(\mu, \nu; q) = \sum_{\pi} q^{\deg \pi} \frac{1}{|\text{Aut}(\pi)|},
\]

(2)

where \( \pi \) ranges over all inequivalent covers (1) with ramification data \( \mu \) and \( \nu \) as above. Note that the degree of any such \( \pi \) is even.

In particular, for \( \mu = \nu = \emptyset \) any connected cover has the form

\[
\pi : \mathbb{T}^2 \to \mathbb{T}^2/\pm
\]

with \( \pi' \) unramified. We have \( |\text{Aut}(\pi)| = 2|\text{Aut}(\pi')| \) corresponding to the lift of \( \pm \).

\(^1\) From first principles, the count of the branched coverings does not change if one replaces two ramification conditions by the product of the corresponding conjugacy classes in the class algebra of the symmetric group. In this way, one can generate complicated ramifications from simpler ones.
\[
Z(\emptyset, \emptyset; q) = \prod_n (1 - q^{2n})^{-1/2}.
\]

By definition, we set
\[
Z'(\mu, \nu; q) = \frac{Z(\mu, \nu; q)}{Z(\emptyset, \emptyset; q)}.
\]

This enumerates covers without unramified connected components. By the usual inclusion-exclusion, one can extract from (3) a generating function for connected covers. This generating function for connected covers will be denoted by \(Z^0(\mu, \nu; q)\).

1.2.2

Recall the classical level 1 Eisenstein series
\[
E_{2k}(q) = \frac{\zeta(1 - 2k)}{2} + \sum_{n=1}^{\infty} \left( \sum_{d|n} d^{2k-1} \right) q^n, \quad k = 1, 2, \ldots.
\]

The algebra they generate is called the algebra \(\mathcal{QM}(\Gamma(1))\) of quasimodular forms for \(\Gamma(1) = SL_2(\mathbb{Z})\); see [16] and also below in Section 3.3.7. It is known that \(E_2, E_4, \) and \(E_6\) are free commutative generators of \(\mathcal{QM}(\Gamma(1))\). The algebra \(\mathcal{QM}(\Gamma(1))\) is naturally graded by weight, where \(\text{wt } E_{2k} = 2k\). Clearly, for any integer \(N\), \(E_{2k}(q^N)\) is a quasimodular form of weight \(2k\) for the group
\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c \equiv 0 \mod N \right\} \subset SL_2(\mathbb{Z}).
\]

The quasimodular forms that will appear in this paper will typically be inhomogeneous, so instead of weight grading we will only keep track of the corresponding filtration. We define the weight of a partition by
\[
\text{wt } \mu = |\mu| + \ell(\mu).
\]

The main result of this paper is the following.

**Theorem 1.** The series \(Z'(\mu, \nu; q)\) is a polynomial in \(E_2(q^2), E_2(q^4), \) and \(E_4(q^4)\) of weight \(\text{wt } \mu + |\nu|/2\).

Several explicit examples of the forms \(Z'(\mu, \nu; q)\) are given in the appendix.

1.2.3

Quasimodular forms occur in nature, for example, as coefficients of the expansion of the odd genus 1 theta-function
\[
\vartheta(x) = (x^{1/2} - x^{-1/2}) \prod_{i=1}^{\infty} \frac{(1 - q^i x)(1 - q^i/x)}{(1 - q^i)^2}
\]
at the origin \(x = 1\). The techniques developed below give a certain formula for (3) in terms of derivatives of \(\vartheta(x)\) at \(x = \pm 1\), from which the quasimodularity follows.
### 1.2.4

The following discussion closely parallels the corresponding discussion for the case of holomorphic differentials in [7, Section 1.2].

Let $Q(\mu, \nu)$ denote the moduli space of pairs $(\Sigma, \phi)$, where $\phi$ is a quadratic differential on a curve $\Sigma$ with zeroes of multiplicities $\{v_i - 2, 2\mu_i - 2\}$. Note that we allow $v_i = 1$; hence our quadratic differentials can have simple poles. For $(\Sigma, \phi) \in Q(\mu, \nu)$, let $\tilde{\Sigma}$ denote the double cover of $\Sigma$ on which the differential

$$\omega = \sqrt{\phi}$$

is well defined. The pair $(\tilde{\Sigma}, \omega)$ belongs to the corresponding space of holomorphic differentials with zeroes of multiplicity

$$\{v_i - 1, \mu_i - 1, \mu_i - 1\}.$$

By construction, $\Sigma$ is the quotient of $\tilde{\Sigma}$ by an involution $\sigma$. Let $P$ denote the set of zeroes of $\omega$; it is clearly stable under $\sigma$. Then $\sigma$ acts as an involution on the relative homology group $H_1(\tilde{\Sigma}, P, \mathbb{Z})$. Let $H^- \subset H_1(\tilde{\Sigma}, P, \mathbb{Z})$ denote the subspace on which $\sigma$ acts as multiplication by $-1$. Choose a basis $\{\gamma_1, \ldots, \gamma_n\}$ for $H^-$, and consider the period map $\Phi : Q(\mu, \nu) \to \mathbb{C}^n$ defined by

$$\Phi(\Sigma, \phi) = \left(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_n} \omega\right).$$

It is known [18] that $\Phi(\Sigma, \phi)$ is a local coordinate system on $Q(\mu, \nu)$ and, in particular, $n = \dim_{\mathbb{C}} H^- = \dim_{\mathbb{C}} Q(\mu, \nu)$.

Pulling back the Lebesgue measure from $\mathbb{C}^n$ yields a well-defined measure on $Q(\mu, \nu)$. However, this measure is infinite since $\phi$ can be multiplied by any complex number. Thus we define $Q_1(\mu, \nu)$ to be the subset satisfying

$$\text{Area}(\tilde{\Sigma}) \equiv \frac{\sqrt{-1}}{2} \int_{\tilde{\Sigma}} \omega \wedge \overline{\omega} = 2.$$

As in the case of holomorphic differentials, the area function is a quadratic form in the local coordinates on $Q(\mu, \nu)$, and thus the image under $\Phi$ of $Q_1(\mu, \nu)$ can be identified with an open subset of a hyperboloid in $\mathbb{C}^n$.

Now let $E \subset Q_1(\mu, \nu)$ be a set lying in the domain of a coordinate chart, and let $C \Phi(E) \subset \mathbb{C}^n$ denote the cone over $\Phi(E)$ with vertex $0$. Then we can define a measure $\rho$ on $Q_1(\mu, \nu)$ via

$$\rho(E) = \text{vol}(C \Phi(E)),$$

where $\text{vol}$ is the Lebesgue measure. The proof of [7, Proposition 1.6] shows the analogue

$$\rho(Q_1(\mu, \nu)) = \lim_{D \to \infty} D^{-\dim_{\mathbb{C}} Q(\mu, \nu)} \sum_{d=1}^{2D} \text{Cov}_d^0(\mu, \nu),$$

where $\text{Cov}_d^0(\mu, \nu)$ is the $d$th covariance of the measure $\rho$. The limit in this expression converges to a finite value, and the analogue of Proposition 1.6 for quadratic differentials is established.
where \( \text{Cov}_d^0(\mu, \nu) \) is the number of inequivalent degree \( d \) connected covers \( C \to \mathcal{P} \). Thus, the volume \( \rho(Q_1(\mu, \nu)) \) can be read off from the \( q \to 1 \) asymptotics of the connected generating function \( Z^0(\mu, \nu; q) \).

Note that the moduli spaces \( Q(\mu, \nu) \) may be disconnected. Ergodic theory applications require the knowledge of volumes of each connected component. Fortunately, connected components of \( Q(\mu, \nu) \) have been classified by E. Lanneau [17] and these spaces turn out to be connected except for hyperelliptic components (whose volume can be computed separately) and finitely many sporadic cases.

1.2.5

The modular transformation

\[
q = e^{2\pi i \tau} \mapsto e^{-2\pi i / \tau}
\]

relates \( q = 0 \) and \( q = 1 \) and thus gives an easy handle on the \( q \to 1 \) asymptotics of (3). This gives an asymptotic enumeration of pillowcase covers and hence computes the volume of the moduli spaces of quadratic differentials.

1.2.6

In spirit, Theorem 1 is parallel to the results of [1, 8, 13]; see also [2, 3, 5] for earlier results in the physics literature. The main novelty is the occurrence of quasimodular forms of higher level. One might speculate whether similar lattice point enumeration in the space of \( N \)th order differentials leads to level \( N \) quasimodular forms. Those spaces, however, do not admit an \( SL_2(\mathbb{R}) \)-action and a natural interpretation of their volumes is not known.

1.2.7

The following enumerative problem is naturally a building block of the enumerative problem that we consider. Consider branched covers of the sphere ramified over 3 points \( 0, 1, \infty \) with profile \((\nu, 2^{d-|\nu|/2}), (2^d), \) and \( \mu \), respectively, where \( \mu \) is an arbitrary partition of \( 2^d \).

The preimage of the segment \([0, 1]\) on the sphere is a graph \( \mathcal{G} \) on a Riemann surface (also known as a ribbon graph) with many 2-valent vertices (that can be ignored) and a few odd valent vertices (namely, with valencies \( v_i \)). The complement of \( \mathcal{G} \) is a union of \( \ell(\mu) \) disks (known as cells) with perimeters \( 2\mu_i \) in the natural metric on \( \mathcal{G} \). The asymptotic enumeration of such combinatorial objects is, almost by definition, given by integrals of \( \psi \)-classes against Kontsevitch’s combinatorial cycles in \( \overline{M}_{g, \ell(\mu)} \); see [15]. There is a useful expression for these integrals in terms of Schur \( Q \)-functions obtained in [4, 11]. In fact, our original approach to the results presented in this paper was based on these ideas.

While the proof that we give here is more direct, it is still interesting to investigate the connection with combinatorial classes further, especially since a natural geometric
interpretation of combinatorial classes is still missing. Perhaps the Gromov–Witten theory of the orbifold $\mathcal{P}$ is the natural place to look for it. This will be further discussed in [22].

2 Character sums

2.1 Characters of near-involutions

2.1.1

There is a classical way to enumerate branched coverings in terms of irreducible characters, which is reviewed, for example, in [10] or in [21]. Specialized to our case, it gives

$$Z(\mu, \nu; q) = \sum_{\lambda} q^{\ell(\lambda)/2} \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 f_{\nu, 2, 2, \ldots} (\lambda) f_{2, 2, \ldots} (\lambda)^3 \prod_i f_{\mu_i} (\lambda)$$  (4)

where summation is over all partitions, $\dim \lambda$ is the dimension of the corresponding representation of the symmetric group, and $f_\eta (\lambda)$ is the central character of an element with cycle type $\eta$ in the representation $\lambda$. Recall that the sum of all permutations with cycle type $\eta$ acts as a scalar operator in any representation $\lambda$ and, by definition, this number is $f_\eta (\lambda)$. In (4), as usual, we abbreviate $f_{k, 1, 1, \ldots}$ to $f_k$.

2.1.2

A lot is known about the characters of the symmetric group $S(2d)$ in the situation when the representation is arbitrary but the support of the permutation is bounded by some number independent of $d$. In particular, explicit formulas exist for the functions $f_k$.

Understanding the function $f_{\nu, 2, 2, \ldots}$ is the key to evaluation of (4). That is, we must study characters of permutations that are a product of a permutation with finite support and a fixed-point-free involution. We call such permutations near-involutions.

2.1.3

By a result of Kerov and Olshanski [14], the functions $f_k$ belong to the algebra $\Lambda^*$ generated by

$$p_k (\lambda) = (1 - 2^{-k}) \xi (-k) + \sum_i \left[ (\lambda_i - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k \right];$$  (5)

moreover, $f_k$ has weight $k + 1$ in the weight filtration on $\Lambda^*$ defined by setting

$$\text{wt } p_k = k + 1.$$

The functions $p_k$ are central characters of certain distinguished elements in the group algebra of symmetric group known as completed cycles. See [21] for the discussion of the relation between $p_k$ and $f_k$ from the viewpoint of Gromov–Witten theory.
2.1.4

Our next goal is to generalize the results of [14] to characters of near-involutions. This will require enlarging the algebra of functions. In addition to the polynomials \( p_k \), we will need quasi-polynomial functions \( \bar{p}_k \) defined in (6) below.

It is convenient to work with the generating function

\[
e(\lambda, z) \overset{\text{def}}{=} \sum_i e^{z(\lambda_i - i + \frac{1}{2})} = \frac{1}{z} + \sum_k p_k(\lambda) \frac{z^k}{k!}.
\]

By definition, set

\[
\bar{p}_k(\lambda) = i k! [z^k] e(\lambda, z + \pi i)
\]

\[
= \sum_i \left[ (-1)^{\lambda_i-i+1} \left( \lambda_i - i + \frac{1}{2} \right)^k - (-1)^{-i+1} \left( -i + \frac{1}{2} \right)^k \right] + \text{const},
\]

where the constant terms are determined by the expansion

\[
\sum_k \frac{z^k}{k!} \bar{p}_k(\emptyset) = \frac{1}{e^{z/2} + e^{-z/2}}.
\]

Up to powers of 2, they are Euler numbers.

2.1.5

Define

\[
\tilde{\Lambda} = \mathbb{Q}[p_k, \bar{p}_k]_{k \geq 1}.
\]

Setting

\[
\text{wt } \bar{p}_k = k
\]

gives the algebra \( \tilde{\Lambda} \) the weight grading. Note that if \( f \) is homogeneous, then

\[
f(\lambda') = (-1)^{\text{wt } f} f(\lambda),
\]

where \( \lambda' \) denotes the conjugate partition.

2.1.6

In the definition of \( \tilde{\Lambda} \), we excluded the function

\[
\bar{p}_0(\lambda) = \frac{1}{2} + \sum_i [(-1)^{\lambda_i-i+1} - (-1)^{-i+1}],
\]

which measures the difference between the number of even and odd numbers among \( \{\lambda_i - i + 1\} \), also known as the 2-charge of a partition \( \lambda \).
Every partition $\lambda$ uniquely defines two partitions $\alpha$ and $\beta$, known as its 2-quotients, such that
\[
\left\{ \lambda_i - i + \frac{1}{2} \right\} = \left\{ 2 \left( \alpha_i - i + \frac{1}{2} \right) + \bar{p}_0(\lambda) \right\} \cup \left\{ 2 \left( \beta_i - i + \frac{1}{2} \right) - \bar{p}_0(\lambda) \right\}.
\]

A partition $\lambda$ will be called balanced if $\bar{p}_0(\lambda) = \frac{1}{2}$.

Several constructions related to 2-quotients will play an important role in this paper. A modern review of these ideas can be found, for example, in [9]. In particular, it is known that the character $\chi_{\lambda\alpha\beta\ldots}$ of a fixed-point free involution in the representation $\lambda$ vanishes unless $\lambda$ is balanced, in which case
\[
|\chi_{2,2,...}^\lambda| = \left( \frac{|\lambda|/2}{|\alpha|, |\beta|} \right) \dim \alpha \dim \beta.
\]

It follows that only balanced partitions contribute to the sum (4).

2.1.7

For a balanced partition $\lambda$, define
\[
g_\nu(\lambda) = \frac{f(\nu, 2, 2, \ldots)(\lambda)}{f(2, 2, \ldots)(\lambda)}.
\]

We will prove that this function lies in $\bar{\Lambda}$ in the following sense.

**Theorem 2.** The ratio (9) is the restriction of a unique function $g_\nu \in \bar{\Lambda}$ of weight $|\nu|/2$ to the set of balanced partitions.

Several examples of the polynomials $g_\nu$ can be found in the appendix.

2.1.8

In view of Theorem 2, it is natural to introduce the pillowcase weight
\[
w(\lambda) = \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 f_{2,2,...}(\lambda)^4.
\]

Theorem 1 follows from (4), Theorem 2, and the following result.

**Theorem 3.** For any $F \in \bar{\Lambda}$, the average
\[
\langle F \rangle_w = \frac{1}{Z(\emptyset, \emptyset; q)} \sum_{\lambda} q^{|\lambda|} w(\lambda) F(\lambda)
\]

is a polynomial in $E_2(q^2)$, $E_2(q^4)$, and $E_4(q^4)$ of weight $\text{wt} F$.

Note that if $F$ is homogeneous of odd weight, then $\langle F \rangle_w = 0$. This can be seen directly from (7). Also note that (10) will not in general be of pure weight even if $F$ is a monomial in the generators $p_k$ and $\bar{p}_k$. This contrast with [1, 8] hints to the existence of a better set of generators of the algebra $\bar{\Lambda}$. Probably such generators are related to descendents of orbifold points in the Gromov–Witten theory of $\mathfrak{P}$. 
2.1.9

It will be convenient to work with the following generating functions for the sums (10):

\[ F(x_1, \ldots, x_n) = \left( \prod_i \exp(\lambda, \ln x_i) \right)_{\Lambda}. \] (11)

The function (11) will be called the \textit{n-point function}.

2.2 Proof of Theorem 2

2.2.1

In the proof of theorems 2 and 3 it will be very convenient to use the fermionic Fock space formalism. This formalism is standard and [12, 20] can be recommended as a reference. A quick review of these techniques can be found, for example, in [21, Section 2]. We follow the notation of [21].

2.2.2

By definition, the space \( \Lambda_{\frac{3}{2}} V \) is spanned by the infinite wedge products

\[ v_{\lambda} = \lambda_1 - \frac{1}{2} \wedge \lambda_2 - \frac{3}{2} \wedge \lambda_3 - \frac{5}{2} \wedge \cdots, \] (12)

where \( k, k \in \mathbb{Z} + \frac{1}{2} \), is a basis of the underlying space \( V \) and \( \lambda \) is a partition. The subscript 0 in \( \Lambda_{\frac{3}{2}} V \) refers to the charge zero condition: the \( i \)th factor in (12) is \(-i + \frac{1}{2}\) for all sufficiently large \( i \).

There is a natural projective representation of the Lie algebra \( \mathfrak{gl}(V) \) on \( \Lambda_{\frac{3}{2}} V \). For us, the following elements of \( \mathfrak{gl}(V) \) will be especially important:

\[ \mathcal{E}_k[f(x)] = f \left( i - \frac{k}{2} \right) i - k, \] (13)

where \( f \) is a function on the real line. To define the action of \( \mathcal{E}_0[f(x)] \) on \( \Lambda_{\frac{3}{2}} V \) one needs to regularize the infinite sum \( \sum_{i<0} f\left( \frac{1}{2} - i \right) \). This regularization is the source of the central extension in the \( \mathfrak{gl}(V) \) action. When \( f \) is an exponential as in

\[ \mathcal{E}_k(z) = \mathcal{E}_k[e^{z x}], \]

this infinite sum is a geometric series and thus has a natural regularization. By differentiation, this leads to the \( \zeta \)-regularization for operators \( \mathcal{E}_k[f] \) with a polynomial function \( f \).
2.2.3
Other very useful operators are
\[ \alpha_k = E_k[1], \quad k \neq 0. \]
The operator \( H \) defined by
\[ H v_\lambda = |\lambda| v_\lambda \]
is known as the energy operator. It differs only by a constant from the operator \( E_0[\chi] \).
The operator \( H \) defines a natural grading on \( \Lambda \frac{\infty}{2} 0 V \) and \( gl(V) \).

2.2.4
A function \( F(\lambda) \) on partitions of \( n \) can be viewed as a vector
\[ \sum_{|\lambda|=n} F(\lambda) v_\lambda \in \Lambda \frac{\infty}{2} 0 V \]
of energy \( n \). For example, the vectors
\[ |\mu\rangle \overset{\text{def}}{=} \frac{1}{\delta(\mu)} \prod \alpha_{-\mu_i} v_\emptyset = \frac{1}{\delta(\mu)} \sum \chi^\lambda v_\lambda, \quad (14) \]
where
\[ \delta(\mu) = |\text{Aut} \mu| \prod \mu_i, \]
correspond to irreducible characters normalized by the order of the centralizer.

2.2.5
The operator \( E_0(z) \) is the generating function
\[ E_0(z) = E_0[e^{zx}] = \frac{1}{z} + \sum_k \frac{z^k}{k!} P_k, \]
for the operators \( P_k \) acting by
\[ P_k v_\lambda = p_k(\lambda) v_\lambda. \]
In parallel to (6), we define operators \( \tilde{P}_k \) by
\[ iE_0(z + \pi i) = \sum_k \frac{z^k}{k!} \tilde{P}_k. \]
Translated into the operator language, the statement of Theorem 2 is the following: the orthogonal projection of \( |\nu, 2^{d-|\nu|/2} \rangle \) onto the subspace spanned by the \( v_\lambda \) with \( \lambda \) balanced is a linear combination of vectors
\[ \prod P_{\mu_i} \prod \tilde{P}_{\tilde{\mu}_i} |2^d \rangle \]
with
\[ \text{wt} \mu + |\tilde{\mu}| \leq |\nu|/2 \]
and coefficients independent of \( d \).
2.2.6

Let us call the span of $v_{\lambda}$ with $\lambda$ balanced the balanced subspace of $\Lambda^\infty_0 V$. A convenient orthogonal basis of it is provided by the vectors

$$|\rho; \bar{\rho}\rangle \overset{\text{def}}{=} \frac{1}{\delta(\rho)\delta(\bar{\rho})} \prod_{\alpha} \alpha_{-\rho_i} \prod_{\bar{\alpha}} \bar{\alpha}_{-\bar{\rho}_i} v_{\emptyset}, \quad \rho_i, \bar{\rho}_i \in 2\mathbb{Z},$$

(16)

where the operators $\bar{\alpha}_k$ are defined by

$$\bar{\alpha}_k = i^{k+1} E_k(\pi i) = \sum_{n} (-1)^{n+\frac{1}{2}} E_{n-k,n} + \frac{\delta_k}{2},$$

(17)

the operators $E_{i,j}$ being the matrix units of $\text{gl}(V)$. From the commutation relations for the operators $E_k(z)$, we compute

$$[\bar{\alpha}_k, \bar{\alpha}_m] = (-1)^k - (-1)^m [\alpha_{k+m} + k(-1)^k \delta_{k+m}$$

(18)

$$[\alpha_k, \bar{\alpha}_m] = \left[1 - (-1)^k \right] \left( \bar{\alpha}_{k+m} + \frac{\delta_{k+m}}{2} \right).$$

(19)

In particular, when both $k$ and $m$ are even, all these operators commute apart from the central term in $[\bar{\alpha}_k, \bar{\alpha}_{-k}]$.

The adjoint of $\bar{\alpha}_k$ is

$$\bar{\alpha}_k^* = (-1)^k \bar{\alpha}_{-k},$$

which gives the inner products

$$\langle \rho; \bar{\rho} \mid \rho'; \bar{\rho}' \rangle = \frac{\delta_{\rho,\rho'} \delta_{\bar{\rho},\bar{\rho}'}}{\delta(\rho)\delta(\bar{\rho})},$$

(20)

provided all parts of all partitions in (20) are even. In particular, the vectors (16) are orthogonal. It is clear that they lie in the balanced subspace and their number equals the dimension of the space. Therefore, they form a basis.

2.2.7

The projection of $|\nu, 2^d-|\nu|/2\rangle$ onto the balanced subspace is given in term of inner products of the form

$$\langle \nu, 2^d-|\nu|/2 | (\rho, 2^d-|\rho|/2-|\bar{\rho}|/2); \bar{\rho} \rangle$$

where all parts of $\nu$ are odd, all parts of $\rho$ and $\bar{\rho}$ are even, and $\rho$ has no parts equal to 2. From the commutation relations (18) and (19) we conclude that this inner product vanishes unless

$$\rho = \emptyset.$$

The nonvanishing inner products are
\[ \langle \nu, 2^k | 2^k; \bar{\rho} \rangle = \frac{2^{\ell(v) - \ell(\bar{\rho})}}{2^{k l_3(v) l_3(\bar{\rho})}} C(v, \bar{\rho}), \] (21)

where the combinatorial coefficient \( C(v, \bar{\rho}) \) equals the number of ways to represent the parts of \( \bar{\rho} \) as sums of parts of \( \nu \). For example,

\[ C((3, 1, 1, 1), (4, 2)) = 3, \quad C((3, 1, 1, 1), (6)) = 1. \]

\[4.2.8\]

The matrix elements

\[ \left\langle 2^d \prod P_{\mu_i} \prod \bar{P}_{\bar{\mu}_i} \mid (\rho, 2^d - |\rho|/2 - |\bar{\rho}|/2); \bar{\rho} \right\rangle, \quad \rho_i \neq 2, \] (22)

describe the decomposition of the vectors (15) in the basis (16). Since

\[ P_1 |2^d\rangle = \left( d - \frac{1}{24} \right) |2^d\rangle, \] (23)

we can also assume that \( \mu_i \neq 1 \).

We claim that (22) vanishes unless

\[ \text{wt } \mu + |\bar{\mu}| \geq \text{wt } \rho/2 + |\bar{\rho}|/2, \] (24)

where \( \rho/2 \) is the partition with parts \( \rho_i/2 \) (recall that all parts of \( \rho \) are even).

\[4.2.9\]

The usual way to evaluate a matrix element like (22) is to use commutation relations to commute all lowering operators to the right until they reach the vacuum (which they annihilate) and, similarly, commute the raising operators to the left.

We will exploit the following property of the operators \( P_k \) and \( \bar{P}_k \): their commutator with enough operators of the form \( \alpha_{-2\rho_i} \) and \( \tilde{\alpha}_{-2\bar{\rho}_i} \) vanishes. All such commutators have the form \( E_k[f] \) with \( f(x) = (\pm 1)^x p(x) \), where \( p(x) \) is a polynomial. Commutation with \( \alpha_{-2\rho_i} \) takes a finite difference of \( p(x) \); commutation with \( \tilde{\alpha}_{-2\bar{\rho}_i} \) additionally flips the sign of \( \pm 1 \).

Since a \((k + 1)\)-fold finite difference of a degree \( k \) polynomial vanishes, the commutator of \( P_k \) with more than \( k + 1 \) operators of the form \( \alpha_{-2\rho_i} \) or \( \tilde{\alpha}_{-2\bar{\rho}_i} \) vanishes. In fact, a \((k + 1)\)-fold commutator may be nonvanishing only because of the central extension term. To pick up this central term, the total energy of all operators involved should be zero and the number of \( \tilde{\alpha} \)s should be even. The same reasoning applies to \( \bar{P}_k \), but now the number of \( \tilde{\alpha} \)s should be odd to produce a nontrivial \((k + 1)\)-fold commutator.
2.2.10

Now look at one of the raising operators involved in (22), say \( \bar{\alpha} - \bar{\rho}_i \). This operator commutes with \( \alpha_2 \) and its adjoint annihilates the vacuum, so only the terms involving the commutator of \( \bar{\alpha} - \bar{\rho}_i \) with one of the \( P_{\mu_i} \) or \( \bar{P}_{\bar{\mu}_i} \) give a nonzero contribution to (22). The commutator \([P_{\mu_i}, \bar{\alpha} - \bar{\rho}_i]\) has energy \((-\rho_i)\) and so its adjoint again annihilates the vacuum. The same is true for the commutation with \( \bar{P}_{\bar{\mu}_i} \). To bring these commutators back to zero energy, one needs to commute it \( \bar{\rho}_i/2 \) times with \( \alpha_2 \). Given the above bounds on how many commutators we can afford, this implies (24).

2.2.11

When the bound (24) is saturated, then a further condition

\[ \ell(\rho) + \ell(\bar{\rho}) \geq \ell(\mu) + \ell(\bar{\mu}) \]

is clearly necessary for nonvanishing of (22). The unique nonzero coefficient saturating both bounds corresponds to

\[ \rho = 2\mu, \quad \bar{\rho} = 2\bar{\mu}. \]

Moreover, when divided by the norm squared of the vector \( |(\rho, 2^{d-\ell(\rho)/2-\ell(\bar{\rho})/2}; \bar{\rho})| \), this coefficient is independent of \( d \).

2.2.12

For general \( \rho \) and \( \bar{\rho} \), the similarly normalized coefficient will be a polynomial in \( d \) of degree

\[ \frac{1}{2} (\text{wt } \mu + |\bar{\mu}| - \text{wt } \rho/2 - |\bar{\rho}|/2) \]  

(25)

because so many operators \( \alpha_2 \) can commute with \( P_{\mu_i} \)s or \( \bar{P}_{\bar{\mu}_i} \)s instead of commuting directly with \( \alpha_2 \)s.

By induction on weight and length, we can express the basis vectors (16) in terms of (15) with \( \mu_i \neq 1 \) and coefficients being polynomial in \( d \) of degree at most minus the difference (25). By (23), to have \( d \)-dependent coefficients and \( \mu_i \neq 1 \) is the same as to allow \( \mu_i = 1 \) and make the coefficients independent of \( d \). The bound of degree in \( d \) ensures that this transition preserves weight. This concludes the proof of Theorem 2.

3 Proof of Theorem 3

3.1 The pillowcase operator

3.1.1

Consider the operator