CONTINUOUS-TIME SIGNALS

Continuous-Time Signals

by

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To my family

Preface

As far back as the 1870s, when two American inventors Elisha Gray and Graham Bell independently designed devices to transmit speech electrically (the telephone), and the 1890s, when the Russian scientist Aleksandr Popov and the Italian engineer Guglielmo Marconi independently demonstrated the equipment to transmit and receive messages wirelessly (the radio), the theory of electrical signal was born. However, the idea of signals has been employed by mankind all through history, whenever any message was transmitted from a far point. Circles on water indicating that some disturbance is present in the area give a vivid example of such messages. The prehistory of electrical signals takes us back to the 1860s, when the British scientist James Clerk Maxwell predicted the possibility of generating electromagnetic waves that would travel at the speed of light, and to the 1880s, when the German physicist Heinrich Hertz demonstrated this radiation (hence the word "radio").

As a time-varying process of any physical state of an object that serves for representation, detection, and transmission of messages, a modern electrical *signal*, in applications, possesses many specific properties including:

- A flow of information, in information theory;
- Disturbance used to convey information and information to be conveyed over a communication system;
- An asynchronous event transmitted between one process and another;
- An electrical transmittance (either input or output) that conveys information;
- Form of a radio wave in relation to the frequency, serving to convey intelligence in communication;
- A mechanism by which a process may be notified by the kernel of an event occurring in the system;
- A detectable impulse by which information is communicated through electronic or optical means, or over wire, cable, microwave, laser beams, etc;
- A data stream that comes from electrical impulses or electromagnetic waves;

- Any electronic visual, audible, or other indication used to convey information;
- The physical activity of the labeled tracer material that is measured by a detector instrument; the signal is the response that is measured for each sample;
- A varying electrical voltage that represents sound.

How to pass through this jungle and understand the properties of signals in an optimum way? Fundamental knowledge may be acquired by learning the continuous-time signals, for which this book offers five major steps:

1. Observe applications of signals in electronic systems, elementary signals, and basic canons of signals description (Chapter 1).
2. Consider the representation of signals in the frequency domain (by Fourier transform) and realize how the spectral density of a single waveform becomes that of its burst and then the spectrum of its train (Chapter 2).
3. Analyze different kinds of amplitude and angular modulations and note a consistency between the spectra of modulating and modulated signals (Chapter 3).
4. Understand the energy and power presentations of signals and their correlation properties (Chapter 4).
5. Observe the bandlimited and analytic signals, methods of their description, transformation (by Hilbert transform), and sampling (Chapter 5).

This book is essentially an extensive revision of my Lectures on Radio Signals given during a couple of decades in Kharkiv Military University, Ukraine, and several relevant courses on Signals and Systems as well as Signal Processing in the Guanajuato University, Mexico, in recent years. Although, it is intended for undergraduate and graduate students, it may also be useful in postgraduate studies.

Salamanca, Mexico *Yuriy S. Shmaliy*

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Introduction

Signals and *processes* in electronic systems play a fundamental role to transfer information from one point or space to any other point or space. Their description, transformation, and conversion are basic in *electrical engineering*. Therefore, it becomes possible to optimize systems with highest efficiency both in the time and frequency domains. This is why the theory of signals is fundamental for almost all electrical engineering fields; Mechanical, chemical, physical, biological, and other systems exploit fundamentals of this theory whenever waves and waveforms appear. Our purpose in this chapter is to introduce a concept and necessary fundamental canons of *signals*, thereby giving readers food for learning the following chapters.

1.1 Signals Application in Systems

The word "signal" has appeared from the Latin term *signum* meaning "sign" and occupied a wide semantic scope in various ranges of science and engineering. It is defined as follows:

> ***Signal***: A *signal* is a time-varying process of any physical state of any object, which serves for representation, detection, and transmission of messages.

□

In electrical engineering, time variations of electric currents and voltages in electronic systems, radio waves radiated by a transmitter in space, and noise processes in electronic units are examples of signals. Application of signals in several most critical electronic systems are illustrated below.

1.1.1 Radars

The *radar* is usually called a device for determining the presence and location of an object by measuring the time for the echo of a radio wave to return from it and the direction from which it returns. In other words, it is a measuring instrument in which the echo of a pulse of microwave radiation is used to detect and locate distant objects. A radar pulse-train is a type of amplitude modulation of the radar frequency carrier wave, similar to how carrier waves are modulated in communication systems. In this case, the information signal is quite simple: a single pulse repeated at regular intervals.

Basic operation principle of radars is illustrated in Fig. 1.1. Here transmitter generates radio frequency (RF) impulse signal that is reflected from the target (moving or stationary object) and is returned to receiver. Conventional ("monostatic") radar, in which the illuminator and receiver are on the same platform, is vulnerable to a variety of countermeasures. Bistatic radar, in which the illuminator and receiver are widely separated, can greatly reduce the vulnerability to countermeasures such as jamming and antiradiation weapons, and can increase slow moving target detection and identification capability by "clutter tuning" (receiver maneuvers so that its motion compensates for the motion of the illuminator; creates zero Doppler shift for the

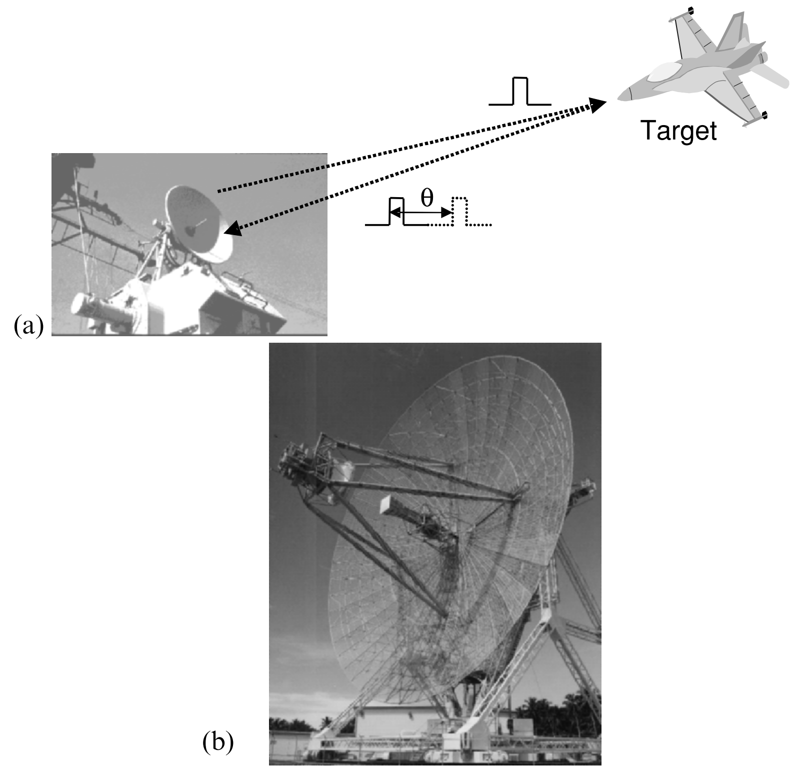

Fig. 1.1 Operation principle of radars: (**a**) pulse radar and (**b**) long-range radar antenna.

area being searched). The transmitter can remain far from battle area, in a "sanctuary." The receiver can remain "quiet."

At the early stage, radars employed simple single pulses to fulfill requirements. With time, for the sake of measuring accuracy, the pulses with frequency modulation and pulse-coded bursts were exploited. The timing and phase coherent problems can be orders of magnitude more severe in bistatic than in monostatic radar, especially when the platforms are moving. The two reference oscillators must remain synchronized and synchronized during a mission so that the receiver knows when the transmitter emits each pulse, so that the phase variations will be small enough to allow a satisfactory image to be formed. Low noise crystal oscillators are required for short-term stability. Atomic frequency standards are often required for long-term stability.

1.1.2 Sonar

Sonar (acronym for *SOund NAvigation and Ranging*) is called a measuring instrument that sends out an acoustic pulse in water and measures distances in terms of the time for the echo of the pulse to return. This device is used primarily for detection and location of underwater objects by reflecting acoustic waves from them, or by interception of acoustic waves from an underwater, surface, or above-surface acoustic source. Note that sonar operates with acoustic waves in the same way that radar and radio direction-finding equipment operate with electromagnetic waves, including use of the Doppler effect, radial component of velocity measurement, and triangulation.

1.1.3 Remote Sensing

Remote sensing is the science — and to some extent, art — of acquiring information about the Earth's surface without actually being in contact with it. This is done by sensing and recording reflected or emitted energy and processing, analyzing, and applying that information. Two kinds of remote sensing are employed. In *active remote sensing*, the object is illuminated by radiation produced by the sensors, such as radar or microwaves (Fig. 1.2a). In *passive remote sensing*, the sensor records energy that is reflected or emitted from the source, such as light from the sun (Fig. 1.2b). This is also the most common type of system.

1.1.4 Communications

Analog and digital communications are likely the most impressive examples of efficient use of signals. In analog communications, an analog method of modulating radio signals is employed so that they can carry information such as voice or data. In digital communications, the carrier signal is modulated digitally by encoding information using a binary code of "0" and "1". Most

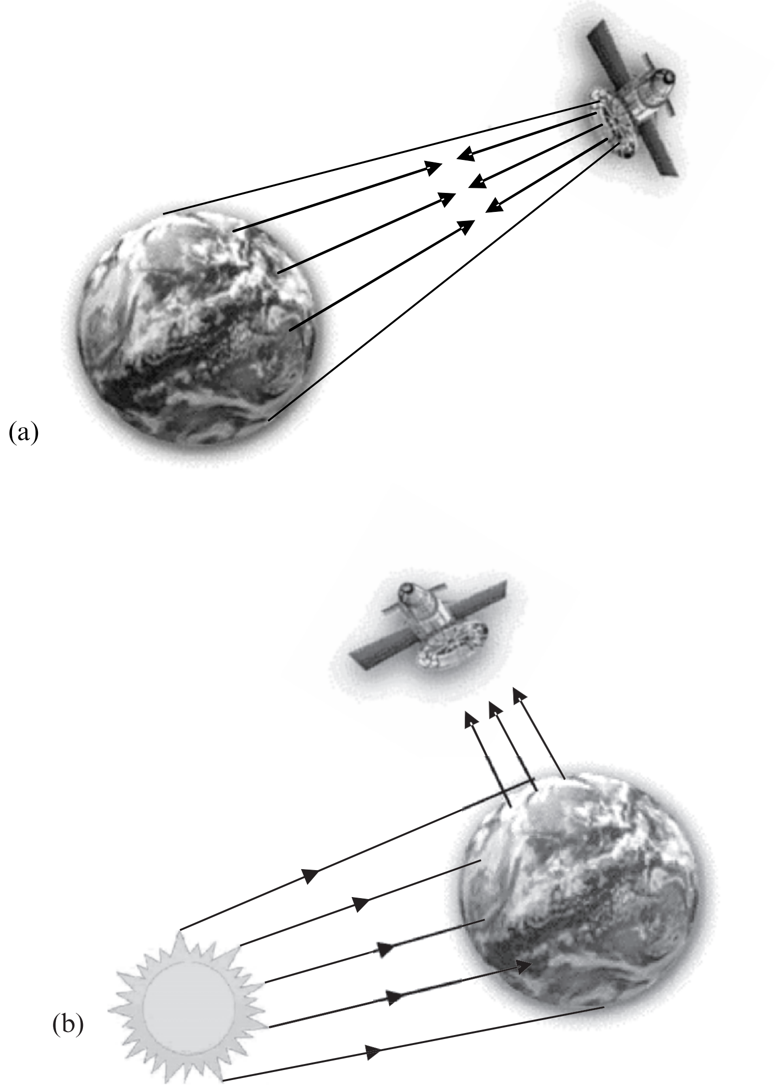

Fig. 1.2 Remote sensing operation principle: (**a**) active and (**b**) passive.

newer wireless phones and networks use digital technology and one of the most striking developments of the past decade has been the decline of public service broadcasting systems everywhere in the world. Figure 1.3 illustrates the basic principle of two-way satellite communications.

To transfer a maximum of information for the shortest possible time duration, different kinds of modulation had been examined for decades at different

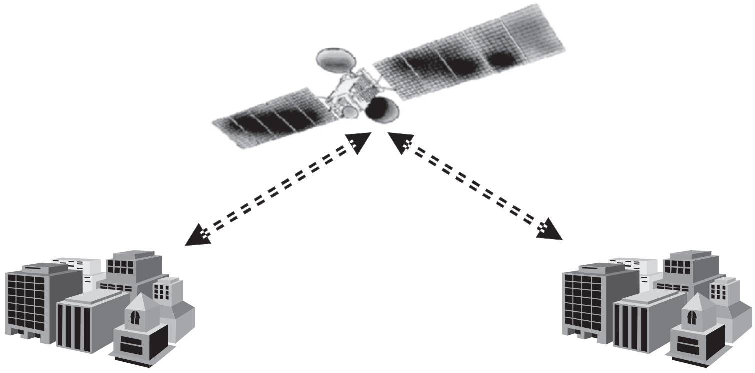

Fig. 1.3 Two-way satellite communications.

carrier frequencies. In digital transmission, either a binary or M-ary keying is used in amplitude, phase, and frequency providing the commercially available resources with a minimum error.

Historically, as the number of users of commercial two-way radios have grown, channel spacing have been narrowed, and higher-frequency spectra have had to be allocated to accommodate the demand. Narrower channel spacings and higher operating frequencies necessitate tighter frequency tolerances for both the transmitters and the receivers. In 1949, when only a few thousand commercial broadcast transmitters were in use, a 500 ppm (ppm = 10^{-6}) tolerance was adequate. Today, the millions of cellular telephones (which operate at frequency bands above 800 MHz) must maintain a frequency tolerance of 2.5 ppm. The 896–901 MHz and 935–940 MHz mobile radio bands require frequency tolerances of 0.1 ppm at the base station and 1.5 ppm at the mobile station. The need to accommodate more users will continue to require higher and higher frequency accuracies. For example, NASA concept for a personal satellite communication system would use walkie-talkie-like hand-held terminals, a 30 GHz uplink, a 20 GHz downlink, and a 10 kHz channel spacing. The terminals' frequency accuracy requirement is few parts in 10^{-8}.

1.1.5 Global Positioning System

Navigation systems are used to provide moving objects with information about their positioning. An example is the satellite-based global positioning system (GPS) that consists of (a) a constellation of 24 satellites in orbit 11,000 nmi above the Earth, (b) several on-station (i.e., in-orbit) spares, and (c) a ground-based control segment. Figure 1.4 gives an example of the GPS use in ship navigation. Each space vehicular (SV) transmits two microwave carrier signals (Fig. 1.5). The L1 frequency (1575.42 MHz) carries the navigation message and the standard positioning service (SPS) code signals. The L2 frequency (1227.60 MHz) is used to measure the ionospheric delay by precise

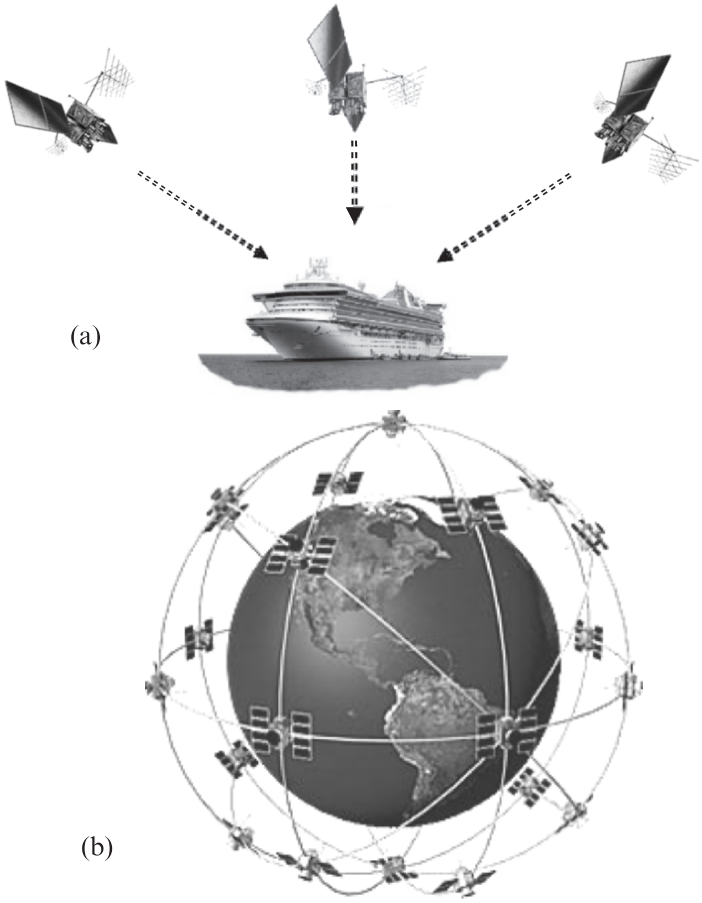

Fig. 1.4 GPS system: (**a**) ship navigation and (**b**) GPS constellation.

positioning service (PPS) equipped receivers. Three binary codes shift the L1 and/or L2 carrier phase:

- The coarse acquisition (C/A) code modulates the L1 carrier phase. The C/A code is a repeating 1 MHz pseudorandom noise (PRN) code. This noise-like code modulates the L1 carrier signal, "spreading" the spectrum over a 1 MHz bandwidth. The C/A code repeats every 1023 bits (one millisecond). There is a different C/A code PRN for each SV. GPS satellites are often identified by their PRN number, the unique identifier for each PRN code. The C/A code that modulates the L1 carrier is the basis for the civil SPS.
- The precision (P) code modulates both the L1 and the L2 carrier phases. The P code is a very long (7 days) 10 MHz PRN code. In the antispoofing (AS) mode of operation, the P code is encrypted into the Y code. The

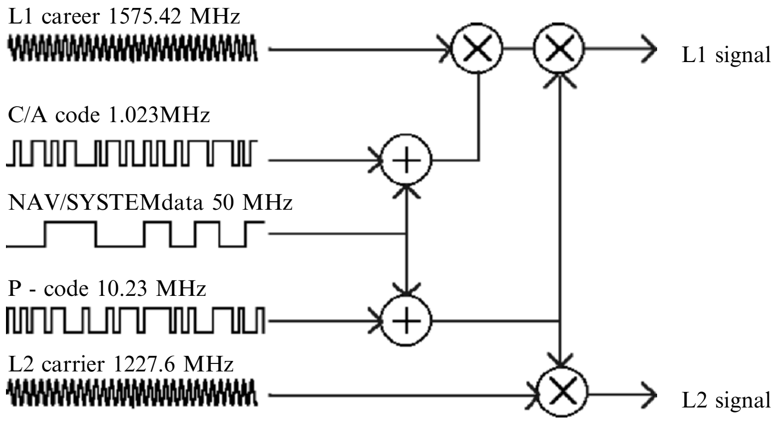

Fig. 1.5 GPS satellite signals.

encrypted Y code requires a classified AS module for each receiver channel and is for use only by authorized users with cryptographic keys. The P(Y) code is the basis for the PPS.

- The navigation message also modulates the L1-C/A code signal. The navigation message is a 50 Hz signal consisting of data bits that describe the GPS satellite orbits, clock corrections, and other system parameters.

Any navigation system operates in time. Therefore, to obtain extremely accurate 3-D (latitude, longitude, and elevation) global navigation (position determination), precise time (time signals) must also be disseminated. These signals are used in what is called timekeeping.

Historically, navigation has been a principal motivator in man's search for better clocks. Even in ancient times, one could measure latitude by observing the stars' position. However, to determine longitude, the problem became one of timing. This is why GPS-derived position determination is based on the arrival times, at an appropriate receiver, of precisely timed signals from the satellites that are above the user's radio horizon. On the whole, in the GPS, atomic clocks in the satellites and quartz oscillators in the receivers provide nanosecond-level accuracies. The resulting (worldwide) navigational accuracies are about 10 m and some nanoseconds. Accordingly, GPS has emerged as the leading methodology for synchronization not only for communication but also for transport, navigation, commercial two-way radio, space exploration, military requirements, Doppler radar systems, science, etc.

1.2 Signals Classification

Classification of signals may be done for a large number of factors that mostly depend on their applications in systems.

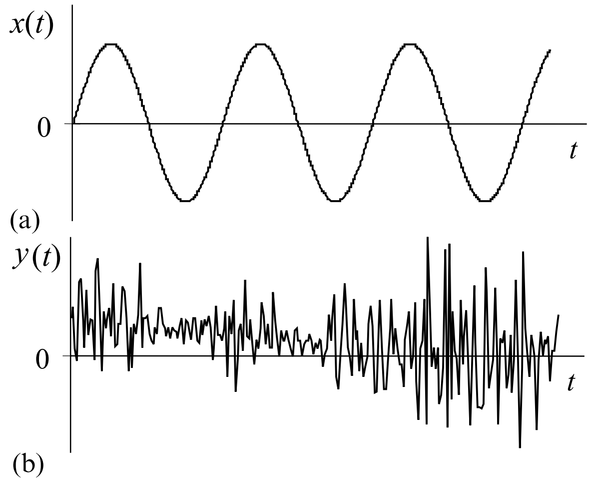

Fig. 1.6 Example of signals: (**a**) deterministic and (**b**) random.

1.2.1 Regularity

Most commonly, the signals are separated into two big classes: *deterministic* (Fig. 1.6a) (*regular* or *systematic* in which a random amount is insignificant) and *random* (*noisy*) (Fig. 1.6b).

- *Deterministic signals* are precisely determined at an arbitrary time instant; their simulation implies searching for proper analytic functions to describe them explicitly or with highest accuracy.
- *Random signal* cannot be described analytically at an arbitrary time instant owing to its stochastic nature; such signals cannot be described in deterministic functions or by their assemblage and are subject to the probability theory and mathematical statistics.

It is important to remember that a recognition of signals as deterministic and random is conditional in a sense. Indeed, in our life there are no deterministic physical processes at all, at least because of noise that exists everywhere. The question is, however, how large is this noise? If it is negligible, as compared to the signal value, then the signal is assumed to be deterministic. If not, the signal is random, and stochastic methods would be in order for its description.

1.2.2 Causality

The signals produced by physical devices or systems are called *causal*. It is assumed that such a signal exists only at or after the time the signal generator is turned on. Therefore, the casual signal $y(t)$ satisfies $y(t) = x(t)$, if $t \geqslant 0$ and

$y(t) = 0$, if $t < 0$. Signals that are not causal are called *noncausal*. Noncausal signals representation is very often used as a mathematical idealization of real signals, supposing that $x(t)$ exists with $-\infty \leq t \leq \infty$.

1.2.3 Periodicity

Both deterministic and random signals may be either *periodic* (Fig. 1.7a) or *single* (*nonperiodic*) (Fig. 1.7b).

- *Periodic signals* (Fig. 1.7a) reiterate their values through the equal time duration T called a period of repetition. For such signals the following equality holds true:

$$x(t) = x(t \pm nT) \tag{1.1}$$

  where $x(t)$ is a signal and $n = 0, 1, 2, ...$ It seems obvious that simulation of (1.1) implies that a signal may be described only on the time interval T and then repeated n times with period T.
- *Single signals* or *nonperiodic signals* (Fig. 1.7b) do not exhibit repetitions on the unlimited time interval and therefore an equality (1.1) cannot be applied.
- *Impulse signals.* A special class of signals unites the *impulse* signals. A single impulse signal is the one that exists only during a short time. Impulse signals may also be periodic. Two types of impulse signals are usually distinguished:
  - *Video pulse signal*, also called *waveform*, is an impulse signal $x(t)$ without a carrier (Fig. 1.8a).
  - *Radio frequency (RF) pulse* signal $y(t)$ is a video pulse signal $x(t)$ filled with the carrier signal $z(t)$ (Fig. 1.8b).

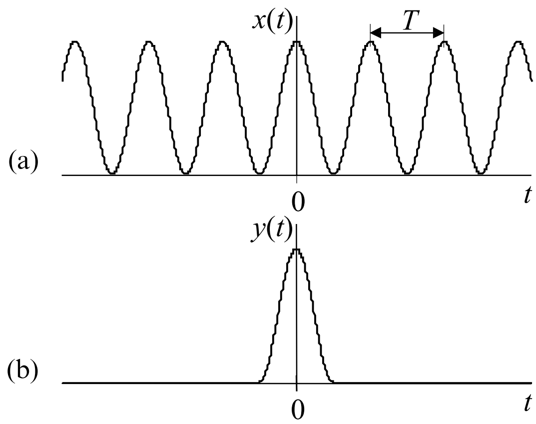

Fig. 1.7 Example of signals: (**a**) $x(t)$ is periodic and (**b**) $y(t)$ is nonperiodic.

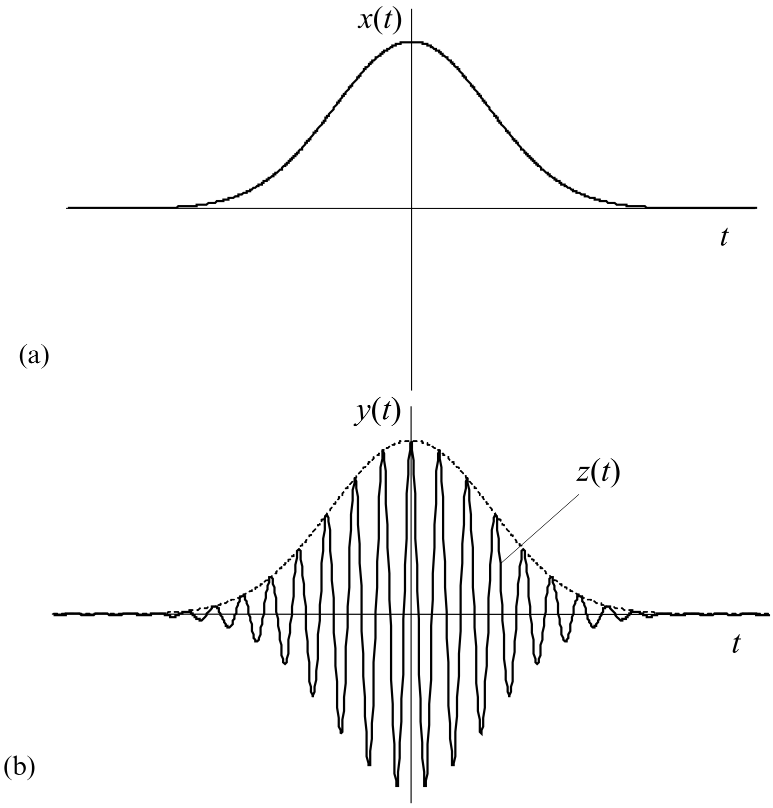

Fig. 1.8 Impulse signals: (**a**) video pulse and (**b**) radio pulse.

1.2.4 Dimensionality

Both periodic and single signals may depend on different factors and exist in the same timescale. Accordingly, they may be *one-dimensional* and *multidimensional*:

- *One-dimensional (scalar) signal* is a function of one or more variables whose range is 1-D. A scalar signal is represented in the time domain by means of only one function. Examples are shown in Figs. 1.6 and 1.7. A physical example is an electric current in an electronic unit.
- *Multidimensional (vector) signal* is a vector function, whose range is 3-D or, in general, N-dimensional (N-d). A vector signal is combined with an assemblage of 1-D signals. An N-d signal is modelled as a vector of dimensions $N \times 1$

$$\mathbf{x} \equiv \mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T \tag{1.2}$$

where an integer N is said to be its *order* or *dimensionality.* An example of a multidimensional signal is several voltages on the output of a multipole. An example of 2-D signals is an electronic image of the USA and Mexico obtained by NASA with a satellite remote sensing at some instant t_1 (Fig. 1.9a). An example of 3-D signals is fixed at some instant t_2, a cosine wave attenuated with a Gaussian[1] envelope in the orthogonal directions (Fig. 1.9b).

1.2.5 Presentation Form

Regarding the form of presentation, all signals may be distinguished to fall within three classes:

- An *analog signal* or *continuous-time signal* is a signal $x(t)$, which value may be determined (measured) at an arbitrary time instant (Fig. 1.10a).
- *Discrete-time* signal is a signal $x(t_n)$, where n is an integer that represents an analog signal by discrete values at some time instants t_n, usually with a constant sample time $\Delta = t_{n+1} - t_n$ (Fig. 1.10b).
- *Digital* signal is a signal $x[n]$, which is represented by discrete values at discrete points n with a digital code (binary, as a role) (Fig. 1.10c). Therefore, basically, $x[n] \neq x(t_n)$ and the quantization error depends on the resolution of the analog-to-digital converter.

1.2.6 Characteristics

Every signal may be explicitly described either in the time domain (by time functions) or in the frequency domain (by spectral characteristics). Signals presentations in the time and frequency domains are interchangeable to mean that any signal described in the time domain may be translated to the frequency domain and come back to the time domain without errors. The following characteristics are usually used to describe signals:

- In the time domain: *effective duration*, *covariance function*, *peak amplitude*, *period of repetition*, *speed of change*, *correlation time*, *time duration*, etc.
- In the frequency domain:
  - Spectrum of periodic signals is represented by the Fourier[2] series with the *magnitude spectrum* and *phase spectrum.*

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[1] Johann Carl Friedrich Gauss, German mathematician, 30 April 1777–23 February 1855.

[2] Jean Baptiste Joseph Fourier, French mathematician, 21 March 1768–16 May 1830.

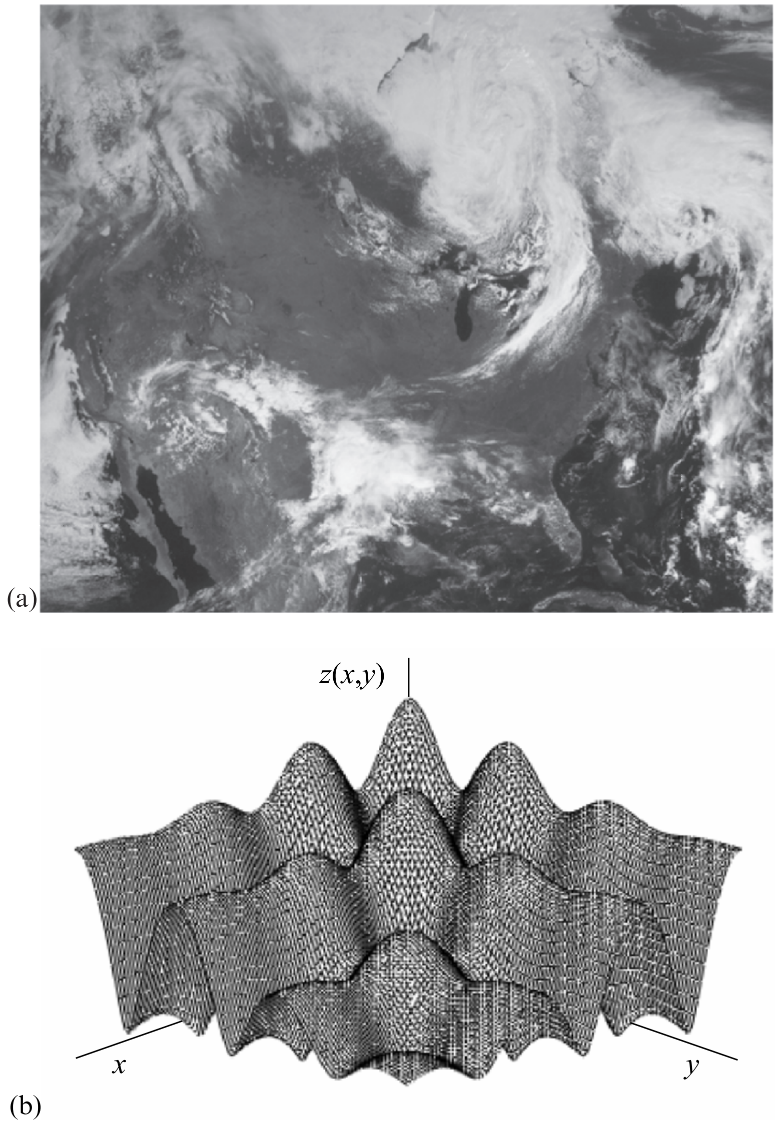

Fig. 1.9 Multidimensional signals: (**a**) 2-D satellite electronic image and (**b**) 3-D Gaussian radio pulse.

- Spectral density of nonperiodic signals is represented by the Fourier transform with the *magnitude spectral density* and *phase spectral density*.
- Both the spectrum and spectral density are characterized with the *signal energy*, *signal power*, *spectral width*, *spectral shape*, etc.

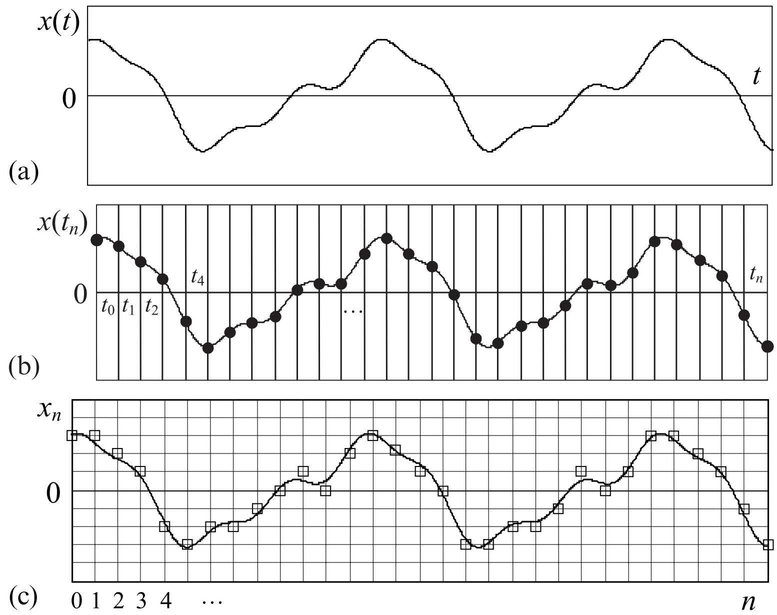

Fig. 1.10 Types of signals: (**a**) continuous-time, (**b**) discrete-time, and (**c**) digital.

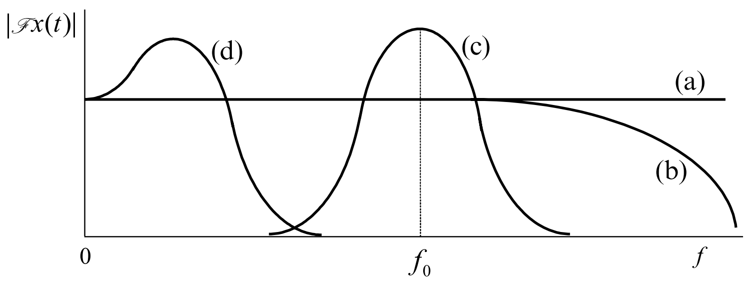

Fig. 1.11 Types of signals: (a) broadband, (b) bandlimited, (c) narrowband, and (d) baseband.

1.2.7 Spectral Width

In the frequency domain, all signals may be classified as follows:

- A *broadband* signal is the one, which spectrum is distributed over a wide range of frequencies as it is shown in Fig. 1.11a.
- A *bandlimited* signal is limited in the frequency domain with some maximum frequency as it is shown in Fig. 1.11b.
- A *narrowband* signal has a spectrum that is localized about a frequency f_0 that is illustrated in Fig. 1.11c.

- A *baseband* signal has a spectral contents in a narrow range close to zero (Fig. 1.11d). Accordingly, a spectrum beginning at 0 Hz and extending contiguously over an increasing frequency range is called a *baseband spectrum.*

1.2.8 Power and Energy

Every signal bears some energy and has some power. However, not each signal may be described in both terms. An example is a constant noncausal signal that has infinite energy.

An instantaneous power of a real signal $x(t)$ is defined by

$$P_x(t) = x^2(t). \tag{1.3}$$

In applications, however, it is much more important to know the signal energy or average power over some time bounds $\pm T$. Accordingly, two types of signals are recognized:

- *Energy signal* or *finite energy signal* is a signal, which energy

$$E_x = \|x\|_2^2 = \lim_{T\to\infty} \int_{-T}^{T} P_x(t)\mathrm{d}t = \lim_{T\to\infty} \int_{-T}^{T} x^2(t)\mathrm{d}t < \infty \tag{1.4}$$

  is finite. The quantity $\|x\|_2$ used in (1.4) is known as the L_2-norm of $x(t)$.

- *Power signal* or *finite power signal* is a signal which average power

$$P_x = \langle x^2(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\mathrm{d}t < \infty \tag{1.5}$$

  is finite. If $x(t)$ is a periodic signal with period T then the limit in (1.5) is omitted.

Example 1.1. Given a harmonic noncausal signal $x(t) = A_0 \cos\ \omega_0 t$, which energy is infinite, $E_x = A_0^2 \int_{-\infty}^{\infty} \cos^2 \omega_0 t\mathrm{d}t = A_0^2 \infty$. Thus, it is not an energy signal. However, its average power is finite, $P_x(t) = \frac{A_0^2}{2T} \int_{-T}^{T} \cos^2 \omega_0 t\mathrm{d}t = \frac{1}{2} A_0^2 < \infty$. Hence, it is a power signal.

□

1.2.9 Orthogonality

In the correlation analysis and transforms of signals, *orthogonal signals* play an important role.

- Two real signals $x(t)$ and $y(t)$ are said to be *orthogonal*, $x(t) \perp y(t)$, on the interval $[a, b]$ if their inner (scalar) product (and so the joint energy) is zero:

$$\langle x, y \rangle = \int_a^b x(t)y(t)\mathrm{d}t = 0\,. \quad (1.6)$$

- Two same signals are called *orthonormal* if

$$\langle x, y \rangle = \begin{cases} 1, & x(t) = y(t) \\ 0, & \text{otherwise} \end{cases}.$$

In other words, if a function (signal) $x(t)$ has a zero projection on some other function (signal) $y(t)$, then their joint area is zero and they are orthogonal. Such an important property allows avoiding large computational burden in the multidimensional analysis.

Example 1.2. Given three signals:

$$x(t) = A_0 \cos \omega_0 t\,,$$
$$y(t) = A_0 \sin \omega_0 t\,,$$
$$z(t) = A_0 \cos(\omega_0 t + \pi/4)\,.$$

It follows, by (1.6), that two first signals are orthogonal and that no other pair of these signals satisfies (1.6).

□

We have already classified the signals with many characteristics. Even so, this list is not exhaustive and may be extended respecting some new methods of signals generation, transmitting, formation, and receiving. Notwithstanding this fact, the above given classification is sufficient for an overwhelming majority of applied problems.

1.3 Basic Signals

Mathematical modeling of signals very often requires its presentation by simple elementary signals, which properties in the time and frequency domains are well studied. Indeed, if we want to describe, for example, a rectangular pulse-train, then a linear combination of gained and shifted elementary unit-step functions will certainly be the best choice. We may also want to describe some continuous function that may be combined with elementary harmonic functions in what is known as the Fourier series. So, basic elementary functions play an important role in the signals theory.

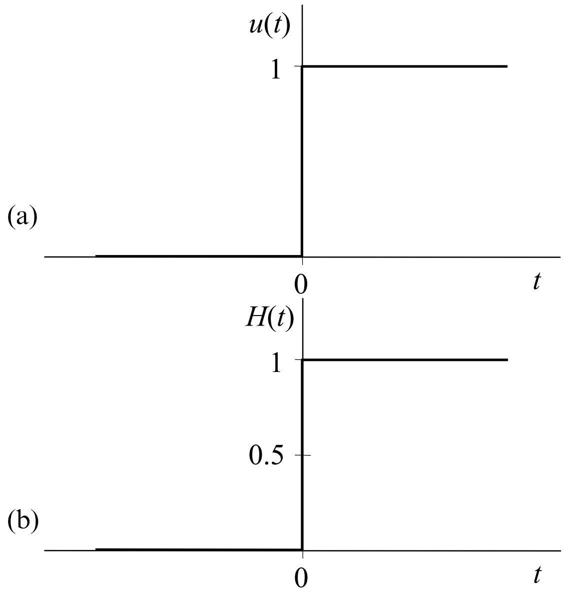

Fig. 1.12 Unit step: (**a**) Unit-step function and (**b**) Heaviside unit-step function.

1.3.1 Unit Step

A *unit-step function* (Fig. 1.12a) is defined by

$$u(t) = \begin{cases} 1, & t \geqslant 0 \\ 0, & t < 0 \end{cases} \tag{1.7}$$

and is usually used in signals to model rectangular waveforms and in systems to define the step response.

The other presentation of a unit step was given by Heaviside[3] in a conventionally continuous form. The *Heaviside unit-step function* (Fig. 1.12b) is performed as

$$H(t) = \begin{cases} 1, & t > 0 \\ 0.5, & t = 0 \\ 0, & t < 0 \end{cases} \tag{1.8}$$

and may be modeled by the function

$$v(t, \xi) = \begin{cases} 1, & t > \xi \\ 0.5(\frac{t}{\xi} + 1), & -\xi \leqslant t \leqslant \xi \\ 0, & t < -\xi \end{cases}, \tag{1.9}$$

[3] Oliver Heaviside, English physicist, 18 May 1850–3 February 1925.

once $H(t) = \lim_{\xi \to 0} v(t, \xi)$. This is not the only way to model the unit step. The following function may also be useful:

$$v(t, n) = \frac{1}{1 + e^{-nt}} \,. \tag{1.10}$$

It follows from (1.10) that tending n toward infinity makes the function to be more and more close to the Heaviside step function, so that one may suppose that $H(t) = \lim_{n \to \infty} v(t, n)$.

Example 1.3. Given a rectangular impulse signal (Fig. 1.13a). By (1.7), it is described to be $x(t) = 8.5[u(t-1) - u(t-3)]$.

□

Example 1.4. Given a truncated ramp impulse signal (Fig. 1.13b). By (1.7) and (1.9), we go to the model $x(t) = 8.5[v(t-2, 1) - u(t-3)]$.

□

Example 1.5. Given an arbitrary continuous signal (Fig. 1.13c). By (1.7), this signal is described as

$$x(t) = \sum_{i=-\infty}^{\infty} x(iT)[u(t - iT) - u(t - iT - T)] \,,$$

where a sample time T should be chosen to be small enough to make the approximation error negligible.

□

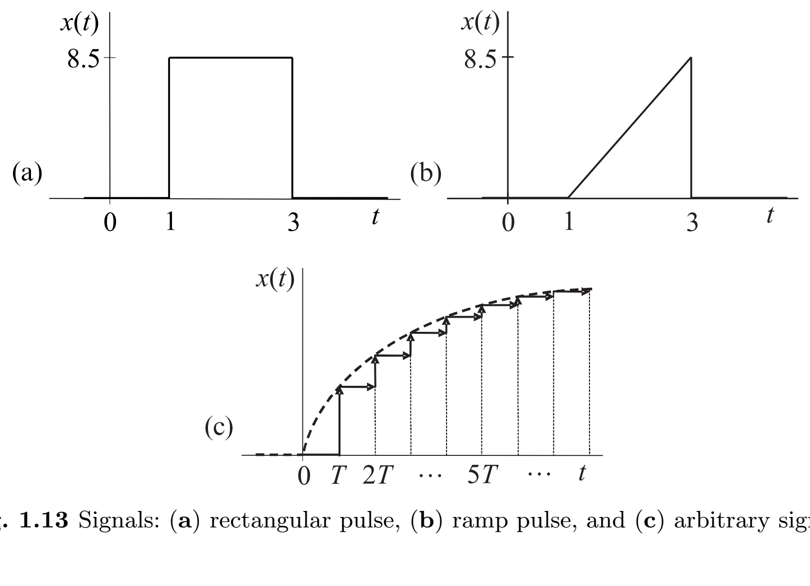

Fig. 1.13 Signals: (**a**) rectangular pulse, (**b**) ramp pulse, and (**c**) arbitrary signal.

1.3.2 Dirac Delta Function

The *Dirac*[4] *delta function*, often referred to as the *unit impulse*, *impulse symbol*, *Dirac impulse*, or *delta function*, is the function that defines the idea of a unit impulse, having the fundamental properties

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}, \tag{1.11}$$

$$\int\limits_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1\,. \tag{1.12}$$

Mathematically, $\delta(t)$ may be defined by the derivative of the unit-step function,

$$\delta(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}\,. \tag{1.13}$$

In an equivalent sense, one may also specify the unit step by integrating the delta function,

$$u(t) = \int\limits_{-\infty}^{t} \delta(t)\mathrm{d}t\,. \tag{1.14}$$

The fundamental properties of the delta function, (1.11) and (1.12), are also satisfied if to use the following definition:

$$\delta(t) = \lim_{\xi \to 0} \frac{\mathrm{d}H(t,\xi)}{\mathrm{d}t}\,. \tag{1.15}$$

Therefore, the unit impulse is very often considered as a rectangular pulse of the amplitude $1/2\xi$ (Fig. 1.14). Following (1.11), it needs to set $\xi = 0$ in (1.15) and Fig. 1.14a, and thus the delta function is not physically realizable.

The *Kronecker*[5] *impulse* (or *symbol*) is a discrete-time counterpart of the delta function; however, it is physically realizable, as $\xi \neq 0$ in the discrete scale. Both the delta function (Fig. 1.14b) in the continuous time and the Kronecker impulse in the discrete time are used as test functions to specify the system's impulse response.

The following properties of $\delta(t)$ are of importance.

1.3.2.1 Sifting

This property is also called *sampling property* or *filtering property* . Since the delta function is zero everywhere except zero, the following relations hold true:

$$x(t)\delta(t) = x(0)\delta(t) \quad \text{and} \quad x(t)\delta(t-\theta) = x(\theta)\delta(t-\theta)\,,$$

[4] Paul Adrien Maurice Dirac, English mathematician, 8 August 1902–20 October 1984.

[5] Leopold Kronecker, German mathematician, 7 December 1823–29 December 1891.

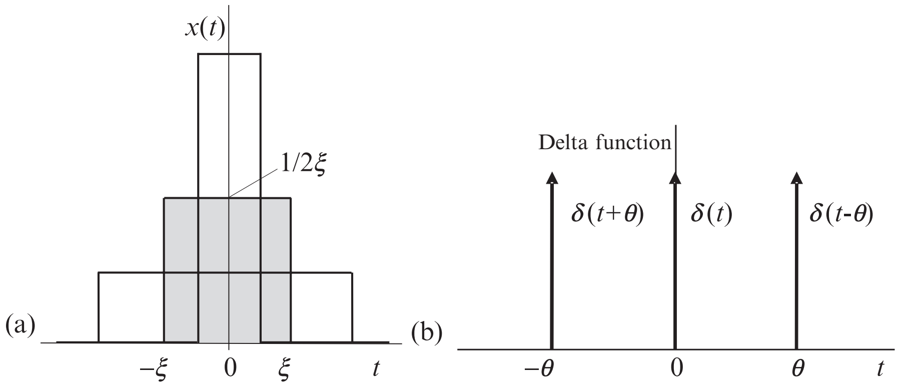

Fig. 1.14 Unit impulse: (**a**) rectangular model and (**b**) positions, by a time-shift $\pm\theta$.

allowing us to write

$$\int_{-\infty}^{\infty} x(t)\delta(t-\theta)\mathrm{d}t = \int_{-\infty}^{\infty} x(\theta)\delta(t-\theta)\mathrm{d}t$$
$$= x(\theta)\int_{-\infty}^{\infty} \delta(t-\theta)\mathrm{d}t = x(\theta)\,. \tag{1.16}$$

So, if to multiply any continuous-time function with the delta function and integrate this product in time, then the result will be the value of the function exactly at the point where the delta function exists. In a case of $\theta = 0$, (1.16) thus degenerates to

$$\int_{-\infty}^{\infty} x(t)\delta(t)\mathrm{d}t = x(0)\,. \tag{1.17}$$

Alternatively, the sifting property also claims that

$$\int_{a}^{b} x(t)\delta(t)\mathrm{d}t = \begin{cases} x(0), & a < 0 < b \\ 0, & a < b < 0 \text{ or } 0 < a < b \\ x(0)\delta(0), & a = 0 \quad\text{ or } b = 0 \end{cases}. \tag{1.18}$$

It is important to remember that both (1.17) and (1.18) are symbolic expressions and should not be considered as an ordinary Riemann[6] integral. Therefore, $\delta(t)$ is often called a *generalized function* and $x(t)$ is then said to be a *testing function.*

[6] Georg Friedrich Bernhard Riemann, German mathematician, 17 September 1826–20 July 1866.