NEARRINGS AND NEARFIELDS

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Proceedings of the Conference on Nearrings and Nearfields, Hamburg, Germany
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Foreword

This present volume is the Proceedings of the 18th International Conference on Nearrings and Nearfields held in Hamburg at the Universität der Bundeswehr Hamburg from July 27 to August 03, 2003. This Conference was organized by Momme Johs Thomsen and Gerhard Saad from the Universität der Bundeswehr Hamburg and by Alexander Kreuzer, Hubert Kiechle and Wen-Ling Huang from the Universität Hamburg.

It was already the second Conference on Nearrings and Nearfields in Hamburg after the Conference on Nearrings and Nearfields at the same venue from July 30 to August 06, 1995.

The Conference was attended by 57 mathematicians and many accompanying persons who represented 16 countries from all five continents.

The first of these conferences took place 35 years earlier in 1968 at the Mathematische Forschungsinstitut Oberwolfach in the Black Forest in Germany. This was also the site of the second, third, fifth and eleventh conference in 1972, 1976, 1980 and 1989. The other twelve conferences held before the second Hamburg Conference took place in nine different countries. For details about this and, moreover, for a general historical overview of the development of the subject we refer to the article "On the beginnings and developments of near-ring theory" by Gerhard Betsch [3] in the proceedings of the 13th Conference in Fredericton, New Brunswick, Canada.

During the last fifty years the theory of nearrings and related algebraic structures like nearfields, nearmodules, nearalgebras and seminearrings has developed into an extensive branch of algebra with its own features. In its position between group theory and ring theory, this relatively young branch of algebra has not only a close relationship to these two more well-known areas of algebra, but it also has, just as these two theories, very intensive connections to many further branches of mathematics.

Thanks to the foresight of the early workers in the field, a comprehensive classified bibliography was established and is updated regularly in the Nearring Newsletter. The latest version [9] appeared at the end of

2003. It listed 2485 publications contributed by a total of 708 authors. Within this large number of papers is reflected the great diversity of the subject.

That the development of nearrings and nearfields has matured to a substantial theory with numerous applications can now be best retraced by studying the five existing books on the subject. They are written by the authors G. Pilz [11], J.D.P. Meldrum [10], H. Wähling [15], J.R. Clay [4], C. Cotti Ferrero and G. Ferrero [6].

This present volume is the ninth proceedings of a nearring conference following the proceedings [13], [5], [1], [2], [12], [7], [14] and [8]. It contains the written version of five invited lectures followed by 13 contributed papers. All papers in the volume have been refereed.

This Proceedings opens with the invited paper by Wen-Fong Ke which reports on some recent developments of planar nearrings and points out several possible research directions in this area for the future.

The second paper is an expanded version of the invited survey talk by Carl J. Maxson on nearrings of mappings which mentions several open questions in this field of research. This paper is a continuation of the invited survey paper by the same author on nearrings of homogeneous functions in the Proceedings of the first Hamburg Conference of 1995.

Next we have the invited paper by John D.P. Meldrum which presents an account of some of the work on group nearrings, emphasizing the parallels with matrix nearrings and the most recent developments.

The invited paper of Silvia Pianta on loop-nearrings considers a generalization of the notion of nearring by relaxing the associativity of the addition. Then for these loop-nearrings several generalizations of planarity and corresponding Ferrero pairs are investigated.

Just as eight years before, Stuart R. Scott brought to Hamburg from the opposite side of our planet the by far longest paper of this Proceeding "The Z-Constrained Conjecture". For an overview of this substantial work, we refer to its first twelve pages.

The topics of the 13 contributed papers are so diverse that, for an overview, we refer to the Table of Contents and the abtracts or introductions at the beginning of each paper.

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Momme Johs Thomsen
Hamburg, November 2004

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Many thanks are due to the Wilhelm-Blaschke-Gedächtnis-Stiftung for financial support.

We are especially indebted to our co-organizers Wen-ling Huang and Gerhard Saad who shared in the organization from the beginnings. Later Marco Möller and Sebastian Rudert joined the effort.

Many thanks go to Wolfgang Löbnitz who designed and implemented the web-pages.

Very helpful was the secretarial work provided by Corina Flegel of the HSU and of Elisabeth Himmler of the UniHH.

The enormous engagement of Elisabeth Himmler was really far above and beyond the call of duty.

We are also especially indebted to Marta Thomsen for arranging a welcome party at the Thomsen's home including the catering for almost hundred people, for helping to organize the joint excursion of all those people to Lübeck and Ratzeburg, and also for planning and implementation of numerous further activities for the accompanying persons.

Concerning the Proceedings, we express our gratitude to all the authors. Special thanks go to the referees who took the pain of examining the manuscripts.

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I

INVITED ADDRESSES

ON RECENT DEVELOPMENTS OF PLANAR NEARRINGS

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1. Introduction

Since the first appearance of planar nearrings in 1968, there has been plenty of research results attributed to the understanding and applications of them. However, it appears to us that, at this stage, we have just begun to unearth this beautiful mathematical object.

In February 2002, a two year international joint project on research of planar nearrings was established between the research group in Kepler University, Linz and that in National Cheng Kung University, Tainan. This project was supported by the Austrian Science Foundation (FWF) and the National Science Council, R.O.C. (NSC), and has been proven fruitful.

The goal of this survey article is to report some recent developments of planar nearrings, and point out some possible research directions for future researches. Some of the directions have been under investigations by the Linz-Tainan cooperation. The materials presented in this article are organized based on the one-hour-talk the author gave at the International Conference on Nearrings and Nearfields, Universität der Bundeswehr Hamburg and Universität Hamburg, 27 July–3 August 2003. We would also like to point out that there is a whole chapter in Clay's book *Nearrings: Genesis and Applications* (reference item [10]) devoted to this subject that one would like to go through and refer back from time to time.

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H. Kiechle et al. (eds.), Nearrings and Nearfields, 3–23.

2. Definitions and examples

Let $(N,+,\cdot)$ be a (left) nearring. An equivalence relation $\equiv_m$ can be defined on N by

$$a \equiv_m b \Leftrightarrow ax = bx \quad \text{for all } x \in N.$$

We say that $(N,+,\cdot)$ is *planar* if $|N/{\equiv_m}| \geq 3$, and for each triple $a,b,c \in N$ with $a \not\equiv_m b$, the equation $ax = bx + c$ has a unique solution for x in N. It is custom to denote by N^* the set of elements not multiplicative equivalent to 0, i.e. $N^* = \{x \in N \mid x \not\equiv_m 0\}$. On the other hand, the set of "zero-multipliers" is denoted by A, i.e. $A = \{x \in N \mid x \equiv_m 0\}$.

Certainly all fields are planar nearrings. It is also true that all finite nearfields are planar nearrings (cf. [10, Theorem 4.26]). The first three nontrivial examples of planar nearrings were given in [1] which we record again in the following.

Consider the field of complex numbers $\mathbb{C}$. For $\boldsymbol{a},\boldsymbol{b} \in \mathbb{C}$, where $\boldsymbol{a} = a_1 + \boldsymbol{i}a_2$ with $a_1, a_2 \in \mathbb{R}$ and $\boldsymbol{i}^2 = -1$, define

$$\begin{aligned}
\boldsymbol{a} *_1 \boldsymbol{b} &= \begin{cases} a_1 \cdot \boldsymbol{b} & \text{if } a_1 \neq 0, \\ a_2 \cdot \boldsymbol{b} & \text{if } a_1 = 0; \end{cases} \\
\boldsymbol{a} *_2 \boldsymbol{b} &= |\boldsymbol{a}| \cdot \boldsymbol{b}; \\
\boldsymbol{a} *_3 \boldsymbol{b} &= \begin{cases} \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|} \cdot \boldsymbol{b} & \text{if } \boldsymbol{a} \neq \boldsymbol{0}, \\ \boldsymbol{0} & \text{if } \boldsymbol{a} = \boldsymbol{0}. \end{cases}
\end{aligned}$$

Then $(\mathbb{C},+,*_1)$, $(\mathbb{C},+,*_2)$, and $(\mathbb{C},+,*_3)$ are planar nearrings which are not rings. These three examples have served as models for many researches on planar nearrings since.

It is natural at this point to ask for more examples of planar nearrings. Indeed, one can construct planar nearrings (somewhat) freely when the relationship between planar nearrings and *Ferrero pairs* is studied.

3. Planar nearrings and Ferrero pairs

It was shown by G. Ferrero in 1970 [13] that every planar nearring N gives rise to a group of automorphisms Φ of the additive group $(N,+)$ having specific properties (to be discussed below). On the other hand, if an additive group $(G,+)$ is given together with a group of automorphisms of G satisfying the specific properties, then G can be turned into a planar nearring through a fixed process.

3.1 From planar nearrings to Ferrero pairs

Let N be a planar nearring. For $a \in N^*$, define $\varphi_a : N \to N; x \mapsto ax$ for all $x \in N$. Then

- $\varphi_a \in \mathrm{Aut}\,(N,+)$, and $\varphi_a \neq 1$ if and only if $a \not\equiv_m 1$;
- $\varphi_a(x) = x$ if and only if $\varphi_a = 1$ or $x = 0$;
- $-1 + \varphi_a$ is surjective if $\varphi_a \neq 1$.

Thus, $\Phi = \{\varphi_a \mid a \in N,\, a \not\equiv_m 0\}$ is a regular group of automorphisms of $(N,+)$ with the property that $-1+\varphi_a$ is surjective if $\varphi_a \neq 1$. We call (N,Φ) a *Ferrero pair.*

In general, if Φ is a group acting on another group N as an automorphism group, and for $\varphi \in \Phi \setminus \{1\}$, $-1+\varphi$ is bijective, then (N,Φ) is called a *Ferrero pair.*

3.2 From Ferrero pairs to planar nearrings

Given a Ferrero pair (N,Φ), where N is an additive group. Let C be a complete set of orbit representatives of Φ in N. Let $E \subseteq C$ such that $0 \notin E$ and $|E| \geq 2$. Then

$$N = \Big(\cup_{e\in E}\, \Phi(e) \Big) \bigcup \Big(\cup_{e'\in C\setminus E}\, \Phi(e') \Big),$$

here for an $a \in N$, $\Phi(a) = \{\phi(a) \mid \phi \in \Phi\}$ is the orbit of Φ in N determined by a. Now, define a binary operation $*_E$ on N by

$$\varphi(e) *_E y = \begin{cases} \varphi(y) & e \in E,\, \varphi \in \Phi,\, y \in N, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(N,+,*_E)$ is a planar nearring. Notice that

(1) the elements in E are exactly the left identities of N, and

(2) N is an integral planar nearring if and only if $E = C \setminus \{0\}$.

Remark 3.1. (1) Since Φ is a regular group of automorphisms of N, $\Phi(a)$ and Φ have the same cardinality for all nonzero $a \in N$.

(2) The set E given above is exactly the set of left identities of the planar nearring $(N,+,*_E)$.

(3) For each $e \in E$, $N_e = \Phi(e)$ is a subgroup of the multiplicative semigroup $(N,*_E)$ have e as the identity element. Actually, $(N_e,*_E)$ is isomorphic to Φ. (Cf. [10, (4.9)].)

Thus, once we have a Ferrero pair, we can easily obtain many planar nearrings.

3.3 Some examples

First, let us look at the three planar nearrings $(\mathbb{C}, +, *_i)$, $i = 1, 2, 3$. Here we find that

(1) the corresponding Ferrero pair of $(\mathbb{C}, +, *_1)$ is $(\mathbb{C}, \widehat{\mathbb{R}^*})$, where $\widehat{\mathbb{R}^*}$ is the group $\{\varphi_r \mid r \in \mathbb{R} \setminus \{0\}\}$;

(2) the corresponding Ferrero pair of $(\mathbb{C}, +, *_2)$ is $(\mathbb{C}, \widehat{\mathbb{R}^+})$, where $\widehat{\mathbb{R}^+}$ is the group $\{\varphi_r \mid r > 0\}$; and

(3) the corresponding Ferrero pair of $(\mathbb{C}, +, *_3)$ is $(\mathbb{C}, \widehat{C})$, where $\widehat{C}$ is the group $\{\varphi_{\boldsymbol{c}} \mid |\boldsymbol{c}| = 1\}$.

From the above examples, one immediately obtain the following generalization. Let F be a field. Take $U \leq F^* = F \setminus \{0\}$ and put $\widehat{U} = \{\varphi_a \mid a \in U\} \leq \mathrm{Aut}\,(F, +)$ where each $\varphi_a : F \to F$ is the left multiplication by a. Then $(F, \widehat{U})$ is a Ferrero pair. Any planar nearring constructed from $(F, \widehat{U})$ is referred to as *field generated*.

Yet, one can generalize this ideal to certain rings. So, if R is a ring with unity, and let $\mathcal{U}$ be the group of (multiplicative) invertible elements. If A is a subgroup of $\mathcal{U}$ with $|A| \geq 2$ and $-1 + a \in \mathcal{U}$ for all $a \in A \setminus \{1\}$, then the group $\widehat{A} = \{\varphi_a \mid a \in A\}$, where each $\varphi_a : R \to R$ is the left multiplication by a, is a regular group of automorphisms of $(R, +)$. Moreover, $(R, \widehat{A})$ is a Ferrero pair. Any planar nearring defined using $(R, \widehat{A})$ will be said to be *ring generated*.

4. Isomorphism problem

For a given Ferrero pair (N, Φ), there are many ways to choose the set E of orbit representatives of Φ. Each choice of E give rise to a planar nearring. Naturally, one wonders that whether all of these planar nearrings are isomorphic or not? The answer is "no" even if the planar nearrings are integral. Then, the second question would be that "is there a way to distinguish the planar nearrings constructed from (N, Φ)?" The answer to this question is "yes!"

Theorem 4.1 ([4])**.** *Let (M, Ψ) and (N, Φ) be Ferrero pairs and let E_1 and E_2 be sets of orbit representatives of Ψ and Φ in M and N, respectively, with $|E_1| \geq 2$. Let $(M, +, \cdot)$ and $(N, +, \star)$ be the planar nearrings defined on M and N using E_1 and E_2, respectively. Then an additive isomorphism σ from $(M, +)$ to $(N, +)$ is an isomorphism of the planar nearrings $(M, +, \cdot)$ and $(N, +, \star)$ if and only if $\sigma(E_1) = E_2$ and $\sigma\Psi\sigma^{-1} = \Phi$.*

In particular, if $(M, \Psi) = (N, \Phi)$*, then* $\sigma \in \operatorname{Aut}(N, +)$ *is an isomorphism of* $(N, +, *_{E_1})$ *and* $(N, +, *_{E_2})$ *if and only if* $\sigma(E_1) = E_2$ *and* σ *normalizes* Φ*.*

As an illustration, we consider the Ferrero pair $(\mathbb{C}, \widehat{C})$ of the planar nearring $(\mathbb{C}, +, *_3)$. Let E_1 and E_2 be two complete sets of orbit representatives of $\widehat{C}$ in $\mathbb{C} \setminus \{0\}$. Let $\sigma \in \operatorname{Aut}(\mathbb{C}, +)$. If σ is an isomorphism of the two planar nearrings, then $\sigma(C) = C$. It can be shown that, in this case, σ is either a rotation of the complex plane about the origin or the reflection of the complex plane about a line through the origin. Therefore, $(\mathbb{C}, +, *_{E_1})$ and $(\mathbb{C}, +, *_{E_2})$ are isomorphic as integral planar nearrings if and only if $E_2 = e^{i\theta}E_1$ or $E_2 = e^{i\theta}\overline{E}_1$ for some $\theta \in \mathbb{R}$, where $\overline{E}_1$ denotes the complex conjugate of E_1.

Remark 4.2. Theorem 4.1 is valid for a more general class of nearring constructions called *Ferrero nearrings*. To describe what a Ferrero nearring is, we start with a group G and $\Phi \leq \operatorname{Aut} G$. Let A be a complete set of orbit representatives of Φ in G. Suppose that $E \subseteq A$. If $E = \varnothing$, then we have trivial multiplication on G. If $E \neq \varnothing$, we want E to satisfy

$$\varphi(e) \neq e \quad \text{for all } \varphi \in \Phi \setminus \{1\} \text{ and } e \in E.$$

Put $A^{\circ} = A \setminus E$ and $G^{\circ} = \Phi(A^{\circ})$. For $x, y \in G$, define

$$x * y = \begin{cases} 0 & \text{if } x \in G^{\circ}, \\ \varphi(y) & \text{if } x = \varphi(e) \in \Phi(E). \end{cases}$$

Then $(G, +, *)$ is called a Ferrero nearring. Note that $(G, +, *)$ is a planar nearring if and only if (G, Φ) is a Ferrero pair.

Problem 4.3. Study the structure of the planar nearrings constructed from the Ferrero pair $(\mathbb{C}, \widehat{C})$.

Problem 4.4. Note that if E is a complete set of orbit representatives of $\widehat{C}$ in $\mathbb{C} \setminus \{0\}$, then the planar nearring $(\mathbb{C}, +, *_E)$ is a topological nearring if and only if E is the graph of a continuous curve in $\mathbb{C}$. Is there a way to characterize them?

5. Characterizations of Planar Nearrings

There have been some results on the characterizations of planar nearrings other than the Ferrero pair construction.

Theorem 5.1 ([3]). *Let* N *be a zero-symmetric* 3*-prime nearring. Let* L *be an* N*-subgroup of* N*. Then there is an* $e = e^2 \in N$ *such that* $L = eN$*. Let* $\Phi = eNe \setminus \{0\}$*, then* (L, Φ) *is a Ferrero pair, and* L *is a planar nearring.*

Theorem 5.2 ([27]). *Let N be a nearring. Then the following are equivalent:*

(1) *N is planar.*

(2) *There exists a zero-symmetric nearring M and a left invariant subnearring P of M such that M acts 2-primitively on P via nearring multiplication, $(P, \mathrm{Aut}_M(P))$ is a Ferrero pair, and $N \cong P$.*

In [28] it is shown that a planar nearring is a centralizer nearring in the usual sense, but multiplication is not the usual function composition but rather composition of functions with a suitable sandwich function in between.

6. Algebraic structure of planar nearrings

The structure of radicals of planar nearrings was completely determined in [14], also lots of facts about ideals in planar nearrings can be found there. For example,

Theorem 6.1 ([14, Teorema 1]). *Let N be a planar nearring. Then there exists a greatest proper ideal D, which is the sum of all proper left ideals.*

Using this result, Wendt determines the ideal structure of planar nearrings completely.

Theorem 6.2 ([29]). *Let N be a planar nearring and D its greatest proper ideal. Then the proper left ideals of N are precisely the additive normal subgroups of N contained in D.*

Denote by $P(N)$ and $N(N)$, respectively, the prime and the nil radicals of a nearring N. Also, let $J_1(N)$ and $J_2(N)$, respectively, be the J_1 and J_2 radicals of N.

Theorem 6.3 ([14]). *Let N be a planar nearring. Then $P(N) = N(N) = J_1(N) = D$, D the greatest ideal properly contained in N. In case that $J_2(N) \neq N$, we have that $P(N) = N(N) = J_1(N) = J_2(N)$.*

Planar nearrings are very often 2-primitive (without identity).

Theorem 6.4 ([29] and also [14]). *A planar nearring N is 2-primitive if and only if A does not contain nontrivial subgroups of N.*

It is shown in [15] that the (nontrivial) homomorphic images of a finite planar nearring N is again planar. Also, if a planar nearring has a distributive element, it has a very special structure.

Theorem 6.5 ([29]). *If a planar nearring N has a distributive element $d \not\equiv_m 0$, then A is an ideal of N and N/A is a nearfield.*

7. Combinatorial designs from planar nearrings

Finite planar nearrings have a natural connection with combinatorial objects called tactical configurations. The three planar nearrings $(\mathbb{C}, +, *_i)$, $i = 1, 2, 3$, provide raw models for such connection. One particular group of tactical configurations receives most attentions from researchers because they are both structural for studies and practical for real life applications. These tactical configurations are referred to as BIBDs in short.

Definition 7.1. A finite set X with v elements together with a family $\mathbf{S}$ of k-subsets of X is called a *balanced incomplete block design* (*BIBD*) if

(i) each element belongs to exactly r subsets, and

(ii) each pair of distinct elements belongs to exactly λ subsets.

The k-subsets in $\mathbf{S}$ are called *blocks*, and the integers $v, b = |\mathbf{S}|, r, k, \lambda$ are referred to as the *parameters* of the BIBD.

7.1 $\mathbf{B}$, $\mathbf{B}^-$ and $\mathbf{B}^*$

Let $(N, +, \cdot)$ be a finite planar nearring with corresponding Ferrero pair (N, Φ). Denote $\Phi^0 = \Phi \cup \{0\}$ and $\Phi^- = \Phi \cup (-\Phi) \cup \{0\}$. Let

$$\begin{aligned} \mathbf{B} &= \{N \cdot a + b \mid a, b \in N, a \neq 0\} = \{\Phi^0(a) + b \mid a, b \in N, a \neq 0\}, \\ \mathbf{B}^- &= \{(N \cdot a + b) \cup (N \cdot (-a)) + b \mid a, b \in N, a \neq 0\} \\ &= \{\Phi^-(a) + b \mid a, b \in N, a \neq 0\}, \\ \mathbf{B}^* &= \{N^* \cdot a + b \mid a, b \in N, a \neq 0\} = \{\Phi(a) + b \mid a, b \in N, a \neq 0\}. \end{aligned}$$

We usually denote the set $\mathbf{B}^*$ as $\mathbf{B}_\Phi$ to emphasize the role of Φ.

Remark 7.2. These sets get their grounds from the geometrical considerations of the three examples $(\mathbb{C}, +, *_i)$, $i = 1, 2, 3$: $\mathbf{B}$ and $\mathbf{B}^-$ are the set of straight lines of the complex plane obtained in $(\mathbb{C}, +, *_1)$ and $(\mathbb{C}, +, *_2)$, respectively, while $\mathbf{B}^*$ is the set of circles of the complex plane obtained in $(\mathbb{C}, +, *_3)$.

Now, it is known that $(N, \mathbf{B})$ and $(N, \mathbf{B}^-)$ are sometimes BIBDs, and $(N, \mathbf{B}^*)$ is always a BIBD (cf. [9, Theorems 5.5, 7.14, and 7.99]).

Since $(N, \mathbf{B}_\Phi)$ is always a BIBD, it seems natural to investigate the automorphism group of it. Obviously, every normalizer of Φ in the group $\mathrm{Aut}\,(N, +)$ serves as an automorphism of the design. It is conjectured that the converse is also true.

Conjecture 7.3 (Modisett). The automorphism group of $(N, \mathbf{B}_\Phi)$ is $N \rtimes N_{\mathrm{Aut}\,(N,+)}(\Phi)$, where $N_{\mathrm{Aut}\,(N,+)}(\Phi)$ is the normalizer of Φ in $\mathrm{Aut}\,(N, +)$.

Since N has an additive group structure and $\mathbf{B}_\Phi$ is obtained from additive translations, it is natural to consider $(N, +, \mathbf{B}_\Phi)$ as a *design group*. Namely, N has a group structure, and each of the translations $\rho_a : N \to N; x \to x + a$, $a \in N$, is an automorphisms of the design. In this case, a mapping $N \to N$ is called an *automorphism* of the design group if it is at the same time an automorphism of the group as well as of the design. With this condition added for abelian N and Φ, or in case that either N or Φ is not abelian, N is large enough, we see that Modisett's conjecture has affirmative answer.

Theorem 7.4 ([17]). *Let (N, Φ) be a finite Ferrero pair such that N and Φ are abelian with $|\Phi| < |N| - 1$. Then* $\mathrm{Aut}\,(N, +, \mathbf{B}_\Phi)$ *is the normalizer of Φ in* $\mathrm{Aut}\,(N, +)$.

Theorem 7.5 ([6]). *Let (M, Ψ) and (N, Φ) be finite Ferrero pair and let σ be an isomorphism from $(M, \mathbf{B}_\Psi, +)$ to $(N, \mathbf{B}_\Phi, +)$. Let $|\Phi| = k$ and set $s = 2k^2 - 6k + 7$. If $|N/[N, N]| > s$, then $\sigma\Psi\sigma^{-1} = \Phi$. In particular, if $(M, \Psi) = (N, \Phi)$, then σ is a normalizer of Φ.*

The requirement that N is large enough in case when Φ is not abelian is necessary as the next example shows (cf. [6]).

Example 7.6. Let $F = \mathrm{GF}(7^3)$ and $\kappa : F \to \mathrm{Aut}\,(F)$ a coupling on F such that $F^\kappa := (F, +, \circ)$ is a proper nearfield with $a \circ b := a \cdot \kappa_a(b)$. Let $\Phi \le F^*$ of index 2. Since Φ is characteristic, $\Phi^\kappa := (\Phi, \circ)$ is a subgroup of $(F^\kappa)^*$. Then Φ^κ is nonabelian, and so Φ and Φ^κ are not isomorphic; therefore Φ and Φ^κ cannot be conjugate to each other. But $(F, \mathbf{B}_\Phi) = (F, \mathbf{B}_{\Phi^\kappa})$.

7.2 Segments

The planar nearring $(\mathbb{C}, +, *_2)$ inspires yet another possible construction of interesting geometric objects: the segments. For those who wonder how the segments can be interesting, the paper of Clay [11] offers a surprising construction of "triangles" with measurement of "angles" of the triangles within fields, and an analog result of the classical Euclidean geometry that the sum of the three angles of a triangle is π.

Let (N, Φ) be a Ferrero pair. For distinct $a, b \in N$, define

$$\overline{a, b} = \big(\Phi^0(b - a) + a\big) \cap \big(\Phi^0(a - b) + b\big),$$

and call it a *segment* with endpoints a and b. Let

$$\mathbf{S} = \{\overline{a, b} \mid a, b \in N,\ a \ne b\}.$$

Note that if one puts $S = \Phi^0 \cap (1 - \Phi^0)$, then $1 - S = S$ and $\overline{a, b} = (b - a)S + a$.

Theorem 7.7 ([25])**.** *If N is a nearfield or a ring, then $\overline{a,b} = \overline{c,d}$ if and only if $(a,b) = (c,d)$.*

The use of the set S in a finite field generated Ferrero pair suggests us to look at a more general construction from finite fields. Let F be a finite field and S a subset of F with $|S| \geq 2$. Consider

$$\mathbf{S} = \{Sa + b \mid a, b \in F,\ a \neq 0\}.$$

Then $(F, \mathbf{S})$ is always a BIBD. When $|S| = 3$, we are able to compute that full automorphism group of the design without the assumption of the design group structure. Our data also suggests that even for larger S, the full automorphism group of $(F, \mathbf{S})$ should obey this theorem.

Theorem 7.8 ([7])**.** *If $|S| = 3$, then the $(F, \mathbf{S})$ is a 2-$(q, 3, \lambda)$ design with $\lambda \in \{1, 2, 3, 6\}$. Let $U = \big\{r \mid \{0, 1, r\} \in \mathbf{S}\big\}$, and let $K = \langle U, +, \cdot\rangle$ be the subfield of F generated by U. Then under some mild condition, we have that $f \in \mathrm{Aut}\,(F, \mathbf{S})$ if and only if $f(x) = T(\alpha(x)) + b$ $(x \in F)$ for some $b \in F$, $\alpha \in \mathrm{Aut}_K(F)$, and $T \in \mathcal{L}(F, K)$ (= linear transformations of the vector space F over K).*

The last along this line of applications of planar nearrings and Ferrero pairs is to construct partial balanced incomplete block designs, PBIBD in short. We record the definition of a PBIBD from Clay's book [10, Defition 7.107].

Definition 7.9. Start with a finite tactical configuration $(N, \mathbf{T}, \in)$ and let $\mathcal{P} = \{A \mid A \subseteq N,\ |A| = 2\}$. Suppose $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ is a partition of $\mathcal{P}$. Then $\mathcal{A}$ is an *association scheme* on N if, given $\{x, y\} \in A_h$, the number of $z \in N$ such that $\{x, z\} \in A_i$ and $\{y, z\} \in A_j$ depends only upon h, i, and j, and *not* upon x and/or y. That is, there is a number p^h_{ij} such that for $\{x, y\} \in A_h$, there are exactly p^h_{ij} distinct elements $z \in N$ such that $\{x, z\} \in A_i$ and $\{y, z\} \in A_j$. Association schemes with $m = 1$ or $m = v(v-1)/2$, where $v = |N|$, are declared 'uninteresting'.

Suppose $(N, \mathbf{T}, \in, \mathcal{A})$ is a finite tactical configuration with association scheme $\mathcal{A}$. This structure is a *partially balanced incomplete block design* (PBIBD) if:

(a) to each $A_i \in \mathcal{A}$, there is a number n_i such that for each $x \in N$, there are exactly n_i distinct elements $y \in N$ such that $\{x, y\} \in A_i$;

(b) to each $A_i \in \mathcal{A}$, there is a number ℓ_i such that $\{x, y\} \in A_i$ implies x and y belong to exactly ℓ_i blocks of $\mathbf{T}$.

Following the above definition, Clay invited his readers to take out pens and paper to construct examples of PBIBDs using Hall's method (in eight steps). But here, without the mysterious eight steps, one can have many examples using ring generated planar nearrings.

Let $(R, +, \cdot)$ be a finite ring with unity and denote by $\mathcal{U}$ the group of units of R. Suppose that Φ is a subgroup of $\mathcal{U}$ with $-1 \in \Phi$. Let $\{s_1, \ldots, s_m\}$ be a complete set of orbit representatives of Φ in $R\backslash\{0\}$. For each i, let $A_i = \{\{x, y\} \mid x - y \in \Phi(s_i)\}$, and set $\mathcal{A} = \{A_i \mid 1 \le i \le m\}$.

Theorem 7.10 ([26])**.** (1) $(R, \mathcal{A})$ *is an associative scheme.* (2) *For any proper subset S of R with $|S| \ge 2$, denote $\mathbf{S} = \{aS + b \mid a \in \Phi, b \in R\}$. Then $(R, \mathbf{S}, \mathcal{A})$ is a PBIBD.*

8. Circularity and graphs

Clay came up with the concept of circular planar nearrings in [9] when he studied the planar nearring $(\mathbb{C}, +, *_3)$. We first give the definition of circular planar nearrings and Ferrero pairs. (Note that this definition is slightly different from the one given in [10, (5.1)].)

Definition 8.1. Let $(N, +, \cdot)$ be a planar nearring. If for $a, c, b, d \in N$, $a \not\equiv_m 0$ and $c \not\equiv_m 0$, it holds that $N^*a + b \ne N^*c + d$ implies that $|(N^*a + b) \cap (N^*c + d)| \le 2$, then we say that N is *circular.* If (N, Φ) is the corresponding Ferrero pair, then $N^*a = \Phi(a) = \{\varphi(a) \mid \varphi \in \Phi\}$. So N is circular if $|(\Phi(a) + b) \cap (\Phi(c) + d)| \le 2$ for all $a, b, c, d \in N$ with $a \ne 0$, $c \ne 0$ and $\Phi(a) + b \ne \Phi(c) + d$. We also say that the Ferrero pair (N, Φ) is *circular* in this manner.

For example, the planar nearring $(\mathbb{C}, +, *_3)$ is a circular planar nearring since for nonzero $a, b \in \mathbb{C}$, $\mathbb{C}^*a + b$ is simply the circle which passes through the point $a + b$ and centers at b. Actually, this example was the source for the definition of circularity of planar nearrings, and also was the inspiration for the connection between circular planar nearrings and graphs.

Surprisingly enough, the combinatorial condition of circularity imposed on planar nearrings selects a well-behaved class of nearrings.

8.1 Characterization of finite circular Ferrero pairs

First we consider Ferrero pairs (N, Φ) with Φ abelian. There are abundant circular Ferrero pairs to be found from finite fields.

Example 8.2. Let $F = \mathrm{GF}(p^2)$, p a prime, and let Φ_{p+1} be the subgroup of F^* order $p + 1$. Then the Ferrero pair (F, Φ_{p+1}) is circular. Consequently, if $k \ge 3$ and p is a prime with $k \mid (p+1)$, then the Ferrero (F, Φ_k) is circular.

The above examples pave the path to the following characterization of circular Ferrero pairs (F, Φ) with F a finite field. It turns the testing for circularity from "combinational" to "numerical." An algorithm for computing the finite sets $\mathcal{P}_k$ in the theorem was also provided in the cited paper. An improved method for computing $\mathcal{P}_k$ can be derived from [2], and we shall state it after the theorem.

Theorem 8.3 ([22]). *For each $k \geq 3$, there is a nonempty finite subset $\mathcal{P}_k$ of prime numbers with the following property: Let $q = p^s$, a power of some prime p, be such that $k \mid (q-1)$. Then there is a subgroup Φ_k of the multiplicative group $\mathrm{GF}(q)^*$ of order k, and the Ferrero pair $(\mathrm{GF}(q), \Phi_k)$ is circular if and only if $p \notin \mathcal{P}_k$.*

Here, we give our algorithm for computing the sets $\mathcal{P}_k$. Let $\zeta = e^{2\pi i/k} \in \mathbb{C}$. For u, v, s, t with $1 \leq u < v \leq s \leq k-1$, $1 \leq t \leq k-1$, and $v \neq t$ and $s \neq t$, define $\varphi_{u,v,s,t} = (\zeta^u - 1)(\zeta^t - 1) - (\zeta^v - 1)(\zeta^s - 1) \in \mathbb{Z}[\zeta]$. Then $\varphi_{u,v,s,t}$ is nonzero and has integer norm $N_{u,v,s,t} = N_{\mathbb{Q}(\zeta):\mathbb{Q}}(\varphi_{u,v,s,t})$. It can be seen that if F is a finite field of characteristic p and (F, Φ) is a field generated Ferrero pair, then (F, Φ) is circular if and only if the norms $N_{u,v,s,t}$ are nonzero when considered as elements of F. Thus, the set $\mathcal{P}_k$ consists of the prime factors of all such norms $N_{u,v,s,t}$.

For practical applications (e.g. to construct codes or cryptosystems), a sharp upper bound of $\mathcal{P}_k$ ($k \geq 3$) in terms of k may be useful. The above method provides us a trivial bound, namely the maximum of all possible $N_{u,v,s,t}$. Since $\varphi_{u,v,s,t}$ expands to 6 summands of powers of ζ, we see that the norm is less than or equal $6^{\varphi(k)}$, where $\varphi(k)$ is the Euler totient function giving the number of automorphisms of the kth cyclotomic field.

Conjecture 8.4. For $k \geq 3$, $N_{\mathbb{Q}(\zeta):\mathbb{Q}}(\varphi_{u,v,s,t}) \leq (8\sqrt{3}/3)^{\varphi(k)}$.

Problem 8.5. Find a better bound for $\mathcal{P}_k$, $k \geq 3$.

Yet another question one can ask is

Problem 8.6. For $k \geq 3$, what is the size of $\mathcal{P}_k$?

Next, we consider circular Ferrero pairs (N, Φ) with finite nonabelian Φ. Since Φ is a regular group of automorphisms of N, the Sylow p-subgroups of Φ are either cyclic or generalized quaternion (cf. [24, (12.6.15) and (12.6.17))]. The circularity of (N, Φ) excludes the second possibility. Namely, if (N, Φ) is circular, then Φ is metacyclic [2, (3.2)].

We note that the converse of the above assertion is not true. One can find Ferrero pairs (N, Φ) with metacyclic Φ while (N, Φ) is not circular.

Similar to that of $\mathcal{P}_k$, there is a finite set $\mathcal{P}_\Phi$ of primes which can be used to determine the circularity of (N, Φ) numerically.

Theorem 8.7 ([2]). *Let (N, Φ) be a circular Ferrero pair with finite Φ. Then there is a nonempty finite subset $\mathcal{P}_\Phi$ of prime numbers with the following property: Let M be a finite group such that (M, Φ) is a Ferrero pair. Then (M, Φ) is circular if and only if $p \notin \mathcal{P}_\Phi$ for all prime divisors p of $|M|$.*

Remark 8.8. The assumption that (N, Φ) is circular Ferrero pair is used to guarantee the finiteness of $\mathcal{P}_\Phi$ (see [2] for details). Thus, we said that a given group Φ

- is a *group without fixed points* if there is a group N such that (N, Φ) is a Ferrero pair, and
- is a *circular group without fixed points* if there is a group M such that (M, Φ) is a circular Ferrero pair.

We have just seen that if Φ is a finite group without fixed points and Φ is circular, then Φ is metacyclic. Conversely, all finite metacyclic groups are groups without fixed points, *but not all of them are circular.*

Problem 8.9. Let Φ be finite metacyclic group. (Thus Φ is a group without fixed points.) Under what conditions is Φ circular?

Metacyclic groups have very nice presentations as generators and relations. The following is one of the presentations.

Theorem 8.10 (Zassenhaus 1936). *Let Φ be a metacyclic group. Then*

$$\Phi \cong \langle A, B \mid A^m = B^n = 1,\ B^{-1}AB = A^r \rangle,$$

where $m > 0$, $\gcd(m, (r-1)n) = 1$, and $r^n \equiv 1 \pmod{m}$. If d is the order of r modulo m, then all irreducible complex representation of Φ are of degree d.

Using an extra assumption on Φ so that it can be more easily handled, there is a partial answer to Problem 8.9.

Theorem 8.11 ([5]). *Let Φ be a metacyclic group with a presentation as in Theorem 8.10 and $d = 2$. If Φ is embeddable as a subgroup of the multiplicative group of some skew field, then Φ is circular.*

The data we have at hand suggests that this theorem should hold in general.

Conjecture 8.12. If Φ is a metacyclic group embeddable as a subgroup of the multiplicative group of some skew field, then Φ is circular.

8.2 Graphs of circular Ferrero pairs

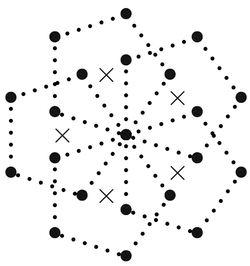
Another idea of Clay on studying the structure of circular planar nearrings is to connect them to graphs. Let (N, Φ) be a Ferrero pair. We will assume that Φ is finite. For $r, c \in N \setminus \{0\}$, define

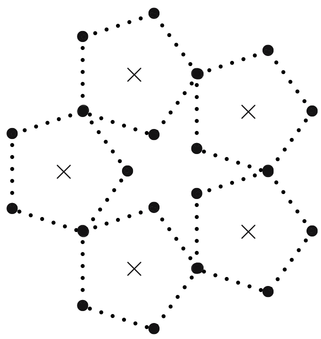
$$E_c^r = \{\Phi(r) + b \mid b \in \Phi(c)\}.$$

Then the set E_c^r is a partition of $N \setminus \{0\}$. In fact, it was from an equivalence relation defined on the set $\{\Phi(a) + b \mid a, b \in N,\ a \neq 0\}$, that Clay came up with this partition.

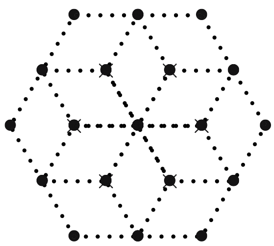
Example 8.13. We consider the Ferrero pair $(\mathbb{C}, \widehat{C})$. For an integer $k \geq 3$, the subgroup of $\widehat{C}$ of order k is denoted by Φ_k.

(1) Here are two E_c^r's in $(\mathbb{C}, \Phi_5)$ (each $\times$ indicates the center of the circle that the five points (vertices of a pentagon) of a $\Phi_5 r$ inscribed):





(2) Here is an E_c^r in $(\mathbb{C}, \Phi_6)$:

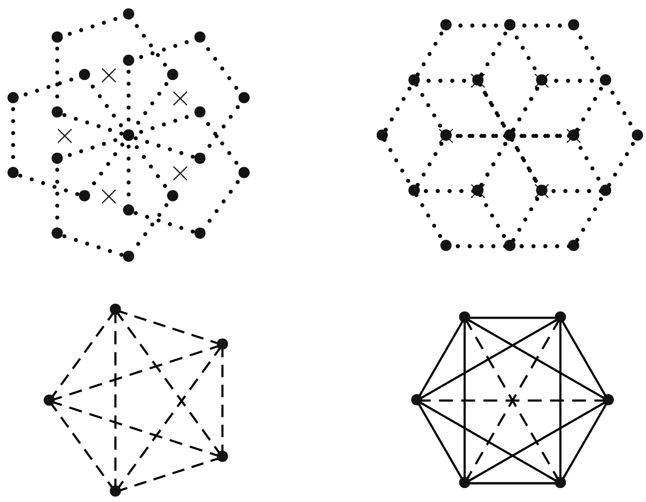


To see the structure of an E_c^r, a graph $G(E_c^r) = (\mathcal{V}, \mathcal{E})$ is assigned to it: the vertex set $\mathcal{V} = \Phi(c)$ and the edge set $\mathcal{E}$ is

$$\{c_1c_2 \mid c_1, c_2 \in \Phi(c), c_1 \neq c_2, \text{ and } (\Phi(r) + c_1) \cap (\Phi(r) + c_2) \neq \varnothing\}.$$

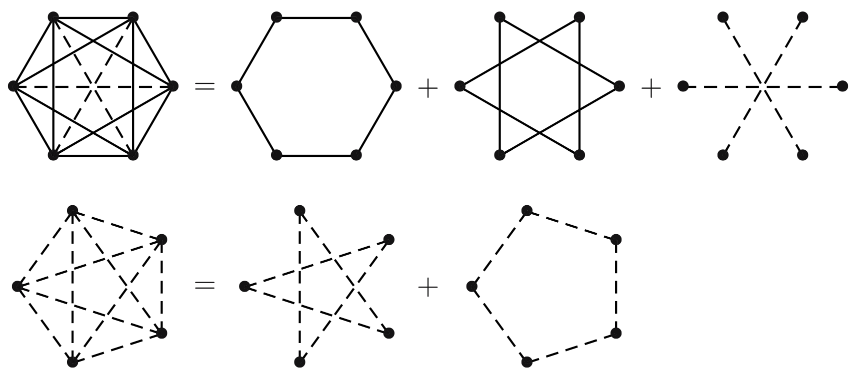
An edge c_1c_2 is *even* if $|(\Phi(r) + c_1) \cap (\Phi(r) + c_2)| = 2$, and is *odd* if $|(\Phi(r) + c_1) \cap (\Phi(r) + c_2)| = 1$. The following pictures show the two E_c^r

we have seen above and the graphs they defined.



Some properties of the graphs $G(E_c^r)$ can be observed immediately. The most obvious one is that every $G(E_c^r)$ is a regular graph, i.e. all the vertices have the same number of edges connected to them.

On the other hand, some properties require some detailed analysis of the graphs. If $G(E_c^r)$ has nonnull edges, then it is a union of even and/or odd *basic graphs*. Here we illustrate this by two examples.



So the leftmost graph on the first line is the "disjoint union" of two odd graphs while the leftmost graph on the second line is the "disjoint union" of two even graphs and an odd graph.

Some special arrangements of the graphs produce pictures like David's stars and Prisms (see [10]). After investigation on many examples, we found that there are numbers about the basic graphs that depends on k alone.

So, let (N, Φ) be a ring generated Ferrero pair with $|\Phi| = k$. Fix an $r \in N \setminus \{0\}$ and consider all $G(E_c^r)$, where $c \in N$, such that $G(E_c^r)$ has some edges. Then each of such graphs is either a basic graph or a "union" of basic graphs. Let γ_j denote the total number of the appearances the odd jth basic graphs in these nonnull graphs, and π_j the total number of the appearances of the even jth basic graphs.

Theorem 8.14 ([19]). *If* $2 \mid k$, *then* $\gamma_j = 1$ *and* $\pi_j = k/2 - 1$ *for any* $j \in \{1, 2, \ldots, k/2\}$.

The reason for j to stop at $k/2$ in the statement of the above theorem is that the jth basic graph and the $(k - j)$th basic graph are identical.

Here is a somewhat surprising application of the above counting of basic graphs to find solutions of certain equations over finite fields.

Let $F = \mathrm{GF}(q)$ be the Galois field of order q. Let $k \mid (q - 1)$ be such that (F, Φ_k) is a circular Ferrero pair. Put $m = (q - 1)/k$. Denote by n the number of solutions of the equation

$$x^m + y^m - z^m = 1$$

in F, and by n' the number of solutions with $xyz \neq 0$.

Theorem 8.15 ([18]). (1) *If k is even, then*

$$n = \begin{cases} 3(k-1)m^3 + 6m^2 + 3m & \text{if } 6 \mid k; \\ 3(k-1)m^3 + 3m^2 + 3m & \text{if } p = 3; \\ 3(k-1)m^3 + 3m & \text{otherwise;} \end{cases}$$

and $n' = 3(k - 1)m^3$.

(2) *If k is odd, and if* $(\mathrm{GF}(q), \Phi_{2k})$ *is also circular, then* $n = (2k - 1)m^3 + 2m$ *and* $n' = (2k - 1)m^3$.

Actually, one can explicitly write down the solutions. This was done by Kiechle in [21].

Next, we observed that when (N, Φ) is a ring generated Ferrero pair with cyclic Φ, some of the basic graphs always appear together in some $G(E_c^r)$ (referred as *overlapped* graphs). A complete understanding of such behavior is the key to count the total number of graphs $G(E_c^r)$. Using a theorem of *vanishing sums* [12, Theorem 6], we have a complete description of this phenomenon in the case of $(\mathbb{C}, \Phi_k)$, $\Phi_k \leq \widehat{C}$ and $|\Phi_k| = k \geq 3$. (See [20].)

Finally, we have also noticed that the graphs of E_c^r's occur in the finite field generated case and the complex plane case are the same when the field has large enough characteristic. For small characteristic, there are more overlapped basic graphs. To explain partly this phenomenon, we note that the overlaps of the basic graphs are in one-one correspondence with the solutions (u, v, s, t) of the equations

$$\frac{\zeta^u - 1}{\zeta^v - 1} = \zeta^w \frac{\zeta^s - 1}{\zeta^t - 1}$$

where ζ is a primitive kth root of unity, $1 \leq u < v \leq s \leq k-1$, $1 \leq t \leq k-1$, $v \neq t$, $s \neq t$, and $1 \leq w \leq k-1$.

Now, consider $\zeta = e^{2\pi i/k} \in \mathbb{C}$ as before, and put the set $\mathcal{OP}_k$ the prime factors of the norms of

$$(\zeta^u - 1)(\zeta^t - 1) - \zeta^w(\zeta^s - 1)(\zeta^v - 1)$$

for all suitable u, v, s, t, w. Then each $\mathcal{OP}_k$ is a finite set. When p is a prime larger than any of that in $\mathcal{OP}_k$ and $k \,|\, (p^\ell - 1)$ for some positive integer ℓ, the overlaps of the graphs of the E_c^r's from $(\mathrm{GF}(p^s), \Phi_k)$ and that from $(\mathbb{C}, \Phi_k)$ are the same.

Problem 8.16. As we have mentioned, an E_c^r is simply an equivalence class of a block $\Phi(r) + b$. Are there any other equivalence on the set $\{\Phi(a)+b \mid a, b \in N, a \neq 0\}$ which will give use interesting (and hopefully manageable) equivalence classes?

9. List of ongoing research problems on planar nearrings

We would like to invite more people to join us on exploring the fascinating world of planar nearrings, circular or not. In the following, a list of problems concerning planar nearrings are given. This list came from "Group Discussions" when the author visited Linz in the summers of 2002 and 2003. One realizes easily by scanning through the list that there are much more of planar nearrings to be uncovered!

(1) **The complex number field $\mathbb{C}$.**

- What to study in each individual planar nearring?
- Characterize all fixed point free automorphism groups Φ on $(\mathbb{C}, +)$: $\Phi \leq (\mathbb{C}^*, \cdot)$, or $\mathbb{C}$ as a $\mathbb{R}^2$, or $\mathbb{C}$ as a vector space over $\mathbb{Q}$. Note that the descriptions of finite Φ's can be found in [30].
- Is being algebraically closed important for the study? How continuity may come into play?
- Are there other constructions similar to Jim Clay's hyperbolas?

(2) **The real number field $\mathbb{R}$ and the rational field $\mathbb{Q}$.**