Handbook of Continued Fractions for Special Functions

Annie Cuyt · Vigdis Brevik Petersen ·
Brigitte Verdonk · Haakon Waadeland ·
William B. Jones

Handbook of Continued Fractions for Special Functions

With contributions by

Franky Backeljauw · Catherine Bonan-Hamada

Verified numerical output

Stefan Becuwe · Annie Cuyt

Springer

Annie Cuyt
University of Antwerp
Department of Mathematics and Computer Science
BE-2020 Antwerpen
Belgium

Vigdis Brevik Petersen
Sør-Trøndelag University College
Faculty of Teacher and Interpreter Education
NO-7004 Trondheim
Norway

Brigitte Verdonk
University of Antwerp
Department of Mathematics and Computer Science
BE-2020 Antwerpen
Belgium

Haakon Waadeland
Norwegian University of Science and Technology
Department of Mathematical Sciences
NO-7491 Trondheim
Norway

William B. Jones
University of Colorado
Department of Mathematics
Boulder, CO 80309-0395
USA

ISBN: 978-1-4020-6948-2 e-ISBN: 978-1-4020-6949-9

Library of Congress Control Number: 2007941383

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

TABLE OF CONTENTS

Part II: NUMERICS

PREFACE

The idea to write a *Handbook of Continued fractions for Special functions* originated more than 15 years ago, but the project only got started end of 2001 when a pair of Belgian and a pair of Norwegian authors agreed to join forces with the initiator W.B. Jones. The book splits naturally into three parts: *Part I* discussing the concept, correspondence and convergence of continued fractions as well as the relation to Padé approximants and orthogonal polynomials, *Part II* on the numerical computation of the continued fraction elements and approximants, the truncation and round-off error bounds and finally *Part III* on the families of special functions for which we present continued fraction representations.

Special functions are pervasive in all fields of science and industry. The most well-known application areas are in physics, engineering, chemistry, computer science and statistics. Because of their importance, several books and websites (see for instance `functions.wolfram.com`) and a large collection of papers have been devoted to these functions. Of the standard work on the subject, the *Handbook of mathematical functions with formulas, graphs and mathematical tables* edited by Milton Abramowitz and Irene Stegun, the American National Institute of Standards and Technology claims to have sold over 700 000 copies (over 150 000 directly and more than fourfold that number through commercial publishers)! But so far no project has been devoted to the systematic study of continued fraction representations for these functions. This handbook is the result of such an endeavour. We emphasise that only 10% of the continued fractions contained in this book, can also be found in the Abramowitz and Stegun project or at the Wolfram website!

The fact that the Belgian and Norwegian authors could collaborate in pairs at their respective home institutes in Antwerp (Belgium) and Trondheim (Norway) offered clear advantages. Nevertheless, most progress with the manuscript was booked during the so-called handbook workshops which were organised at regular intervals, three to four times a year, by the first four authors A. Cuyt, V. B. Petersen, B. Verdonk and H. Waadeland. They got together a staggering 16 times, at different host institutes, for a total of 28 weeks to compose, streamline and discuss the contents of the different chapters.

The Belgian and Norwegian pair were also welcomed for two or more weeks at the MFO (Oberwolfach, Germany), CWI (Amsterdam, The Netherlands), University of La Laguna (Tenerife, Spain), the University of Stellenbosch (South-Africa) and last, but certainly not least, the University of

Antwerp and the Norwegian University of Science and Technology. Without the inspiring environment and marvellous library facilities offered by our supportive colleagues G.-M. Greuel, N. Temme, P. Gonzalez-Vera and J.A.C. Weideman a lot of the work contained in this book would not have been possible. In addition, three meetings were held at hotels, in 2002 in Montelupo Fiorentino (Italy) and in 2003 and 2005 in Røros (Norway). At the occasion of the first two of these meetings W.B. Jones joined his European colleagues. In addition to his input and encouragement, his former student Cathy Bonan-Hamada contributed to the handbook as a principal author of *Chapter* 5 and to a lesser extent in a few chapters on special functions.

Several collaborators at the University of Antwerp have also been extremely helpful. The authors have greatly benefitted from the input of S. Becuwe with respect to several TeX-issues, the spell checking, the proof reading and especially, the generation of the tables and numerical verification of all formulas in the book. For the latter, use was made of a Maple library for continued fractions developed by F. Backeljauw [BC07]. Thanks are due to T. Docx for the help with the graphics, for which software was made available by J. Tupper [BCJ+05]. My daughter A. Van Soom was an invaluable help with the entering and management of almost 4600 BibTeX entries, from which only a selection is printed in the reference list.

Financial support was received from the FWO-Flanders (Fonds voor Wetenschappelijk Onderzoek, Belgium) and its Scientific Research Network *Advanced numerical methods for mathematical modelling*, the Department of Mathematics of the Norwegian University of Science and Technology (Trondheim), the Sør Trondelag University College (Trondheim), the Royal Norwegian Society of Science and Letters, and the National Science Foundation (USA).

Thanks are also due to our patient publisher: after many promises the team finally met its own requirements and turned in the manuscript. We apologise to our dear readers: any mistakes found in the book are ours and we take joint responsibility for them.

Annie Cuyt
February 2007
University of Antwerp
Belgium

NOTATION

AS	continued fraction also available in [AS64]
	relative truncation error is tabulated
	error is reliably graphed
$\circ$	composition
$\equiv$	equivalent continued fractions
$\not\equiv$	not identically equal to
$\approx$	asymptotic expansion
$\lfloor \cdot \rfloor$	floor function
$\|\| \cdot \|\|$	norm
$\langle \cdot, \cdot \rangle$	inner product
$\{\cdot_n\}$	sequence
$\| \cdot \|_s$	signed modulus
A_n	n^{th} numerator
a_m	m^{th} partial numerator
$(a)_k$	Pochhammer symbol
$(a;q)_k$	generalised Pochhammer symbol
$\operatorname{Arg} z$	argument, $-\pi < \operatorname{Arg} z \le \pi$
$\arg z$	$\operatorname{Arg} z \pm 2k\pi, k \in \mathbb{N}_0$
(a, b)	open interval $a < x < b$
$[a, b]$	closed interval $a \le x \le b$
$B(a, b)$	beta function
$B_q(a, b)$	q-beta function
$B_x(a, b)$	incomplete beta function
B_n	n^{th} denominator
b_m	m^{th} partial denominator
$\mathbb{C}$	set of complex numbers
$\widehat{\mathbb{C}}$	$\mathbb{C} \cup \{\infty\}$
$C(z)$	Fresnel cosine integral
$\operatorname{Ci}(z)$	cosine integral
$C_n^{(\alpha)}(x)$	Gegenbauer (or ultraspherical) polynomial
$\hat{C}_n^{(\alpha)}(x)$	monic Gegenbauer polynomial
cdf	cumulative distribution function
CMP, CSMP, CHMP	classical moment problems
$\Gamma(z)$	gamma function
$\Gamma(a, z)$	complementary incomplete gamma function
$\Gamma_q(z)$	q-gamma function
$\gamma(a, z)$	(lower) incomplete gamma function

$D_\nu(z)$	parabolic cylinder function
∂	degree
$\mathrm{Ei}(z)$	exponential integral
$\mathrm{Ein}(z)$	exponential integral
$E_n(z)$	exponential integral ($n \in \mathbb{N}_0$)
$E_\nu(z)$	exponential integral ($\nu \in \mathbb{C}$)
$E[X]$	expectation value of X
$\mathrm{erf}(z)$	error function
$\mathrm{erfc}(z)$	complementary error function
$\mathbb{F}, \mathbb{F}(\beta, t, L, U)$	set of floating-point numbers
${}_pF_q(\ldots, a_p; \ldots, b_q; z)$	hypergeometric series
${}_2F_1(a, b; c; z)$	Gauss hypergeometric series
${}_1F_1(a; b; z)$	confluent hypergeometric function
${}_2F_0(a, b; z)$	confluent hypergeometric series
${}_0F_1(; b; z)$	confluent hypergeometric limit function
$\mathtt{F}_n(z; w_n)$	computed approximation of $f_n(z; w_n)$
$f_n, f_n(z)$	n^{th} approximant
$f_n(w_n), f_n(z; w_n)$	n^{th} modified approximant
$f_n^{(M)}$	n^{th} approximant of M^{th} tail
$f^{(n)}, g^{(n)}, \ldots$	n^{th} tail
FLS	formal Laurent series
FPS, FTS	formal power series, formal Taylor series
$\Phi(t)$	distribution function
$\phi(t)$	weight function
${}_r\phi_s(\ldots, a_r; \ldots, b_s; q; z)$	basic hypergeometric series
${}_2\phi_1(q^\alpha, q^\beta; q^\gamma; q; z)$	Heine series
$\varphi_\ell[z_0, \ldots, z_\ell]$	inverse difference
$g_\nu^{(1)}(z), g_\nu^{(2)}(z)$	modified spherical Bessel function 3^{rd} kind
$H_n(x)$	Hermite polynomial
$\hat{H}_n(x)$	monic Hermite polynomial
$H_k^{(m)}(c)$	Hankel determinant for the (bi)sequence c
$H_\nu^{(1)}(z), H_\nu^{(2)}(z)$	Hankel function, Bessel function 3^{rd} kind
$h_\nu^{(1)}(z), h_\nu^{(2)}(z)$	spherical Bessel function 3^{rd} kind
$I_k(x)$	repeated integral of the probability integral
$I_x(a, b)$	regularised (incomplete) beta function
$I_\nu(z)$	modified Bessel function 1^{st} kind
$I^k \mathrm{erfc}(z)$	repeated integral of $\mathrm{erfc}(z)$ for $k \geq -1$
$i_\nu(z)$	modified spherical Bessel function 1^{st} kind
i	imaginary number $\sqrt{-1}$
$\Im z$	imaginary part of z

$J(z)$	Binet function
$J_\nu(z)$	Bessel function 1^{st} kind
$j_\nu(z)$	spherical Bessel function 1^{st} kind
$K_\nu(z)$	modified Bessel function 2^{nd} kind
$\mathrm{K}\,(a_m/b_m)$	continued fraction
$k_\nu(z)$	modified spherical Bessel function 2^{nd} kind
$\mathrm{Ln}(z)$	principal branch of natural logarithm
$L_n^{(\alpha)}(x)$	generalised Laguerre polynomial
$\hat{L}_n^{(\alpha)}(x)$	monic generalised Laguerre polynomial
$\mathrm{li}(x)$	logarithmic integral
$\lambda(L)$	order of FPS $L(z)$
$\Lambda_0(f) = f_{(0)}(z)$	Laurent expansion in deleted neighbourhood of 0
$\Lambda_\infty(f) = f_{(\infty)}(z)$	Laurent expansion in deleted neighbourhood of ∞
$M(a, b, z)$	Kummer function 1^{st} kind
$M_{\kappa,\mu}(z)$	Whittaker function
μ_k	k^{th} moment
μ_k'	k^{th} central moment
$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	$\{0, 1, 2, 3, \ldots\}$
$N(\mu, \sigma^2)$	normal distribution
$[n]_q$	q-analogue of n
$[n]_q!$	q-factorial
$P_n(x)$	Legendre polynomial
$\hat{P}_n(x)$	monic Legendre polynomial
$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomial
$\hat{P}_n^{(\alpha,\beta)}(x)$	monic Jacobi polynomial
pdf	probability density function
$\mathcal{P}_n(L)$	partial sum of degree n of FTS $L(z)$
$\psi_k(z)$	polygamma functions $(k \geq 0)$
$\mathbb{R}$	set of real numbers
$\mathbb{R}[x]$	ring of polynomials with coefficients in $\mathbb{R}$
$R(x)$	Mills ratio
$\Re z$	real part of z
$r_{m,n}(z)$	Padé approximant
$r_{k,\ell}^{(2)}(z)$	two-point Padé approximant
$\rho_\ell[z_0, \ldots, z_\ell]$	reciprocal difference
$S(z)$	Fresnel sine integral
$\mathrm{Si}(z)$	sine integral
$S_n(w_n), S_n(z; w_n)$	modified approximant
$s_n(w_n), s_n(z; w_n)$	linear fractional transformation

SSMP, SHMP	strong moment problems
σ	standard deviation
σ^2	variance
$T_n(x)$	Chebyshev polynomial 1^{st} kind
$\hat{T}_n(x)$	monic Chebyshev polynomial 1^{st} kind
TMP	trigonometric moment problem
$U_n(x)$	Chebyshev polynomial 2^{nd} kind
$\hat{U}_n(x)$	monic Chebyshev polynomial 2^{nd} kind
$U(a, b, z)$	Kummer function 2^{nd} kind
`ulp`	unit in the last place
$\overline{V}$	set closure
V_n	value set
$W_{\kappa,\mu}(z)$	Whittaker function
$w_n(z)$	n^{th} modification for $\mathrm{K}_{m=1}^{\infty}(a_m/1)$
$\tilde{w}_n(z)$	n^{th} modification for $\mathrm{K}_{m=1}^{\infty}(a_m/b_m)$
$w_n^{(1)}(z)$	improved n^{th} modification for $\mathrm{K}_{m=1}^{\infty}(a_m/1)$
$\tilde{w}_n^{(1)}(z)$	improved n^{th} modification for $\mathrm{K}_{m=1}^{\infty}(a_m/b_m)$
$Y_\nu(z)$	Bessel function 2^{nd} kind
$y_\nu(z)$	spherical Bessel function 2^{nd} kind
$\overline{z}$	complex conjugate of z
$\mathbb{Z}$	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Z}^-$	$\{-1, -2, -3, \ldots\}$
$\mathbb{Z}_0^-$	$\{0, -1, -2, -3, \ldots\}$
$\zeta(z)$	Riemann zeta function

0

General considerations

The purpose of this chapter is to explain the general organisation of the book, despite the fact that we hope the handbook is accessible to an unprepared reader. For the customary mathematical notations used throughout the book we refer to the list of notations following the preface.
To scientists novice in the subject of continued fractions we recommend the following order of reading in *Part I* and *Part II*:

- first the *Chapters* 1 through 3 on the fundamental theory of continued fractions,
- then *Chapter* 6, with excursions to *Chapter* 4, on algorithms to construct continued fraction representations,
- and finally the *Chapters* 7 and 8, with *Chapter* 5 as background material, for truncation and round-off error bounds.

0.1 Part one

Part I comprises the necessary theoretic background about continued fractions, when used as a tool to approximate functions. Its concepts and theorems are heavily used later on in the handbook. We deal with three term recurrence relations, linear fractional transformations, equivalence transformations, limit periodicity, continued fraction tails and minimal solutions. The connection between continued fractions and series is worked out in detail, especially the correspondence with formal power series at 0 and ∞.
The continued fraction representations of functions are grouped into several families, the main ones being the S-fractions, C-fractions, P-fractions, J-fractions, T-fractions, M-fractions and Thiele interpolating continued fractions. Most classical convergence results are given, formulated in terms of element and value regions. The connection between C- and P-fractions and Padé approximants on the one hand, and between M-fractions and two-point Padé approximants on the other hand is discussed. To conclude,

several moment problems, their link with Stieltjes integral transform representations and the concept of orthogonality are presented.

0.2 Part two

In *Part II* the reader is offered algorithms to construct different continued fraction representations of functions, known either by one or more formal series representations or by a set of function values. The qd-algorithm constructs C-fractions, the $\alpha\beta$- and FG-algorithms respectively deliver J- and T-fraction representations, and inverse or reciprocal differences serve to construct Thiele interpolating fractions. Also Thiele continued fraction expansions can be obtained as a limiting form.

When evaluating a continued fraction representation, only a finite part of the fraction can be taken into account. Several algorithms exist to compute continued fraction approximants. Each of them can make use of an estimate of the continued fraction tail to improve the convergence. A priori and a posteriori truncation error bounds are developed and accurate round-off error bounds are given.

0.3 Part three

The families of special functions discussed in the separate chapters in *Part III* are the bulk of the handbook and its main goal. We present series and continued fraction representations for several mathematical constants, the elementary functions, functions related to the gamma function, the error function, the exponential integrals, the Bessel functions and also several probability functions. All can be formulated in terms of either hypergeometric or confluent hypergeometric functions. We conclude with a brief discussion of the q-hypergeometric function and its continued fraction representations.

Each chapter in *Part III* is more or less structured in the same way, depending on the availability of the material. We now discuss the general organisation of such a chapter and the conventions adopted in the presentation of the formulas.

All tables and graphs in *Part III* are labelled and preceded by an extensive caption. Detailed information on their use and interpretation is given in the *Sections* 9.2 and 9.3, respectively.

Definitions and elementary properties. The nomenclature of the special functions is not unique. In the first section of each chapter the reader is presented with the different names attached to a single function. The variable z is consistently used to denote a complex argument and x for a real argument.

In a function definition the sign := is used to indicate that the left hand side denotes the function expression at the right hand side, on the domain given in the equation:

$$J(z) := \mathrm{Ln}(\Gamma(z)) - \left(z - \frac{1}{2}\right) \mathrm{Ln}(z) + z - \ln(\sqrt{2\pi}).$$

Here the principal branch of a multivalued complex function is indicated with a capital letter, as in Ln, while the real-valued and multivalued function are indicated with lower case letters, as in ln. The function definition is complemented with symmetry properties, such as mirror, reflection or translation formulas:

$$\mathrm{Ln}(\bar{z}) = \overline{\mathrm{Ln}(z)}.$$

Recurrence relations. Continued fractions are closely related to three-term recurrence relations, also called second order linear difference equations. Hence these are almost omnipresent, as in:

$$\begin{aligned} A_{-1} &:= 1, \qquad A_0 := 0, \\ A_n &:= a_n A_{n-1} + b_n A_{n-2}, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

or

$$\begin{aligned} {}_2F_1(a, b; c+1; z) = &-\frac{c(c-1-(2c-a-b-1)z)}{(c-a)(c-b)z} \, {}_2F_1(a, b; c; z) \\ &- \frac{c(c-1)(z-1)}{(c-a)(c-b)z} \, {}_2F_1(a, b; c-1; z). \end{aligned}$$

The recurrence relations immediately connected to continued fraction theory are listed. Other recurrences may be found in the literature, but may not serve our purpose.

Series expansion. Representations as infinite series are given with the associated domain of convergence. Often these series are power series as in (2.2.2) or (2.2.6). The series in the right hand side and the function in the left hand side coincide, denoted by the equality sign =, on the domain given in the right hand side:

$$\tan(z) = \sum_{k=1}^{\infty} \frac{4^k(4^k - 1)|B_{2k}|}{(2k)!} z^{2k-1}, \qquad |z| < \pi/2.$$

Asymptotic series expansion. Asymptotic expansions of the form (2.2.4) or (2.2.8) are given, if available, with the set of arguments where they are valid. Now the equation sign is replaced by the sign $\approx$:

$$J(z) \approx z^{-1} \sum_{k=0}^{\infty} \frac{B_{2k+2}}{(2k+1)(2k+2)} z^{-2k}, \qquad z \to \infty, \quad |\arg z| < \frac{\pi}{2}.$$

Stieltjes transform. For functions that can be represented as Stieltjes integral transforms, or equivalently as convergent S-fractions, positive T-fractions or real J-fractions, specific sharp truncation error estimates exist and the relative round-off error exhibits a slow growth rate when evaluating the continued fraction representation of the function by means of the backward algorithm.
Hence, if possible, the function under consideration or a closely related function is written as a Stieltjes integral transform:

$$\frac{\Gamma(a,z)}{z^a e^{-z}} = \frac{1}{\Gamma(1-a)} \int_0^{\infty} \frac{e^{-t} t^{-a}}{z+t}\, dt, \qquad |\arg z| < \pi, \quad -\infty < a < 1.$$

The conditions on the right hand side of the integral representation, here $|\arg z| < \pi$, $-\infty < a < 1$, are inherited from the function definition.

S-fraction, regular C-fraction and Padé approximants. S-fraction representations are usually found from the solution of the classical Stieltjes moment problem:

$$e^z E_n(z) = \frac{1/z}{1} + \mathop{\mathrm{K}}_{m=2}^{\infty} \left(\frac{a_m/z}{1} \right), \qquad a_{2k} = n+k-1, \quad a_{2k+1} = k,$$
$$|\arg z| < \pi, \quad n \in \mathbb{N}.$$

The equality sign $=$ between the left and right hand sides here has to be interpreted in the following way. The convergence of the continued fraction in the right hand side is uniform on compact subsets of the given convergence domain, here $|\arg z| < \pi$, excluding the poles of the function in the left hand side. When the convergence domain of the continued fraction in the right hand side is larger than the domain of the function in the left hand side, it may be regarded as an analytic continuation of that function.
C-fractions can be obtained for instance, by dropping some conditions that ensure the positivity of the coefficients a_m:

$$e^z E_\nu(z) = \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m(\nu) z^{-1}}{1} \right), \qquad |\arg z| < \pi, \quad \nu \in \mathbb{C},$$
$$a_1(\nu) = 1, \quad a_{2j}(\nu) = j + \nu - 1, \quad a_{2j+1}(\nu) = j, \quad j \in \mathbb{N}.$$

A C-fraction is intimately connected with Padé approximants, since its successive approximants equal Padé approximants on a staircase in the Padé table. When available, explicit formulas for the Padé approximants in part or all of the table are given. With the operator $\mathcal{P}_k$ defined as in (15.4.1),

$$r_{m+1,n}(z) = \frac{z^{-1}\mathcal{P}_{m+n}\left({}_2F_0(\nu,1;-z^{-1})\ {}_2F_0(-\nu-m,-n;z^{-1})\right)}{{}_2F_0(-\nu-m,-n;z^{-1})}, \qquad m+1 \geq n.$$

T-fraction, M-fraction and two-point Padé approximants. M-fractions correspond simultaneously to series expansions at 0 and at ∞. For instance, the fraction in the right hand side of

$$\frac{{}_1F_1(a+1;b+1;z)}{{}_1F_1(a;b;z)} = \frac{b}{b-z} + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{(a+m)z}{b+m-z}\right), \qquad z\in\mathbb{C}, \quad a\in\mathbb{C}, \quad b\in\mathbb{C}\setminus\mathbb{Z}_0^-$$

corresponds at 0 to the series representation of the function in the left hand side and corresponds at $z=\infty$ to the series representation of

$$-\frac{b}{z}\frac{{}_2F_0(a+1,a-b+1;-1/z)}{{}_2F_0(a,a-b+1;-1/z)}.$$

The two-point Padé approximants $r^{(2)}_{n+k,n-k}(z)$ corresponding to the same series at $z=0$ and at $z=\infty$, are given by

$$r^{(2)}_{n+k,n-k}(z) = \frac{P_{n-1,k}(\infty,a+1,b,z)}{P_{n,k}(\infty,a,b,z)}, \qquad 0\leq k\leq n,$$

where

$$\begin{aligned} P_{n,k}(\infty,b,c,z) &:= \lim_{a\to\infty} P_{n,k}(a,b,c,z/a), \qquad 0\leq k\leq n, \\ &= \mathcal{P}_n({}_1F_1(b;c;z)\ {}_1F_1(-b-n;1-c-k-n;-z)), \end{aligned}$$

for $P_{n,k}(a,b,c,z)$ given by (15.4.9) and the operator $\mathcal{P}_n$ defined in (15.4.1).

Real J-fraction and other continued fractions. Contractions of some continued fractions may result in J-fraction representations. Or minimal solutions of some recurrence relation may lead to yet another continued fraction representation. If closed formulas exist for the partial numerators and denominators of these fractions, these are listed after the usual families of S-, C- and T- or M-fractions. In general, we do not list different equivalent forms of a continued fraction.

Significant digits. Traditionally, the goal in designing mathematical approximations for use in function evaluations or implementations is to minimise the computation time. Our emphasis is on accuracy instead of speed. Therefore our numerical and graphical illustrations essentially focus on the presentation of the number of significant digits achieved by the series and continued fraction approximants. All output is reliable and correctly rounded.
By the presentation of tables and graphs for different approximants, also the speed of convergence in different regions of the complex plane is illustrated. More information on the tables and graphs in this handbook can be found in *Chapter* 9.

Reliability. All series and continued fraction representations in the handbook were verified numerically. So when encountering a slightly different formula from the one given in the original reference, it was corrected because the original work most probably contained a typo.

Further reading

- Similar formula books for different families of functions are [AS64; Ext78; SO87; GR00].
- Books discussing some of the special functions treated in this work are [Luk75; Luk69; AAR99].

Part I

BASIC THEORY

1

Basics

We develop some basic tools to handle continued fractions with complex numbers as elements. These include recurrence relations, equivalence transformations, the Euler connection with series, and a study of the tail behaviour of continued fractions which is quite different from that of series. Starting *Section* 1.10 we also deal with continued fractions in which the elements depend on a complex variable z. The representation of functions is further developed from *Chapter* 2 on.

1.1 Symbols and notation

The expression

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{1.1.1a}$$

is called a continued fraction, where a_m and b_m are complex numbers and $a_m \neq 0$ for all m. More recently, for convenience, other symbols are used to denote the same continued fraction. These include the following:

$$b_0 + \frac{a_1 |}{| b_1} + \frac{a_2 |}{| b_2} + \frac{a_3 |}{| b_3} + \ldots, \tag{1.1.1b}$$

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \ldots \tag{1.1.1c}$$

and

$$b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{a_m}{b_m} \right), \tag{1.1.1d}$$

or for short

$$b_0 + \mathrm{K} \left(\frac{a_m}{b_m} \right). \tag{1.1.1e}$$

The symbol K in (1.1.1d) and (1.1.1e) for (infinite) fraction, from the German word Kettenbruch, is the analogue of Σ for (infinite) sum. Correspondingly the n^{th} approximant f_n of the continued fraction is expressed by

$$f_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}}, \tag{1.1.2a}$$

$$f_n = b_0 + \frac{a_1 |}{| b_1} + \frac{a_2 |}{| b_2} + \frac{a_3 |}{| b_3} + \cdots + \frac{a_n |}{| b_n}, \tag{1.1.2b}$$

$$f_n = b_0 + \frac{a_1}{b_1 +}\,\frac{a_2}{b_2 +}\,\frac{a_3}{b_3 +} \cdots \frac{a_n}{+\, b_n} \tag{1.1.2c}$$

and

$$f_n = b_0 + \mathop{\mathrm{K}}_{m=1}^{n} \left(\frac{a_m}{b_m}\right). \tag{1.1.2d}$$

Only the symbols (1.1.1c), (1.1.1d), (1.1.1e) and (1.1.2c), (1.1.2d) are used in the present book.
The continued fraction (1.1.1) is more than just the sequence of approximants $\{f_n\}$ or the limit of this sequence, if it exists. In fact, the continued fraction is the mapping of the ordered pair of sequences $(\{a_m\}, \{b_m\})$ onto the sequence $\{f_n\}$. This concept is made more precise in the definition of continued fraction in the following section.

1.2 Definitions

The complex plane is denoted by $\mathbb{C}$ and the extended complex plane by

$$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

The symbols $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets

$$\mathbb{N} := \{1, 2, 3, \ldots\}, \qquad \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}.$$

Continued fraction. An ordered pair of sequences $(\{a_m\}_{m\in\mathbb{N}}, \{b_m\}_{m\in\mathbb{N}_0})$ of complex numbers, with $a_m \neq 0$ for $m \geq 1$, gives rise to sequences $\{s_n(w)\}_{n\in\mathbb{N}_0}$ and $\{S_n(w)\}_{n\in\mathbb{N}_0}$ of *linear fractional transformations*

$$s_0(w) := b_0 + w, \quad s_n(w) := \frac{a_n}{b_n + w}, \quad n = 1, 2, 3, \ldots, \tag{1.2.1a}$$

$$S_0(w) := s_0(w), \quad S_n(w) := S_{n-1}(s_n(w)), \quad n = 1, 2, 3, \dots \tag{1.2.1b}$$

and to a sequence $\{f_n\}$, given by

$$f_n = S_n(0) \in \widehat{\mathbb{C}}, \qquad n = 0, 1, 2, \dots. \tag{1.2.2}$$

The ordered pair [Hen77, p. 474]

$$((\{a_m\}, \{b_m\}), \{f_n\}) \tag{1.2.3}$$

is the *continued fraction* denoted by the five symbols in (1.1.1). The numbers a_m and b_m are called m^{th} *partial numerator* and *partial denominator*, respectively, of the continued fraction. The value f_n is called the n^{th} *approximant* and is denoted by the four symbols (1.1.2). Some authors use the term *convergent* where we use approximant. A common name for partial numerator and denominator is *element*.

The linear fractional transformation $S_n(w)$ can be expressed as

$$S_n(w) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n + w}}}}, \tag{1.2.4a}$$

or more conveniently as

$$S_n(w) = b_0 + \frac{a_1}{b_1} \genfrac{}{}{0pt}{}{}{+} \frac{a_2}{b_2} \genfrac{}{}{0pt}{}{}{+} \frac{a_3}{b_3} \genfrac{}{}{0pt}{}{}{+} \genfrac{}{}{0pt}{}{}{\cdots} \genfrac{}{}{0pt}{}{}{+} \frac{a_{n-1}}{b_{n-1}} \genfrac{}{}{0pt}{}{}{+} \frac{a_n}{b_n + w}\,. \tag{1.2.4b}$$

Equivalently,

$$S_n(w) = s_0 \circ s_1 \circ s_2 \circ \cdots \circ s_n(w), \tag{1.2.5}$$

where $\circ$ denotes *composition* such as in

$$s_0 \circ s_1(w) := s_0(s_1(w))\,.$$

In particular,

$$s^n(w) := \underbrace{s \circ \cdots \circ s}_{n \text{ times}}(w)\,.$$

For a given sequence $\{w_n\}_{n \in \mathbb{N}_0}$, the number

$$S_n(w_n) \in \widehat{\mathbb{C}} \tag{1.2.6}$$

is called an n^{th} *modified approximant.*

Convergence. A continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ is said to *converge* if and only if the sequence of approximants $\{f_n\} = \{S_n(0)\}$ converges to a limit $f \in \widehat{\mathbb{C}}$. In this case f is called the *value* of the continued fraction. Note that convergence to ∞ is accepted. If the continued fraction is convergent to f, then the symbols (1.1.1) are used to represent both the ordered pair (1.2.3) and the value f. That is, we may write

$$f = \lim_{n\to\infty} S_n(0) = b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m}{b_m}\right). \tag{1.2.7}$$

Sometimes (1.2.7) is called *classical convergence.*

General convergence. A continued fraction *converges generally* [Jac86; LW92, p. 43] to an extended complex number f if and only if there exist two sequences $\{v_n\}$ and $\{w_n\}$ in $\widehat{\mathbb{C}}$ such that

$$\liminf_{n\to\infty} d(v_n, w_n) > 0$$

and

$$\lim_{n\to\infty} S_n(v_n) = \lim_{n\to\infty} S_n(w_n) = f.$$

Here $d(z,w)$ denotes the *chordal metric* defined by

$$d(z,w) := \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, \qquad z,w \in \mathbb{C}$$

and

$$d(\infty,w) := \frac{1}{\sqrt{1+|w|^2}}, \qquad w \in \mathbb{C}\,.$$

The value f is unique. Convergence to f implies general convergence to f since

$$S_n(\infty) = S_{n-1}(0)$$

but general convergence does not imply convergence.

Example 1.2.1: The continued fraction

$$\frac{2}{1} + \frac{1}{1} + \frac{-1}{1} + \frac{2}{1} + \frac{1}{1} + \frac{-1}{1} + \dots$$

diverges. By using the recurrence relations (1.3.1), we find for $n \geq 1$ that

$$\begin{aligned} A_{3n-2} &= 2^n, & A_{3n-1} &= 2^n, & A_{3n} &= 0, \\ B_{3n-2} &= 2^{n+1} - 3, & B_{3n-1} &= 2^{n+1} - 2, & B_{3n} &= 1. \end{aligned}$$

For the modified approximants $S_n(w_n)$ we find from (1.3.2) that

$$S_{3n-2}(w_{3n-2}) = \frac{2^n + w_{3n-2} \cdot 0}{(2^{n+1} - 3) + w_{3n-2} \cdot 1},$$

which converges to $1/2$ if the sequence $\{w_{3n-2}\}$ is bounded. Similarly, we find that the sequence

$$S_{3n-1}(w_{3n-1}) = \frac{2^n + w_{3n-1} \cdot 2^n}{(2^{n+1} - 2) + w_{3n-1}(2^{n+1} - 3)}$$

converges to $1/2$ if the sequence $\{w_{3n-1}\}$ is bounded away from -1 and the sequence

$$S_{3n}(w_{3n}) = \frac{0 + w_{3n} \cdot 2^n}{1 + w_{3n}(2^{n+1} - 2)}$$

converges to $1/2$ if the sequence $\{w_{3n}\}$ is bounded away from 0. Hence we have that the continued fraction converges generally.

1.3 Recurrence relations

The n^{th} *numerator* A_n and the n^{th} *denominator* B_n of a continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ are defined by the *recurrence relations* (second order linear difference equations)

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} := b_n \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} + a_n \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix}, \qquad n = 1, 2, 3, \ldots \tag{1.3.1a}$$

with initial conditions

$$A_{-1} := 1, \quad B_{-1} := 0, \quad A_0 := b_0, \quad B_0 := 1\,. \tag{1.3.1b}$$

The modified approximant $S_n(w_n)$ in (1.2.6) can then be written as

$$S_n(w_n) = \frac{A_n + A_{n-1}w_n}{B_n + B_{n-1}w_n}, \qquad n = 0, 1, 2, \ldots \tag{1.3.2}$$

and hence for the n^{th} approximant f_n we have

$$f_n = S_n(0) = \frac{A_n}{B_n}, \qquad f_{n-1} = S_n(\infty) = \frac{A_{n-1}}{B_{n-1}}\,. \tag{1.3.3}$$

Determinant formula. The n^{th} numerator and denominator satisfy the *determinant formula*

$$\begin{aligned} \begin{vmatrix} A_n & A_{n-1} \\ B_n & B_{n-1} \end{vmatrix} &= A_n B_{n-1} - A_{n-1} B_n \\ &= (-1)^{n-1} \prod_{m=1}^{n} a_m, \qquad n = 1, 2, 3, \ldots. \end{aligned} \tag{1.3.4}$$

Matrix connection with continued fractions. Let $\mathrm{K}\,(a_m/b_m)$ be a given continued fraction with n^{th} numerator A_n and n^{th} denominator B_n. Let

$$s_m(w) := \frac{a_m}{b_m + w}, \qquad x_m := \begin{pmatrix} 0 & a_m \\ 1 & b_m \end{pmatrix}, \qquad m = 1, 2, 3, \ldots.$$

Then the linear fractional transformation $S_n(w)$ given by (1.2.5) and (1.3.2) leads to

$$X_n := x_1 x_2 x_3 \cdots x_n = \begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix}, \qquad n = 1, 2, 3, \ldots.$$

Therefore multiplication of 2×2 matrices can be used to construct the sequences $\{A_n\}, \{B_n\}$ and $\{f_n\}$, where f_n is given by (1.2.2) and (1.3.3). More generally, if

$$t_m(w) := \frac{a_m + c_m w}{b_m + d_m w}, \qquad y_m := \begin{pmatrix} c_m & a_m \\ d_m & b_m \end{pmatrix}, \qquad m = 1, 2, 3, \ldots$$

then

$$T_n(w) := t_1 \circ t_2 \circ t_3 \circ \cdots \circ t_n(w) = \frac{A_n + C_n w}{B_n + D_n w}, \qquad n = 1, 2, 3, \ldots$$

and

$$Y_n := y_1 y_2 y_3 \cdots y_n = \begin{pmatrix} C_n & A_n \\ D_n & B_n \end{pmatrix}, \qquad n = 1, 2, 3, \ldots.$$

1.4 Equivalence transformations

Two continued fractions $b_0 + \mathrm{K}\,(a_m/b_m)$ and $d_0 + \mathrm{K}\,(c_m/d_m)$ are said to be *equivalent* if and only if they have the same sequence of approximants. This is written

$$b_0 + \operatorname*{K}_{m=1}^{\infty} (a_m/b_m) \equiv d_0 + \operatorname*{K}_{m=1}^{\infty} (c_m/d_m). \tag{1.4.1}$$

The equivalence (1.4.1) holds if and only if there exists a sequence of complex numbers $\{r_m\}$, with $r_0 = 1$ and $r_m \neq 0$ for $m \geq 1$, such that

$$d_0 = b_0, \quad c_m = r_m r_{m-1} a_m, \quad d_m = r_m b_m, \quad m = 1, 2, 3, \ldots. \tag{1.4.2}$$

Equations (1.4.2) define an *equivalence transformation.* Since $a_m \neq 0$ for $m \geq 1$, one can always choose

$$r_m = \prod_{k=1}^{m} a_k^{(-1)^{m+1-k}} = \left(\frac{\prod\limits_{k=1}^{\lfloor m/2 \rfloor} a_{2k}}{\prod\limits_{k=1}^{\lfloor (m+1)/2 \rfloor} a_{2k-1}} \right)^{(-1)^{m-1}}, \qquad m = 1, 2, 3, \ldots,$$

which yields the equivalence transformation

$$\begin{aligned} b_0 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{a_m}{b_m} \right) &\equiv b_0 + \operatorname*{K}_{m=1}^{\infty} \left(\frac{1}{d_m} \right) \\ &= b_0 + \frac{1}{b_1/a_1 +} \, \frac{1}{b_2 a_1/a_2 +} \, \frac{1}{b_3 a_2/(a_1 a_3) + \ldots}, \end{aligned}$$

where in general

$$\begin{aligned} d_1 &= \frac{b_1}{a_1}, \\ d_{2m} &= b_{2m} \frac{a_1 a_3 \cdots a_{2m-1}}{a_2 a_4 \cdots a_{2m}}, \qquad m = 1, 2, 3, \ldots, \\ d_{2m+1} &= b_{2m+1} \frac{a_2 a_4 \cdots a_{2m}}{a_1 a_3 \cdots a_{2m+1}}, \qquad m = 1, 2, 3, \ldots. \end{aligned}$$

Hence, in studying continued fractions there is no loss of generality in the restriction to continued fractions $\mathrm{K}\,(1/d_m)$. On the other hand, if

$$b_m \neq 0, \qquad m = 1, 2, 3, \ldots,$$

then one can obtain an equivalence transformation of the form

$$b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{a_m}{b_m}\right) \equiv b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty}\left(\frac{c_m}{1}\right)$$
$$= b_0 + \frac{a_1/b_1}{1} \underset{+}{} \frac{a_2/(b_1 b_2)}{1} \underset{+}{} \frac{a_3/(b_2 b_3)}{1} \underset{+}{} \ldots,$$

where in general

$$r_m = \frac{1}{b_m}, \qquad m = 1, 2, 3, \ldots,$$

$$c_1 = \frac{a_1}{b_1}, \qquad c_m = \frac{a_m}{b_{m-1} b_m}, \qquad m = 2, 3, 4, \ldots.$$

Hence, in studying continued fractions there is only little loss of generality in the restriction to continued fractions $\mathrm{K}\,(c_m/1)$.

1.5 Contractions and extensions

In this section we let A_n, B_n and f_n denote the n^{th} numerator, denominator and approximant, respectively of a continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ and we let C_n, D_n and g_n denote the n^{th} numerator, denominator and approximant, respectively, of a continued fraction $d_0 + \mathrm{K}\,(c_m/d_m)$. Then $d_0 + \mathrm{K}\,(c_m/d_m)$ is called a *contraction* of $b_0 + \mathrm{K}\,(a_m/b_m)$ if and only if there exists a sequence $\{n_k\}$ such that

$$g_k = f_{n_k}, \qquad k = 0, 1, 2, \ldots. \tag{1.5.1}$$

The continued fraction $b_0 + \mathrm{K}\,(a_m/b_m)$ is then called an *extension* of $d_0 + \mathrm{K}\,(c_m/d_m)$.

Canonical contraction. If in addition to (1.5.1) there exists a sequence $\{n_k\}$ such that

$$C_k = A_{n_k}, \qquad D_k = B_{n_k}, \qquad k = 0, 1, 2, \ldots, \tag{1.5.2}$$

then $d_0 + \mathrm{K}\,(c_m/d_m)$ is called a *canonical contraction* of $b_0 + \mathrm{K}\,(a_m/b_m)$.

Even contraction. A continued fraction $d_0 + \mathrm{K}\,(c_m/d_m)$ is called an *even contraction* or *even part* of $b_0 + \mathrm{K}\,(a_m/b_m)$ if and only if

$$g_n = f_{2n}, \qquad n = 0, 1, 2, \ldots$$

and it is called the *even canonical contraction* of $b_0 + \mathrm{K}\,(a_m/b_m)$ if and only if

$$C_n = A_{2n}, \qquad D_n = B_{2n}, \qquad n = 0, 1, 2, \ldots.$$

An even canonical contraction of $b_0 + \mathrm{K}\,(a_m/b_m)$ exists if and only if

$$b_{2k} \neq 0, \qquad k = 1, 2, 3, \ldots .$$

When it exists, the even canonical contraction of $b_0 + \mathrm{K}\,(a_m/b_m)$ is given by

$$d_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m}{d_m} \right) = b_0 + \frac{a_1 b_2}{a_2 + b_1 b_2} - \frac{a_2 a_3 b_4/b_2}{a_4 + b_3 b_4 + a_3 b_4/b_2} - \frac{a_4 a_5 b_6/b_4}{a_6 + b_5 b_6 + a_5 b_6/b_4} - \ldots \tag{1.5.3a}$$

where

$$\begin{aligned} d_0 &= b_0, \quad c_1 = a_1 b_2, \quad d_1 = a_2 + b_1 b_2, \\ c_m &= -\frac{a_{2m-2} a_{2m-1} b_{2m}}{b_{2m-2}}, \qquad m = 2, 3, 4, \ldots, \\ d_m &= a_{2m} + b_{2m-1} b_{2m} + \frac{a_{2m-1} b_{2m}}{b_{2m-2}}, \qquad m = 2, 3, 4, \ldots . \end{aligned} \tag{1.5.3b}$$

Odd contraction. A continued fraction $d_0 + \mathrm{K}\,(c_m/d_m)$ is called an *odd contraction* or *odd part* of $b_0 + \mathrm{K}\,(a_m/b_m)$ if and only if

$$g_n = f_{2n+1}, \qquad n = 0, 1, 2, \ldots$$

and it is called an *odd canonical contraction* if and only if

$$\begin{aligned} C_0 &= \frac{A_1}{B_1}, \qquad D_0 = 1, \\ C_n &= A_{2n+1}, \qquad D_n = B_{2n+1}, \qquad n = 1, 2, 3, \ldots . \end{aligned}$$

An odd canonical contraction of $b_0 + \mathrm{K}\,(a_m/b_m)$ exists if and only if

$$b_{2k+1} \neq 0, \qquad k = 0, 1, 2, \ldots .$$

If it exists, an odd canonical contraction of $b_0 + \mathrm{K}\,(a_m/b_m)$ is given by

$$d_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \left(\frac{c_m}{d_m} \right) = \frac{a_1 + b_0 b_1}{b_1} - \frac{a_1 a_2 b_3/b_1}{b_1(a_3 + b_2 b_3) + a_2 b_3} - \frac{a_3 a_4 b_1 b_5/b_3}{a_5 + b_4 b_5 + a_4 b_5/b_3} - \frac{a_5 a_6 b_7/b_5}{a_7 + b_6 b_7 + a_6 b_7/b_5} - \ldots \tag{1.5.4a}$$