MATHEMATICAL METHODS IN ENGINEERING

Mathematical Methods in Engineering

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Preface

This book contains some of the contributions under five main titles that are carefully selected according to the reports of referees, presented at the International Symposium, MME06 Mathematical Methods in Engineering, held in Çankaya University, Ankara, April 27–29, 2006.

The Symposium provided a setting for discussing recent developments in Fractional Mathematics, Neutrices and Generalized Functions, Boundary Value Problems, Applications of Wavelets, Dynamical Systems and Control Theory.

The members of the organizing committee were Dumitru Baleanu, Ronald A. DeVore, J.A. Tenreiro Machado, Ali H. Nayfeh and Kenan Tas (Chairman). Lecturers of the Symposium were Om P. Agrawal, Brian Fisher, J.A. Tenreiro Machado, Francesco Mainardi, Hans J. Stetter.

The editors of this book are grateful to the President of the board of trustees of Çankaya University Sitki Alp, to the Rector Prof.Dr. Ziya Aktaş for their continuous support of the Symposium activities. We are also obliged to the TUBITAK (The Scientific and Technological Research Council of Turkey) for their co-sponsorship.

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Ankara, August 8, 2006

Kenan Tas, J.A. Tenreiro Machado, Dumitru Baleanu
Editors

Fractional Mathematics

Fractional calculus and regularized residue of infinite dimensional space

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We have proposed regularization of infinite dimensional integral via fractional calculus. It is done on a Hilbert space H equipped with a Schatten class operator G. The ζ-function $\zeta(G,s)$ of G is assumed to be holomorphic at s=0. Regularization is done by using $\zeta(G,s)$. After reviewing this regularization, it is shown regularized Cauchy kernel of a Hilbert space with the determinant bundle exists if and only if $\nu=\zeta(G,0)$ is an integer. Regularized residue on an infinite dimensional space is obtained as an application of regularized Cauchy kernel.

1 Fractional calculus and regularized infinite product

Let $\{H,G\}$ be a pair of a Hilbert space and a positive Schatten class operator G such that $\zeta(G,s)=\mathrm{tr}G^s$ is holomorphic at $s=0$. $\zeta(G,s)$ is assumed to have its first pole at $s=d$. We also set

$$\nu=\zeta(G,0),\quad detG=e^{\zeta'(G,0)},\quad c=\mathrm{Res}_{s=d}\zeta(G,s).$$

We often need integrity of ν. If H is the Hilbert space of square integrable sections of a bundle E over a compact Riemannian manifold X and G is the Green operator of a positive elliptic operator D acting on the sections of E, choosing suitable mass term m and replace D by $D+mI$, ν becomes an integer. Hence integrity of ν is not restrictive for practical use (cf.[Asa04a]).

The complete ortho-normal basis $e_1,e_2,\ldots$, are taken from eigenvectors of G: $Ge_n=\mu_n e_n$, $\mu_1\geq\mu_2\geq\ldots>0$. By using G, we introduce Sobolev metric $\|x\|_k$ by $\|G^{-k}x\|$. The Sobolev space constructed by H and $\|\cdot\|_k$ is denoted by W^k. The complete ortho-normal basis of W^k is given by $e_{1,k},e_{2,k},\ldots$, $e_{n,k}=\mu_n^k e_n$. We set $e_{\infty,k}=\sum_{n=1}^{\infty}\mu_n^{d/2}e_{n,k}$. $e_{\infty,k}$ does not belong to W^k, but belongs to W^l, $l<k$. If $k=0$, we denote e_∞, instead of $e_{\infty,0}$.

Definition 1. *The Hilbert space $W^{k,\natural}$ is $W^k\oplus\mathcal{K}e_{\infty,k}$ with the inner product*

$$\langle e_{n,k},e_{m,k}\rangle=\delta_{n.m},\quad\langle e_{\infty,k},e_n\rangle=0,\quad\langle e_{\infty,k},e_{\infty,k}\rangle=c. \tag{1}$$

Kenan Taş et al. (eds), Mathematical Methods in Engineering, 3–11.

Here $\mathcal{K}$ is $\mathcal{R}$ if H is a real Hilbert space, and $\mathcal{C}$ if H is a complex Hilbert space. If $k=0$, we denote $H^{\natural}$, instead of $W^{0,\natural}$. We identify W^k and $W^k \oplus 0e_{\infty,k} \subset W^{k,\natural}$. Then the above inner product on $W^{k,\natural}$ coincides with the inner product of W^k. While the inner products $\langle e_{\infty,k}, e_{n,k}\rangle$ and $\langle e_{\infty,k}, e_{\infty,k}\rangle$ come from

$$\langle e_{\infty,k}, e_{n,k}\rangle = \lim_{s\downarrow 0}(\sqrt{s}G^{s/2-k}e_{\infty,k}, \sqrt{s}G^{s/2-k}e_{n,k}),$$

where (,) is the inner product of H.

By definition, $x \in W^{k,\natural}$ is uniquely written as $x_f + te_{\infty,k}$. Hence we can write

$$x = x_f + te_{\infty,k} = \sum_{n=1}^{\infty} x_{f,n}e_{n,k} + te_{\infty,k} = \sum_{n=1}^{\infty} x_n e_{n,k}, \quad x_n = x_{f,n} + \mu_n^{d/2}t. \tag{2}$$

Let $I_n^a f$ be the fractional integral $I_n^a f(x_n) = \dfrac{1}{\Gamma(a)}\displaystyle\int_0^{x_n} \frac{f(t)}{(x-t)^{1-a}}dt$. Then we have

$$\lim_{n\to\infty} I_1^{\mu_1^s}\cdots I_n^{\mu_n^s}1 = \prod_{n=1}^{\infty}\Gamma(1+\mu_n^s)\prod_{n=1}^{\infty}x_n^{\mu_n^s}.$$

Since

$$\log\prod_{n=1}^{\infty}\Gamma(1+\mu_n^s) = -\gamma\zeta(G,s) + \sum_{m=2}^{\infty}(-1)^m\frac{\zeta(m)}{m}\zeta(G,ms),$$

taking a path $C = C(s)$; $0 \le s \le 1$ in the right half plane such that $C(0) = 1$ and does not tangent to real and imaginary axes, the analytic continuation of $\prod_{n=1}^{\infty}\Gamma(1+\mu_n^s)$ to $s=0$ along C takes the value 1.

Definition 2. *Let $x = \sum_{n=1}^{\infty} x_n e_n$ be an element of $W^{k,\natural}$. Then we define regularized infinite product* $:\prod_{n=1}^{\infty} x_n :$ *of $x_1, x_2, \dots$ by*

$$:\prod_{n=1}^{\infty} x_n := \prod_{n=1}^{\infty} x_n^{\mu_n^s}|_{s=0}. \tag{3}$$

Here $|_{s=0}$ means analytic continuation to $s=0$.

It is known $:\prod_n x_n:$ is linear in each variable x_n and

$$|:\prod_{n=1}^{\infty} x_n :| =: \prod_{n=1}^{\infty}|x_n| :, \quad (:\prod_{n=1}^{\infty} x_n)^m =: \prod_{n=1}^{\infty} x_n^m : . \tag{4}$$

If $x = x_f + te_{\infty,k} \in W^{k,\natural}$ and $t \neq 0$, we have

$$:\prod_{n=1}^{\infty} x_n := t^{\nu}(detG)^{k+d/2}\prod_{n-1}^{\infty}(1+\frac{\mu_n^{-(k+d/2)}x_{f,n}}{t})^{\mu_n^s}|_{s=0}.$$

Then regarding $W^l, l > k$ to be a subset of $W^k \subset W^{k,\natural}$, and $W^{1,l}$, etc., to be ℓ^1-type subset $\{\sum_n x_n e_{n,l} | \sum |x_n| < \infty\}$, etc., of W^l, etc., we have

Proposition 1. $:\prod_n x_n:$ *is a single valued function if and only if* ν *is an integer, and the followings hold;*

1. *If* $t \neq 0$ *and* $x_f \in W^{1,k+d/2}$*, then* $:\prod_n x_n:$ *exists. If* $x \in W^{k+d/2}$*, it exists if and only if* $\sum_{n=1}^{\infty} \mu_n^{s-(k+d/2)} x_{f,n}$ *is holomorphic at* $s = 0$.
2. $:\prod_n x_n:$ *is analytic on* $W^{k+d/2} \oplus \mathcal{C}e_{\infty,k}$.

2 Regularized determinant

Let T be a densely defined linear operator on H. Then its regularized trace (renormalized trace) with respect to G is defined by

$$tr_G T = \mathrm{tr}(G^s T)|_{s=0},$$

[CDP02, Payc01]. For example, $\mathrm{tr}_G I = \nu$. By using regularized trace, we define

Definition 3. *If* T *has the logarithm* $S = \log T$*;* $T = e^S$*, then we define regularized determinant* $det_G T$ *of* T *with respect to* G *by*

$$det_G T = e^{\mathrm{tr}_G T} = e^{\mathrm{tr}(G^s T)}|_{s=0}. \tag{5}$$

Note 1. Since $\log T$ is not unique, $det_G T$ is not unique in general.

Example 1. If $I = I_x$; $x = (x_1, x_2, \ldots)$, is a scaling operator $I_x e_n = x_n e_n$, then $\log I_x$ is $I_{\log x}$; $\log x = (\log x_1, \log x_2, \ldots)$. Hence we have

$$det_G I_x = e^{\sum_{n=1}^{\infty} \mu_n^s \log x_n}|_{s=0} = \prod_{n=1}^{\infty} x_n^{\mu_n^s}|_{s=0} =: \prod_{n=1}^{\infty} x_n : .$$

Especially, we have

$$det_G G = det G, \quad det_G D = det D, \; G = D^{-1}, \tag{6}$$

where $det D$ is the Ray-Singer determinant of D.

Note 2. We have $det_G(I_x + N) = det_G I_x$, if N is a generalized nilpotent.

On the other hand, we have only $det_G PTP^{-1} = det_{P^{-1}GP} T$ in general. It may different from $det_G T$. For example, if G and T are

$$Ge_{2n-1} = \frac{1}{n} e_{2n-1}, Ge_{2n} = \frac{1}{n+1} e_{2n}, \; Te_{2n-1} = 2e_{2n-1}, Te_{2n} = 3e_{2n},$$

and $Pe_{2n-1} = e_{2n}$, $Pe_{2n} = e_{2n-1}$, then

$$det_G T = 2^{-1/2} 3^{-3/2} \neq det_G PTP^{-1} = 2^{-3/2} 3^{-1/2}.$$

We have $det_G T = det_G PTP^{-1}$ if $P \in GL(\infty)$, where $GL(\infty)$ is the closure of the group of invertible linear operators of the form $I + K$, K is a compact operator.

3 Regularized integral

Let W^k be a real Hilbert space, f a function on W^k which is extended to $W^{k,\natural}$ and expressed as $f = \lim_{n\to\infty} f(x_1, \ldots, x_n)$. Then the regularized integral $\displaystyle\int_{W^{k,\natural}} f : d^\infty x :$ is defined by

$$\int_{W^{k,\natural}} f : d^\infty x := \lim_{n\to\infty} \int_{\mathcal{R}^n} f(x_1, \ldots, x_n) d(x_1^{\mu_1^s}) \cdots d(x_n^{\mu_n^s})|_{s=0}, \tag{7}$$

[Asa04b], cf.[Asa04a, Asa04c]. Regularized integral on

$$W_+^{k,\natural} = \{\sum_n x_{f,n} e_{n,k} + t e_{\infty,k} \in W^{k,\natural} | x_{f,n} \geq 0, n = 1, 2, \ldots, t \geq 0\},$$

is similarly defined.

Regularized integral simplifies the fractional calculus $\lim\limits_{n\to\infty} I_1^{\mu_1^s} \cdots I_n^{\mu_n^s} f|_{s=0}$. It is also interpreted as an application of the weak limit

$$\lim_{N\to\infty} \frac{\partial^N}{\partial x_1 \cdots \partial x_N} : \prod_{n=1}^{\infty} x_n := 1,$$

which is hold on suitable function space [Asa04b].

Theorem 1. *Let I_a, $a = (a_1, a_2, \ldots)$; $I_a : W^k \to W^l$ be a scaling operator, and let $I_a^a st f(x) = f(I_a x)$, f a function on W^l. Then we have*

$$\int_{W^{l,\natural}} f : d^\infty x := \int_{W^{k,\natural}} |det_G I_a|^{-1} I_a^* f : d^\infty x : . \tag{8}$$

If I_a maps $W_+^{k,\natural}$ to $W_+^{l,\natural}$, then we also have

$$\int_{W_+^{l,\natural}} f : d^\infty x := \int_{W_+^{k,\natural}} |det_G I_a|^{-1} f : d^\infty x : .$$

Example 2. To set $e^{-\pi\|x\|^2} = 0$, if $\|x\| = \infty$, we extend $e^{-\pi\|x\|^2}$ to $H^\natural$. If G is the Green operator of an elliptic operator D, we have

$$e^{-\pi(x,Dx)} = I_{\sqrt{D}}^* e^{-\pi\|x\|^2}, \quad I_{\sqrt{D}} e_n = \sqrt{\lambda_n} e_n, \ \lambda_n = \mu_n^{-1}.$$

Hence we get

$$\int_{W^{1/2,\natural}} e^{-\pi(x,Dx)} : d^\infty x := \int_{H^\natural} |det\sqrt{D}|^{-1} e^{-\|x\|^2} : d^\infty x := \frac{1}{\sqrt{detD}}.$$

This justifies physicist's calculation $\displaystyle\int e^{-\pi(x,Dx)} \mathcal{D}x = \frac{1}{\sqrt{detD}}$.

4 Regularized Cauchy kernel

In the rest, we assume H is a complex Hilbert space. In $W^{k,\natural}$, we set

$$\mathcal{T}_r^{\infty,k} = \{\sum_{n=1}^{\infty} z_n e_{n,k} \in W^{k,\natural} | |z_n| = \mu_n^{d/2} r\}. \tag{9}$$

If $r=1$, we denote $\mathcal{T}^{\infty,k}$ instead of $\mathcal{T}_1^{\infty,k}$. Considering $\mathcal{C}^n$ to be $\{\sum_{j=1}^n z_j e_j\}$, we have

$$\mathcal{T}_r^{\infty,k} \cap \mathcal{C}^n = \{\sum_{j=1}^{n} z_j e_j | |z_j| = \mu_j^{k+d/2} r\}.$$

We denote this set by $\mathcal{T}_r^{n,k}$ and set $D_r^{n,k} = \{\sum_{j=1}^n z_j e_j | |z_j| \leq r\mu_j^k\}$. Here k is omitted if $k=-d/2$ and r is omitted if $r=1$.

By the map $w = z^a$, the circle $\{z = e^{i\theta} | 0 \leq \theta < 2\pi\}$ is mapped to $\{w = e^{i\phi} | 0 \leq \phi < 2a\pi\}$. That is we have

$$\frac{(2\pi i)^{a-1}}{a} \int_{|z|=1} \frac{d(z^a)}{z^a} = (2\pi i)^a, \quad \int_{|z|=1} dz = \int_0^{2\pi} i e^{i\theta} d\theta.$$

Hence we have

$$\lim_{n\to\infty} \int_{\mathcal{T}^n} \Big(\frac{(2\pi i)^{\mu_1^s - 1}}{\mu_1^s} \frac{d(z_1^{\mu_1^s})}{z_1^{\mu_1^s}} \cdots \frac{(2\pi i)^{\mu_n^s - 1}}{\mu_1^s} \frac{d(z_n^{\mu_n^s})}{z_n^{\mu_n^s}} \Big)|_{s=0} = (2\pi i)^{\nu}. \tag{10}$$

Here, $\mathcal{T}^n$ is considered to be $\{e^{\theta_1 i} | 0 \leq \theta_1 < 2\pi\} \times \cdots \times \{e^{\theta_n i} | 0 \leq \theta_n < 2\pi\}$. We set $: d^{\infty} z : |_{\mathcal{T}^n} = \prod_{n=1}^{\infty} \Big(\frac{(2\pi i)^{\mu_n^s - 1}}{\mu_n^s} d(z_n^{\mu_n^s}) \Big)|_{s=0}$. Then by (10), we have

$$\int_{\mathcal{T}^{\infty}} \frac{: d^{\infty} z : |_{\mathcal{T}^n}}{: \prod_{n=1}^{\infty} z_n :} = (2\pi i)^{\nu}. \tag{11}$$

This formula is valid if we regard $\mathcal{T}^{\infty} = \{e^{\theta_1 i} | 0 \leq \theta_1 < 2\pi\} \times \{e^{\theta_2 i} | 0 \leq \theta_2 < 2\pi\} \times \cdots$, because $: \prod_n z_n :$ is not single valued unless ν is an integer. But if ν is an integer, we can regard $\mathcal{T}^{\infty}$ to be an ∞-dimensional torus.

On the other hand, since $d(z^a)/z^a = a dz/z$, we have

$$\lim_{a\to 1} \int_{\gamma} f(z) \frac{d(z^a)}{z^a} = \int_{\gamma} f(z) \frac{dz}{z} = 2\pi i f(0),$$

if γ is a closed curve in D^1 surrounding 0 and f is holomorphic on D^1. Hence we have

Theorem 2. *If ν is an integer, f is a holomorphic function on D^{∞} and $\gamma = \gamma_1 \times \gamma_2 \times \cdots$, γ_n is a closed curve in $\{z_n | |z_n| < 1\}$ surrounding 0. Then we have*

$$f(0) = \frac{1}{(2\pi i)^{\nu}} \int_{\gamma} f(z) \frac{: dz^{\infty} : |_{\mathcal{T}^{\infty}}}{: \prod_{n=1}^{\infty} z_n :}. \tag{12}$$

Here, we say a function f on D^∞ to be holomorphic, if $\dfrac{\partial f}{\partial \bar{z}_n} = 0$, $n = 1, 2, \ldots$. In other words, f is holomorphic if it allows Taylor expansion $f(z) = \displaystyle\sum_{i_1,\ldots,i_m} c_{i_1,\ldots,i_m} z_1^{i_1} \cdots z_m^{i_m}$.

Since $dz_{n,k}/z_{n,k} = dz_n/z_n$, $z_{n,k} = \mu_n^{-k} z_n$, (12) is valid if f is holomorphic on $D^{\infty,k}$ and $\gamma \subset D^{\infty,k}$. By (12), if $\gamma_n = \partial\Gamma_n$, $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots$ and $\zeta = (\zeta_1, \zeta_2, \ldots)$, $\zeta_n \in \Gamma_n$, then we have the following *Cauchy's integral expression of a holomorphic function f on $D^{\infty,k}$*

$$f(\zeta) = \frac{1}{(2\pi i)^\nu} \int_\gamma f(z) \frac{: d^\infty z_n : |_{\mathcal{T}^\infty}}{: \prod_{n=1}^\infty (z_n - \zeta_n) :}. \tag{13}$$

Note 3. If ν is an integer, $: \prod_n z_n :$ is an analytic function, but not holomorphic. For this function, we have

$$\frac{1}{(2\pi i)^\nu} \int_{\mathcal{T}^\infty} : \prod_{n=1}^\infty z_n : \frac{: dz^\infty : |_{\mathcal{T}^\infty}}{: \prod_{n=1}^\infty (z_n - c_n)} = 0, \quad |c_n| < 1, \tag{14}$$

$$\frac{1}{(2\pi i)^\nu} \int_{\mathcal{T}^\infty} : \prod_{n=1}^\infty z_n : \frac{: dz^\infty : |_{\mathcal{T}^\infty}}{: \prod_{n=1}^\infty (z_n - c_n)} = \prod_{n=1}^\infty c_n, \quad |c_n| > 1. \tag{15}$$

Therefore $: \prod_n z_n :$ behaves as if the principal part of a meromorphic function.

5 De Rham type cohomology with ∞-degree elements

In the rest of this paper, we assume ν to be an integer.

Existence of regularized Cauchy kernel implies existence regularized volume form $: dv(\mathcal{T}^\infty) :$ on $\mathcal{T}_r^{\infty,k}$. To set $z_n = r_n e^{i\theta_n}$, we may set

$$: dv(\mathcal{T}^\infty) := \prod_{n=1}^\infty i(2\pi i)^{\mu_n^s - 1} d\theta_n |_{s=0}. \tag{16}$$

We also set

$$: dv(\mathcal{T}^{\infty - \{i_1,\ldots,i_p\}}) := \prod_{n \notin \{i_1,\ldots,i_p\}} i(2\pi i)^{\mu_n^s - 1} d\theta_n |_{s=0}, \tag{17}$$

and define

$$d\theta_{j_1} \wedge \cdots \wedge d\theta_{j_q} \wedge : dv(\mathcal{T}^{\infty - \{i_1,\ldots,i_p\}}) :=$$
$$= \begin{cases} \pm : dv(\mathcal{T}^{\infty - \{k_1,\ldots,k_r\}}) : if \{j_1,\ldots,j_q\} \cup \{k_1,\ldots,k_r\} = \{i_1,\ldots,i_p\}, \\ \qquad 0 \qquad\qquad\qquad\qquad\qquad\qquad : otherwise. \end{cases}$$

The cohomology algebra $H^*(\mathcal{T}^\infty, \mathcal{C})$ of $\mathcal{T}^\infty$ is the Grassmann algebra generated by $d\theta_1, d\theta_2, \ldots$. To define Hodge $*$-operator (Poincaré duality) by

$$*(d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_p}) = (-1)^{i_1+\cdots+i_p-p(p-1)/2} : dv(\mathcal{T}^{\infty-\{i_1,\ldots,i_p\}}) :, \tag{18}$$

we obtain a de Rham type cohomology algebra

$$H^{*,*}(\mathcal{T}^\infty, \mathcal{C}) = H^*(\mathcal{T}^\infty, \mathcal{C}) \oplus *\big(H^*(\mathcal{T}^\infty, \mathcal{C})\big). \tag{19}$$

Note 4. Since multiplicative structure of $H^{*,*}(\mathcal{T}^\infty, \mathcal{C})$ depends on ν, it is *not* a topological invariant.

Let $W_*^{k,\natural}$ be $\{\sum_n z_n e_n \in W^{k,\natural} | z_n \neq 0, n = 1, 2, \ldots\}$, and $W_+^{k,\natural}$ is same as in §3. Then we have

$$W_*^{k,\natural} = \mathcal{T}^{\infty,k} \times (W_*^{k,\natural} \cap W_+^{k,\natural}). \tag{20}$$

Hence we can define de Rham type cohomology with infinite degree elements $H^{*,*}(W_*^{k,\natural}, \mathcal{C})$ of $W_*^{k,\natural}$ by the same way. In this case, we denote

$$*dz_{i_1} \wedge \ldots \wedge dz_{i_p} =: dz^{\infty-\{i_1,\ldots,i_p\}} : |_{\mathcal{T}^\infty}. \tag{21}$$

Let $W_{i_1,\ldots,i_p}^{k,\natural}$ be the subspace of $W^{k,\natural}$ defined by $z_{i_1} = 0, \ldots, z_{i_p} = 0$. Then the Cauchy kernel of $W_{i_1,\ldots,i_p}^{k,\natural}$ is $\dfrac{: dz^{\infty-\{i_1,\ldots,i_p\}} : |_{\mathcal{T}^\infty}}{: \prod_{n \notin \{i_1,\ldots,i_p\}} z_n :}$, and we have

$$\begin{aligned} &H^{*,*}(W_*^{k,\natural}, \mathcal{C}) \cong H^{*,*}(\mathcal{T}^\infty, \mathcal{C}). \\ &H^{*,*}(W_*^{k,\natural}, \mathcal{C}) = H^*(W_*^{k,\natural}, \mathcal{C}) \oplus *(H^*(W_*^{k,\natural}, \mathcal{C})). \end{aligned} \tag{22}$$

$H^*(W_*^{k,\natural}, \mathcal{C})$ is isomorphic to $H^*(\mathcal{T}^\infty, \mathcal{C})$. Hence it is an ∞-dimensional Grassmann algebra.

Note 5. Since there is the regularized volume form $: d\omega :$ of the sphere $\hat{S}^\infty$ of $\hat{H}$, Hilbert space added the longitude, we can define the real coefficients de Rham type cohomology $H^{*,*}(\hat{S}^\infty, \mathcal{R})$ of $\hat{S}^\infty$ by

$$\begin{aligned} H^{*,*}(\hat{S}^\infty, \mathcal{R}) &= H^0(\hat{S}^\infty, \mathcal{R}) \oplus H^\infty(\hat{S}^\infty, \mathcal{R}), \\ H^\infty(\hat{S}^\infty, \mathcal{R}) &= *H^0(\hat{S}^\infty, \mathcal{C}) \cong \mathcal{R} : d\omega : . \end{aligned} \tag{23}$$

We conclude this section asking are there any relation between de Rham type cohomology with ∞-degree elements and entire cyclic cohomology, or stochastic de Rham complexes (cf. [Con98, Cun02, Léan03]).

6 Regularized residue

We set $W_{n,*}^{k,\natural} = \{\sum_{m>n} z_m e_m \in W^{k,\natural} | z_m \neq 0, m = n+1, \ldots\}$ and $\mathcal{C}_*^n = \{\sum_m z_m e_m \in \mathcal{C}^n | z_m \neq 0\}$. We also denote $W_n^{k,\natural}$ the subspace of $W^{k,\natural}$ defined by $z_1 = 0, \ldots, z_n = 0$. If $m \leq n$, we regard

$$\mathcal{C}_*^m \times W_n^{k,\natural} \subset W_{n-m}^{k,\natural}, \quad \mathcal{C}^m \times W_{n,*}^{k,\natural} \subset W_{n-m}^{k,\natural}.$$

Then composing the residue maps

$$res : H^p(\mathcal{C}_*^m \times W_n^{k,\natural}, \mathcal{C}) \to H^{p-1}(\mathcal{C}^{m-1} \times W_n^{k,\natural}, \mathcal{C}),$$

the composed residue map [Ler59], cf.[Asa68]

$$res^m : H^m(\mathcal{C}_*^m \times W_n^{k,\natural}, \mathcal{C}) \to H^0(W_n^{k,\natural}, \mathcal{C}) \cong \mathcal{C},$$

is obtained,and we have

$$\int_{|z_1|=\epsilon_1,\ldots,|z_m|=\epsilon_m} \phi = (2\pi i)^m res^m(\phi).$$

Definition 4. *If $p \geq n$, we define the map $res^{\infty-p}$ by*

$$res^{\infty-p} * \left(\frac{: dz^{\infty-\{i_1,\ldots,i_p\}} : |_{\mathcal{T}^\infty}}{: \prod_{n \notin \{i_1,\ldots,i_p\}} z_n :}\right) = \frac{dz_{i_{n+1}}}{z_{i_{n+1}}} \wedge \ldots \wedge \frac{dz_{i_p}}{z_{i_p}}. \tag{24}$$

Since $H^{\infty-n}(W_*^{k,\natural}, \mathcal{C}) = *H^n(W_*^{k,\natural}, \mathcal{C})$, $res^{\infty-p}$ induces the map

$$res^{\infty-p} : H^{\infty-n}(W_*^{k,\natural}, \mathcal{C}) \to H^{p-n}(\mathcal{C}_*^{p-n}, \mathcal{C}) \cong \mathcal{C},$$

and we have the following *regularized residue formula*

$$\int_{\mathcal{T}_r^{\infty-n,k}} *\phi^n = (2\pi i)^{\nu-n+p} \int_{\mathcal{T}^{p-n}} res^{\infty-p}(*\phi^n). \tag{25}$$

Here, $\mathcal{T}_r^{\infty-n,k}$ is the torus in $W_n^{k,\natural}$ defined by $|z_m| = r$, $m \geq n$ and $\mathcal{T}^{p-n}$ is the torus in $\mathcal{C}^{p-n}$ defined by $|z_j| = c_j$, $j = 1, \ldots, p-n$. The integral in the right hand side is done in usual sense, but the the integral in the left hand side is the regularized integral. Cauchy's integral formula on $W^{k,\natural}$ is a consequence of this formula.

By using the map $res^{\infty-p}$, we have the following exact sequence

$$\begin{aligned} H^{\infty-p}(\mathcal{C}^{p-n} \times W_{p-n,*}^{k,\natural}, \mathcal{C}) \longrightarrow^{\iota} H^{\infty-p}(W_*^{k,\natural}, \mathcal{C}) \longrightarrow \\ \longrightarrow^{res^{\infty-p}} H^{p-n}(\mathcal{C}_*^{p-n}, \mathcal{C}) \longrightarrow^{\delta} H^{\infty-p+1}(\mathcal{C}^{p-n} \times W_{p-n,*}^{k,\natural}, \mathcal{C}). \end{aligned}$$

This sequence is not embedded in long exact sequence of de Rham type cohomology groups. Because $res^{\infty-p}$ is a kind of composed residue. But we can not get $res^{\infty-p}$ composing ordinary residue maps.

Note 6. If X is an orientable ∞-dimensional smooth manifold modeled by $\hat{H}$ and Y is an orientable smooth r-dimensional submanifold of X ($r < \infty$), then if $H^{*,*}(X, \mathcal{R})$ and $H^{*,*}(X - Y, \mathcal{R})$ are defined, the *regularized residue map* $res : H^{\infty-p}(X - Y, \mathcal{R}) \to H^{r-p+1}(Y, \mathcal{R})$ may defined and we may have the following *regualrized residue exact sequence* (cf.[1])

$$\begin{aligned} \cdots \longrightarrow H^{\infty-p}(X, \mathcal{R}) \longrightarrow^{\iota} H^{\infty-p}(X - Y, \mathcal{R}) \longrightarrow^{res} \\ \longrightarrow H^{r-p+1}(Y, \mathcal{R}) \longrightarrow^{\delta} H^{\infty-p+1}(X, \mathcal{R}) \longrightarrow \cdots. \end{aligned}$$

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Fractional spaces generated by the positive differential and difference operators in a Banach space

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The structure of the fractional spaces $E_{\alpha,q}(L_q[0,1], A^x)$ generated by the positive differential operator A^x defined by the formula $A^x u = -a(x)\frac{d^2u}{dx^2} + \delta u$, with domain $D(A^x) = \{u \in C^{(2)}[0,1] : \ u(0) = u(1), u'(0) = u'(1)\}$ is investigated. It is established that for any $0 < \alpha < \frac{1}{2}$ the norms in the spaces $E_{\alpha,q}(L_q[0,1], A^x)$ and $W_q^{2\alpha}[0,1]$ are equivalent. The positivity of the differential operator A^x in $W_q^{2\alpha}[0,1](0 \leq \alpha < \frac{1}{2})$ is established. The discrete analogy of these results for the positive difference operator A_h^x a second order of approximation of the differential operator A^x, defined by the formula

$$A_h^x u^h = \left\{ -a(x_k)\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k \right\}_1^{M-1}, u_h = \{u_k\}_0^M, Mh = 1$$

with $u_0 = u_M$ and $-u_2 + 4u_1 - 3u_0 = u_{M-2} - 4u_{M-1} + 3u_M$ is established. In applications, the coercive inequalities for the solutions of the nonlocal boundary-value problem for two-dimensional elliptic equation and of the second order of accuracy difference schemes for the numerical solution of this problem are obtained.

1 Introduction

It is a well-known (see, e.g., [Kre66, Gri84, Fat85]) that the study of the various properties of partial differential equations is based on a positivity property of the differential operator in a Banach space. The positivity of the wider class of differential operators has been studied by many researchers (see [Sol59, Sol60, KZPS76, Ste80]). To prove stability, in a number of works (see [AS94]-[AS84] and the references given therein) difference schemes were treated as operator equations in a Banach space, and the investigation was based on the positivity property of the operator coefficient. Important progress has been made in the

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study of positive operators from the viewpoint of the stability analysis of high order of accuracy difference schemes for partial differential equations. Application of theory of fractional spaces generated by the positive operators in a Banach space permits us to establish the stability and coercive stability of the difference schemes in various norms for partial differential equations specially when we cannot use approaches of a maximum principle and energy method. We introduce the Banach spaces $E_{\alpha,q} = E_{\alpha,q}(E, A)(0 < \alpha < 1)$, consisting of all $v \in E$ for which the following norms are finite:

$$\| v \|_{E_{\alpha,q}} = (\int\limits_0^\infty \| z^\alpha A(z + A)^{-1} v \|_E^q \frac{dz}{z})^{\frac{1}{q}}, 1 \leq q < \infty,$$

$$\| v \|_{E_{\alpha,\infty}} = \sup_{z>0} ||z^\alpha A(z + A)^{-1} v \|_E, q = \infty.$$

The positive operator A commutes with its resolvent $(\lambda + A)^{-1}$ for all $\lambda, \lambda \in (0, \infty)$. Therefore, using the definition of the fractional spaces $E_{\alpha,q} = E_{\alpha,q}(E, A)$, we obtain

$$\| (\lambda + A)^{-1} \|_{E_{\alpha,q} \to E_{\alpha,q}} \leq \| (\lambda + A)^{-1} \|_{E \to E} \tag{1}$$

for all $\alpha, \alpha \in (0, 1)$ and $q, q \in [1, \infty]$.This means that from the positivity of operator A in E it follows the positivity of this operator A in $E_{\alpha,q}$ for all $\alpha, \alpha \in (0, 1)$ and $q, q \in [1, \infty]$.

The investigation of the well-posedness of the various types of boundary value problems for parabolic and elliptic differential and difference equations is based on the positivity of elliptic differential and difference operators A in various Banach spaces E and on the structure of the fractional spaces $E_{\alpha,q}$ generated by these positive operators. Note that an excellent survey of works in the theory of fractional spaces generated by the positive multidimensional difference operators in the space and its applications to partial differential equations parabolic and elliptic types was given in the books [AS94, AS04, Ash92]. Theory and applications of positive operators in Banach spaces have been studied extensively by many researchers (see [Sob71, AS77, AS79], and [SS81]-[AY06] and the references therein). We consider the differential operator A^x defined by the formula

$$A^x u = -a(x) \frac{d^2 u}{dx^2} + \delta u, \tag{2}$$

with domain $D(A^x) = \{u \in C^{(2)}[0, 1] : \ u(0) = u(1), u'(0) = u'(1)\}$. Here $a(x)$ is a smooth function defined on the segment $[0, 1]$ and $a(x) \geq a > 0, \ \delta > 0$.

We introduce the Banach space $C^\beta[0, 1](0 < \beta < 1)$ of all continuous functions $\varphi(x)$ defined on $[0, 1]$ and satisfying a Holder condition and $\varphi(0) = \varphi(1)$ for which the following norm is finite:

$$||\varphi||_{C^\beta[0,1]} = ||\varphi||_{C[0,1]} + \sup_{0\le x<x+\tau\le 1} \frac{|\varphi(x+\tau)-\varphi(x)|}{\tau^\beta},$$

where $C[0,1]$ is the space of all continuous functions $\varphi(x)$ defined on $[0,1]$ and $\varphi(0)=\varphi(1)$ with the usual norm

$$||\varphi||_{C[0,1]} = \max_{0\le x\le 1} |\varphi(x)|.$$

In the paper [AK95] the following two theorems on a structure of fractional spaces $E_\alpha(C[0,1],A^x)$ and on the positivity of A^x in $C^{2\alpha}[0,1](0<\alpha<\frac{1}{2})$ were established.

Theorem 1. *For $0<\alpha<1/2$ the norms of the spaces $E_\alpha(C[0,1],A^x)$ and $C^{2\alpha}$ $[0,1]$ are equivalent.*

Theorem 2. *For all $\lambda\in R_\varphi$, $|\lambda|\ge K_0>0$ and $0<\alpha<1/2$ the resolvent $(\lambda+A^x)^{-1}$ is subject to the bound*

$$\left\|(\lambda+A^x)^{-1}\right\|_{C^{2\alpha}[0,1]\to C^{2\alpha}[0,1]} \le \frac{M(\varphi,\delta)}{\alpha(1-2\alpha)}(1+|\lambda|)^{-1},$$

where $M(\varphi,\delta)$ does not depend on λ and α.

In the papers [AK01] and [AYA05] the positive difference operators A_h^x of a first order of approximation of the differential operator A^x, defined by the formula

$$A_h^x u^h = \left\{-a(x_k)\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}+\delta u_k\right\}_1^{M-1}, u_h=\{u_k\}_0^M \qquad (3)$$

with $u_0=u_M$ and $u_1-u_0=u_M-u_{M-1}$ and of a second order of approximation of the differential operator A^x, defined by the formula

$$A_h^x u^h = \left\{-a(x_k)\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}+\delta u_k\right\}_1^{M-1}, u_h=\{u_k\}_0^M \qquad (4)$$

with $u_0=u_M$ and $-u_2+4u_1-3u_0=u_{M-2}-4u_{M-1}+3u_M$ was presented. It was proved that the spaces $E_\alpha(C_h,A_h^x)$ and $C_h^{2\alpha}$ coincide for any $0<\alpha<\frac{1}{2}$, and their norms are equivalent uniformly in $h, 0<h\le h_0$. The positivity of the difference operators A_h^x in $C_h^{2\alpha}(0\le\alpha<\frac{1}{2})$ was obtained.

In the present paper we study the structure of the fractional spaces $E_{\alpha,q}(L_q[0,1],A^x)$ generated by the positive differential operator A^x defined by the formula(2). It is established that for any $0<\alpha<\frac{1}{2}$ the norms in the spaces $E_{\alpha,q}(L_q[0,1],A^x)$ and $W_q^{2\alpha}[0,1]$ are equivalent. The positivity of the differential operator A^x in $W_q^{2\alpha}[0,1](0 \leq \alpha < \frac{1}{2})$ is established. Here the Banach space $W_q^{\beta}[0,1]$ is the space of the all integrable functions $f(x)$ defined on $[0,1]$, equipped with the norm

$$\| f \|_{W_q^{\beta}[0,1]} = \{ \int_0^1 \int_0^1 \frac{|f(x) - f(x+y)|^q}{|y|^{1+\beta q}} dxdy + \| f \|_{L_q[0,1]} \}^{\frac{1}{q}},$$

$$0 < \beta < 1, 1 \leq q \leq \infty,$$

where $L_q[0,1]$ is the space of the all integrable functions defined on $[0,1]$, equipped with the norm

$$\| f \|_{L_q[0,1]} = \{ \int_0^1 |f(x)|^q dx \}^{\frac{1}{q}}.$$

Moreover, the discrete analogy of these results for the positive difference operator A_h^x defined by the formula (4) is investigated. It is established that the spaces $E_{\alpha,q}(L_{q,h}, A_h^x)$ and $W_{q,h}^{2\alpha}$ coincide for any $0 < \alpha < \frac{1}{2}$, and their norms are equivalent uniformly in h, $0 < h \leq h_0$.The positivity of the difference operator A_h^x in $W_q^{2\alpha}[0,1]_h(0 \leq \alpha < \frac{1}{2})$ is established. In applications, the coercive inequalities for the solutions of the nonlocal boundary-value problem for two-dimensional elliptic equation and of the second order of accuracy difference schemes for the numerical solution of this problem are obtained.

2 The positivity of differential operator A^x.The structure of fractional spaces $E_{\alpha,q}(L_q[0,1], A^x)$

Theorem 3. *For any $0 < \alpha < \frac{1}{2}$ the norms of the spaces $E_{\alpha,q}(L_q[0,1], A^x)$ and $W_q^{2\alpha}[0,1]$ are equivalent.*

The proof of this theorem follows the scheme of the proof of the theorem in [AK95] and it is based on the formulas

$$A^x \left(\lambda + A^x\right)^{-1} f(x) = \frac{\delta}{\lambda + \delta} f(x) + \int_0^1 J(x, s; \lambda + \delta)(f(x) - f(s))ds,$$

$$f(x) = \int_0^1 \int_0^\infty J(x, s; \lambda + t + \delta) A^x \left(\lambda + t + A^x\right)^{-1} f(s) dtds$$

for the positive differential operator A^x and on the pointwise estimates of the Green's function of the resolvent equation

$$A^x u + \lambda u = f$$

or

$$-a(x)\frac{d^2u(x)}{dx^2} + \delta u(x) + \lambda u(x) = f(x), 0 < x < 1, \tag{5}$$

$$u(0) = u(1),\ u'(0) = u'(1)$$

and its derivative.

Theorem 4. *For all* λ, $\lambda \in R_\varphi = \{\lambda : |\arg \lambda| \leq \varphi, \varphi < \pi/2\}, \alpha \in (0, \frac{1}{2})$ *and* $|\lambda| \geq K_0 > 0$ *the resolvent* $(\lambda I + A^x)^{-1}$ *is subject to the bound*

$$\left\|(\lambda I + A^x)^{-1}\right\|_{W_p^{2\alpha}[0,1] \to W_p^{2\alpha}[0,1]} \leq \frac{M(\varphi, \delta)}{\alpha(1-2\alpha)}(1 + |\lambda|)^{-1},$$

where $M(\varphi, \delta)$ *does not depend on* λ.

The proof of this theorem follows the scheme of the proof of the theorem in [AK95] and it is based on the estimate (1) and on the positivity of differential operator A^x in $L_p[0,1]$. The proof of the positivity of differential operator A^x in $L_p[0,1]$ is based on the formula

$$(\lambda + A^x)^{-1} f(x) = \int_0^1 J(x, s; \lambda + \delta) f(s) ds$$

and on the pointwise estimates for the Green's function of the resolvent equation (5) and its derivative.

Now, we consider the nonlocal boundary-value problem for two-dimensional elliptic equation

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - a(x)\frac{\partial^2 u}{\partial x^2} + \delta u = f(t,x), 0 < t < T, 0 < x < 1, \\ u(0,x) = \varphi(x), u(T,x) = \psi(x),\ 0 \leq x \leq 1, \\ u(t,0) = u(t,1),\ u_x(t,0) = u_x(t,1),\ \ 0 \leq t \leq T, \end{cases} \tag{6}$$

where $a(x)$, $\varphi(x)$, $\psi(x)$ and $f(t,x)$ are given sufficiently smooth functions and $a(x) \geq a > 0,\ \delta > 0$ is a sufficiently large number.

Theorem 5. *For the solution of the boundary value problem (1) the following coercive inequalities are valid:*

$$\| \frac{\partial^2 u}{\partial t^2} \|_{L_p([0,T], W_q^{2\alpha}[0,1])} + \| \frac{\partial^2 u}{\partial x^2} \|_{L_p([0,T], W_q^{2\alpha}[0,1])}$$

$$\leq M(q,p,\alpha) \parallel f \parallel_{L_p([0,T],W_q^{2\alpha}[0,1])} + M(\alpha)(||\varphi||_{W_q^{2\alpha}[0,1]} + ||\psi||_{W_q^{2\alpha}[0,1]}),$$

$$1 < p,q < \infty, 0 < \alpha < \frac{1}{2}.$$

Here $M(q,p,\alpha)$ *and* $M(\alpha)$ *are independent of* $f(t,x)$, $\varphi(x)$ *and* $\psi(x)$.

The proof of Theorem 5 is based on the Theorem 3 on the structure of the fractional spaces $E_{\alpha,q}(L_q[0,1], A^x)$ and the Theorem 4 on the positivity of the operator A^x in $W_q^{2\alpha}[0,1]$ and on the following theorems on the structure of the fractional spaces $E_{\alpha,q}(L_q[0,1], (A^x)^{\frac{1}{2}})$[Ash92, Tri78] and on coercivity inequalities in $L_p(E_{\alpha,q})$[AS04] for the solution of the abstract boundary-value problem for differential equation

$$-v''(t) + Av(t) = f(t) \quad (0 \leq t \leq T), v(0) = v_0, v(T) = v_T \tag{7}$$

in an arbitrary Banach space E with the linear positive operator A .

Theorem 6. *The spaces* $E_{\alpha,q}(L_q, A^x)$ *and* $E^*_{2\alpha,q}(L_q, (A^x)^{\frac{1}{2}})$ *coincide for any* $0 < \alpha < \frac{1}{2}$, *and their norms are equivalent.*

Theorem 7. *Let* $1 < p,\ q < \infty$ *and* $0 < \alpha < 1$.*Suppose that* A *is the positive operator in a Banach space* E. *Then problem (7) is well posed in* $L_p(E^*_{\alpha,q})$ *and the coercivity inequality holds:*

$$||v''||_{L_p(E^*_{\alpha,q})} + \|Av\|_{L_p(E^*_{\alpha,q})} \leq \frac{M(q)p^2}{\alpha(1-\alpha)(p-1)}\|f\|_{L_p(E^*_{\alpha,q})}$$

$$+M(||A\varphi||_{E^*_{\alpha,q}} + ||A\psi||_{E^*_{\alpha,q}}),$$

where $M, M(q)$ *do not depend on* $\alpha,\ p,\varphi,\psi$ *and* $f(t)$. *Here, the Banach space* $E^*_{\alpha,q} = E^*_{\alpha,q}(E, A^{\frac{1}{2}})(0 < \alpha < 1, 1 < q < \infty)$ *consists of those* $v \in E$ *for which the norm*

$$\parallel v \parallel_{E^*_{\alpha,q}} = (\int_0^\infty \lambda^{1-\alpha} \parallel A^{\frac{1}{2}} exp\{-\lambda A^{\frac{1}{2}}\} v \parallel_E^q \frac{d\lambda}{\lambda})^{1/q}$$

is finite.

3 The positivity of difference operator A_h^x.The structure of fractional spaces $E_{\alpha,q}(L_q[0,1]_h, A_h^x)$

We denote $L_{q,h} = L_{q,h}[0,1]_h$ and $W^\alpha_{q,h} = W^\alpha_q[0,1]_h, 1 \leq q < \infty$ the Banach spaces of all grid functions $v^h = \{v_k\}_1^{M-1}$defined on $[0,1]_h = \{x_k = kh,\ 0 \leq k \leq M,\ Mh = 1\}$ equipped with the norms

$$\parallel \varphi^h \|_{L_{q,h}} = (\sum_{k=1}^{M-1} |\varphi_k|^q h)^{\frac{1}{q}},$$

$$\| \varphi^h\|_{W^{\beta}_{q,h}} = (\sum_{k=1}^{M-1}\sum_{m=1}^{M-1}\frac{|\varphi_k-\varphi_{k+m}|^q}{|mh|^{1+\beta q}}h^2+\| \varphi^h\|_{L_{q,h}})^{\frac{1}{q}},$$

$$0<\beta<1, 1\leq q<\infty.$$

Note that the Banach space $E'_{\alpha,q}=E'_{\alpha,q}(E,B)(0<\alpha<1)$ consists of those $v\in E$ for which the norm

$$\| v \|_{E'_{\alpha,q}}=(\int_0^\infty [z^{\alpha}\parallel B(z+B)^{-1}v\parallel_E]^q\frac{dz}{z})^{\frac{1}{q}},\ 1\leq q<\infty,$$

$$\| v \|_{E'_{\alpha}}=\| v \|_{E'_{\alpha,\infty}}=\sup_{\lambda>0}\lambda^{\alpha}\parallel B(\lambda+B)^{-1}v\parallel_E$$

is finite.

Theorem 8. *The the spaces $E_{\alpha,q}(L_{q,h},A_h^x)$ and $W_{q,h}^{2\alpha}$ coincide for any $0<\alpha<\frac{1}{2}$, and their norms are equivalent uniformly in $h, 0<h\leq h_0$.*

The proof of this theorem follows the scheme of the proof of the theorem in [AK01] and it is based on the formulas

$$A_h^x\left(\lambda+A_h^x\right)^{-1}f_k=\lambda\sum_{j=1}^{M-1}J(k,j;\lambda+\delta)\left[f_k-f_j\right]h+\frac{\delta}{\lambda+\delta}f_k, 0\leq k\leq M,$$

$$f_k=\int_0^\infty\sum_{j=1}^{M-1}J\left(k,j;t+\lambda+\delta\right)A_h^x\left(t+\lambda+A_h^x\right)^{-1}f_jhdt, 0\leq k\leq M$$

for the positive difference operator A_h^x and on the pointwise estimates for the Green's function of the resolvent equation

$$A_h^xu^h+\lambda u^h=f^h$$

or

$$-a_k\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}+\delta u_k+\lambda u_k=f_k, \tag{8}$$

$$a_k=a(x_k), f_k=f(x_k), x_k=kh, 1\leq k\leq M-1,$$

$$u_0=u_M, -u_2+4u_1-3u_0=u_{M-2}-4u_{M-1}+3u_M$$

and its difference derivative.

Theorem 9. *For all λ, $\lambda\in R_{\varphi}=\{\lambda:|\arg\lambda|\leq\varphi,\varphi<\pi/2\},\alpha\in(0,\frac{1}{2})$ and $|\lambda|\geq K_0>0$ the resolvent $(\lambda I+A_h^x)^{-1}$ is subject to the bound*

$$\left\|(\lambda I+A_h^x)^{-1}\right\|_{W_{p,h}^{2\alpha}\to W_{p,h}^{2\alpha}}\leq\frac{M(\varphi,\delta)}{\alpha(1-2\alpha)}(1+|\lambda|)^{-1},$$

where $M(\varphi,\delta)$ does not depend on λ and h.

The proof of this theorem follows the scheme of the proof of the theorem in [AK01] and it is based on the estimate (1) and on the positivity of difference operator A_h^x in $L_{p,h}$.The proof of the positivity of difference operator A_h^x in $L_{p,h}$ is based on the formula

$$(\lambda + A_h^x)^{-1} f_k = \sum_{j=1}^{M-1} J(k,j;\lambda+\delta) f_j h, 0 \leq k \leq M,$$

$$(\lambda + A^x)^{-1} f(x) = \int_0^1 J(x,s;\lambda+\delta) f(s) ds$$

and on the pointwise estimates for the Green's function of the resolvent equation (8) and its difference derivative.

In applications, we consider the difference scheme of the second order of accuracy

$$\begin{cases} -\frac{1}{\tau^2}(u_{k+1}^n - 2u_k^n + u_{k-1}^n) - a^n \frac{1}{h^2}(u_k^{n+1} - 2u_k^n + u_k^{n-1}) + \delta u_k^n = \varphi_k^n, \\ \varphi_k^n = f(t_k, x_n), \quad a^n = a(x_n), \quad t_k = k\tau, \quad x_n = nh, \\ 1 \leq k \leq N-1, 1 \leq n \leq M-1, N\tau = 1, Mh = 1, \\ u_0^n = \varphi^n, u_N^n = \psi^n, \varphi^n = \varphi(x_n), \quad \psi^n = \psi(x_n), x_n = nh, 0 \leq n \leq M, \\ u_k^0 = u_k^M, -u_k^2 + 4u_k^1 - 3u_k^0 = u_k^{M-2} - 4u_k^{M-1} + 3u_k^M, 0 \leq k \leq N \end{cases} \tag{9}$$

for the approximate solution of the nonlocal boundary-value problem (6).

Theorem 10. *Let τ and h be a sufficiently small numbers. For the solution of the difference problem* (9) *the following inequalities are valid:*

$$\| \{\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h)\}_1^{N-1} \|_{L_{p,\tau}(W_{q,h}^{2\alpha})}$$
$$+ \| \{\{h^{-2}(u_k^{n+1} - 2u_k^n + u_k^{n-1})\}_1^{M-1}\}_1^{N-1} \|_{L_{p,\tau}(W_{q,h}^{2\alpha})}$$
$$\leq M(p,q,\alpha) \| \{\varphi_k^h\}_1^{N-1} \|_{L_{p,\tau}(W_{q,h}^{2\alpha})}$$
$$+M(p,\alpha)(\| \{h^{-2}(\varphi^{n+1} - 2\varphi^n + \varphi^{n-1})\}_1^{M-1} \|_{W_{q,h}^{2\alpha}}$$
$$+ \| \{h^{-2}(\psi^{n+1} - 2\psi^n + \psi^{n-1})\}_1^{M-1} \|_{W_{q,h}^{2\alpha}}), 1 < p,q < \infty, 0 < \alpha < \frac{1}{2},$$

where $M(p,q,\alpha)$ and $M(p,\alpha)$ do not depend on $\{\varphi_k^h\}_1^{N-1}$,φ^h, ψ^h, h and τ.

The proof of Theorem 10 is based on the Theorem 8 on the structure of the fractional spaces $E_{\alpha,q}(L_{q,h}, A_h^x)$ and the Theorem 9 on the positivity of the operator A_h^x in $W_{q,h}^{2\alpha}$ and on the following theorems on the structure of the fractional spaces $E_{\alpha,q}(L_{q,h}, (A_h^x)^{\frac{1}{2}})$ [Ash92] and on coercivity inequalities in $L_{p,\tau}(E_{\alpha,q})$ [AS04] for the solution of the second order of accuracy difference scheme

$$\begin{cases} -\frac{1}{\tau^2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k = f_k, f_k = f(t_k), t_k = k\tau, \\ \qquad 1 \le k \le N-1, N\tau = 1, u_0 = \varphi, u_N = \psi \end{cases} \tag{10}$$

for the approximate solution of the boundary-value problem (7).

Theorem 11. *The spaces $E_{\alpha,q}(L_{q,h}, A_h^x)$ and $E_{2\alpha,q}(L_{q,h}, (A_h^x)^{\frac{1}{2}})$ coincide for any $0 < \alpha < \frac{1}{2}$, and their norms are equivalent uniformly in h, $0 < h \le h_0$.*

Theorem 12. *Let $1 < p, q < \infty$ and $0 < \alpha < 1$.Suppose that A is the positive operator in a Banach space E. Then problem (10) is well posed in $L_{p,\tau}(E_{\alpha,q})$ and the coercivity inequality holds:*

$$||\{\frac{1}{\tau^2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1}\|_{L_{p,\tau}(E_{\alpha,q})} + \|\{Au_k\}_1^{N-1}\|_{L_p(E_{\alpha,q})}$$

$$\le \frac{M(q)p^2}{\alpha(1-\alpha)(p-1)}\|\{f_k\}_1^{N-1}\|_{L_p(E_{\alpha,q})} + M(||A\varphi||_{E_{\alpha,q}} + ||A\psi||_{E_{\alpha,q}}),$$

where $M, M(q)$ do not depend on α, p, φ, ψ, $\{f_k\}_1^{N-1}$ and τ.

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Sub-diffusion equations of fractional order and their fundamental solutions

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The time-fractional diffusion equation is obtained by generalizing the standard diffusion equation by using a proper time-fractional derivative of order $1-\beta$ in the Riemann-Liouville (R-L) sense or of order β in the Caputo (C) sense, with $\beta \in (0,1)$. The two forms are equivalent and the fundamental solution of the associated Cauchy problem is interpreted as a probability density of a self-similar non-Markovian stochastic process, related to a phenomenon of sub-diffusion (the variance grows in time sub-linearly). A further generalization is obtained by considering a continuous or discrete distribution of fractional time-derivatives of order less than one. Then the two forms are no longer equivalent. However, the fundamental solution still is a probability density of a non-Markovian process but one exhibiting a distribution of time-scales instead of being self-similar: it is expressed in terms of an integral of Laplace type suitable for numerical computation. We consider with some detail two cases of diffusion of distributed order: the double order and the uniformly distributed order discussing the differences between the R-L and C approaches. For these cases we analyze in detail the behaviour of the fundamental solutions (numerically computed) and of the corresponding variance (analytically computed) through the exhibition of several plots. While for the R-L and for the C cases the fundamental solutions seem not to differ too much for moderate times, the behaviour of the corresponding variance for small and large times differs in a remarkable way.

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1 Introduction

The main physical purpose for adopting and investigating diffusion equations of fractional order to describe phenomena of *anomalous diffusion* usually met in transport processes through complex and/or disordered systems including fractal media. In this respect, in recent years interesting reviews, see *e.g.* [MK00, MK04, PSW05, Zas02], have appeared, to which (and references therein) we refer the interested reader.

All the related models of random walk turn out to be beyond the classical Brownian motion, which is known to provide the microscopic foundation of the standard diffusion, see *e.g.* [KS05, SK05]. The diffusion-like equations containing fractional derivatives in time and/or in space are usually adopted to model phenomena of anomalous transport in physics, so a detailed study of their solutions is required.

Our attention in this paper will be focused on the time-fractional diffusion equations of a single or distributed order less than 1, which are known to be models for sub-diffusive processes.

Since in the literature we find two different forms for the time-fractional derivative, namely the one in the Riemann-Liouville (R-L) sense, the other in the Caputo (C) sense, we will study the corresponding time-fractional diffusion equations separately. Specifically, we have worked out how to express their fundamental solutions in terms of an integral of Laplace type suitable for a numerical evaluation. Furthermore we have considered the time evolution of the variance for the R-L and C cases. It is known that for large times the variance characterizes the type of anomalous diffusion.

The plan of the paper is as follows.

In Section 2, after having shown the equivalence of the two forms for the time-fractional diffusion equation of a *single order*, namely the R-L form and the C form, we recall the main results for the common fundamental solution, which are obtained by applying two different strategies in inverting its Fourier-Laplace transform. Both techniques yield the fundamental solution in terms of special function of the Wright type that turns out to be self-similar through a definite space-time scaling relationship.

In Section 3 we apply the second strategy for obtaining the fundamental solutions of the time-fractional diffusion equation of *distributed order* in the R-L and C forms, assuming a general order density. We provide for these solutions a representation in terms of a Laplace-type integral of a Fox-Wright function that appears suitable for a numerical evaluation in finite space-time domains. We also provide the general expressions for the Laplace transforms of the corresponding variance.

Then, in Section 4, we consider two case-studies for the fractional diffusion of distributed order: as a discrete distribution we take two distinct orders β_1, β_2 with $0 < \beta_1 < \beta_2 \le 1$; as continuous distribution we take the uniform density with $0 < \beta < 1$. For these cases we provide the graphical