MECHANICS OF GENERALIZED CONTINUA
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Aims and Scope

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MECHANICS OF GENERALIZED CONTINUA

ONE HUNDRED YEARS AFTER THE COSSEARTS

Edited By

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Springer
Foreword

Welcome!
I am especially privileged and honored that Professors Maugin, Metrikine and Erofeyev, the organizers and chairmen of this meeting, the Euromech Colloquium 510 honoring the Cosserats for the 100 year anniversary of the publication of their book, have asked me to say a few words to express my welcome salute to you. Much as I would have liked to do this in person, my physical being is no longer keeping pace with my mental desires and, thus, alas, is denying me this luxury.

Sometime in the past, I remember reading an article whose author’s name has slipped my memory—perhaps it was Marston Morse, Professor Emeritus at the Institute for Advanced Study, who wrote (and I paraphrase):

**Discovery of new mathematical disciplines originates from two criteria:**

1. **Generalization**
2. **Inversion**

Some of the earliest examples for the validity of these criteria are:

(a) The Newton–Leibniz discovery of differentiation and integration, which started calculus; and

(b) The Theory of Elasticity, which was conceived when Robert Hooke, in 1678, published an anagram: “ceiinosssttuu”, which he expressed as “*ut tensio sic vis*”, meaning, the power of any material is in the same proportion within the tension thereof. Presently, this is known as “Hooke’s Law”.

Some 250 years later, “The modern theory of elasticity may be considered to have its birth in 1821, when Navier first gave the equations for the equilibrium and motion of elastic solids, . . .” (Todhunter and Pearson).

Of course, many other scientists, Cauchy, Poisson, Stokes, and others, after 1821, improved and extended the theory to other materials, e.g., viscous fluids, and they investigated atomic and molecular foundations. This is typical—for the maturation of any discipline is the result of the contributions of many scientists and often takes a long time.
Improvements and extensions of the theory of elasticity continued in the nineteenth and early part of the twentieth century: rigorous mathematical theory of non-linear elasticity, relativistic continuum mechanics, magneto-elasticity and other “hyphenated” sister fields, like viscoelasticity and thermoelasticity. Underlying basic postulates (e.g., frame-independence, thermodynamical restrictions, relativistic invariance) were introduced and applied in the development of field equations and admissible constitutive laws. Research in granular and porous elastic solids, composite elastic materials, polymeric materials, and statistical and molecular foundations of continua are but a few examples that still remain as active research fields.

Eugène Maurice Pierre Cosserat and his brother François Cosserat, 100 years ago, cast the seed of Generalized Continua, by publishing a book, in 1909, entitled Théorie des Corps Déformables (Hermann, Paris). The revolutionary contribution of this book is that material points of an elastic solid are considered equipped with directors, which give rise to the concept of couple stress and a new conservation law for the moment of momentum. By means of a variational principle which they called “l’action euclidienne”, they obtained “balance laws of elasticity”. The introduction of the director concept made it possible to formulate anisotropic fluids, e.g., liquid crystals, blood.

The Cosserats did not give constitutive equations. These, and the introduction of the microinertia tensor and the associated conservation law, which are crucial to the dynamic problems in solid and fluent media (e.g., liquid crystals, suspensions, etc.) were introduced later by other scientists.

Over half a century elapsed before the Cosserats’ book was discovered by researchers. After 1960, independent, Cosserat-like theories were published in European countries, the USA and the USSR, under a variety of nomenclature (e.g., couple stress, polar elasticity, asymmetric elasticity, strain gradient theories, micropolar elasticity, multipolar theory, relativistic continua with directors, etc.). I recall a literature search on these subjects that was shown to me by a visiting scholar, Professor Listrov, from the USSR This book contained several hundred entries of papers published by 1970.

The next significant generalizations appear in 1964 and thereafter, in the areas of microelasticity, microfluid mechanics, micropolar continua, micromorphic electrodynamics, and others that constitute the family of micromorphic continua or microstructure theories.

The conception of these theories was based on the query, “Is it possible to construct continuum theories that can predict physical phenomena on the atomic, molecular, or nano scales?” These would require supplying additional degrees of freedom to the material point beyond a director. After all, the molecules that constitute the internal structures of the material points (particles) undergo deformations and rotations arising from the displacement and rotations of their constituent atoms. This supplies twelve degrees of freedom. A body with such an internal structure is called Micromorphic grade 1. Micromorphic continua of grade $N > 1$ have also been formulated.

To understand the difference between the Cosserat and the micromorphic elasticities, it is important to note that micromorphic elasticity gives rise to two different
second-order strain tensors (only one of which is symmetric), and to one third-order microstrain tensor. Correspondingly, the balance laws introduce two second-order stress tensors (only one of which is symmetric), and one third-order microstress (moment-stress) tensor.

In special cases, the Micromorphic Theory leads to other special continuum theories:

**Micromorphic → Microstretch → Micropolar (Cosserats) → Classical**

The next important contributions are the nonlocal continuum theories that generalize constitutive equations for classical and micromorphic continuum theories, by introducing the influence of distant material points, e.g., the stress tensor is a functional of the strain tensors of all material points of a body. In this sense, micromorphic grade 1 is a nonlocal theory with a *short nonlocality* (or discrete nonlocality). Among the many important contributions of nonlocality, I mention that it eliminates the stress singularity (infinite stress) at the crack tip predicted by classical elasticity. Moreover, a natural fracture criterion was born which states that failure occurs when the maximum stress becomes or exceeds the cohesive stress.

**The Present State.** No doubt other generalized continuum theories are in a state of composition. But mathematical theories cannot be considered the truth without experimental verification. Unfortunately, excluding classical theories, the experimental work for all these theories is left wanting. The opportunity is here and now, for experimentalists to determine the material moduli and/or to confirm or challenge the validity of some of these theories.

**A Note on the Future.** Ultimately, all continuum theories must be based on the quantum field theory, or perhaps, on the quantum theory of general relativity (when unified). This offers the greatest challenges to future scientific investigators.

I am pleased to see so many interesting contributions to some of these fields included in this meeting, which are in the spirit of the Cosserats’ work.

I welcome you and send my best wishes for what, I am sure, will be an inspirational and productive meeting.

Littleton, Colorado, May 2009

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Preface

This volume gathers in some organized and edited manner most of the contributions delivered at the EUROMECH Colloquium 510 held in Paris, May 13–16, 2009. The explicit aim of the colloquium was, on the occasion of the centennial of the publication of a celebrated book (Théorie des corps déformables) by the Cosserat brothers, to examine the evolution in time since the Cosserats, and the actuality of the notion of generalized continuum mechanics to which the Cosserats’ work contributed to some important extent. Of course, the Cosserat book belongs to this collection of classics that are more often cited than read. The reason for this is twofold. First, the vocabulary and mathematical symbols have tremendously evolved since the early 1900s, and second, the Cosserat book by itself is an intrinsically difficult reading. As a matter of fact, more than introducing precisely the notion of Cosserat media (a special class of generalized continua), the Cosserats’ book had a wider ambition, that of presenting a reflection on the general framework of continuum mechanics, with the notion of group permeating—not explicitly—its structure (cf. the notion of “action euclidienne”). This is reflected in many of the following contributions.

Overall, the whole landscape of contemporary generalized continuum mechanics was spanned from models to applications to structures, dynamical properties, problems with measurement of new material coefficients, numerical questions posed by the microstructure, and new possible developments (nanomaterials, fractal structures, new geometrical ideas). Remarkably absent were models and approaches using the concept of strong nonlocality (constitutive equations that are functionals over space). This is a mark of a certain evolution.

An interesting comparison can be made with the contents of the landmark "IUTAM Symposium" gathered in 1967 in Stuttgart-Freudenstadt under the chairmanship of the late E. Kröner. Most of the models presented at that meeting by luminaries such as Noll, Eringen, Rivlin, Green, Sedov, Mindlin, Nowacki, Stojanovic, and others were essentially of the Cosserat type and, still in their infancy, had a much questioned usefulness that is no longer pondered. Most of the contributions were either American or German. With the present EUROMECH we witnessed an enlargement of the classes of models with a marked interest in gradient-type theories. Also, because the political situation has drastically changed within forty tears, we
realize now the importance of the Russian school. The latter was, in fact, very much ignored in the 1960s and 1970s while some Russian teams were ahead of their Western colleagues in acknowledging their debt to the Cosserats and other scientists such as Leroux, Le Corre and Laval in France. Of these heroic Soviet times, E. Aero and V. Palmov, both from St. Petersburg, who published on the subject matter in the early 1960s, were present in Paris. Professor A.C. Eringen (he also in Freundenstadt in 1967), unable to attend, kindly sent us a Welcome address that is reproduced here in the way of a Foreword.

Unfortunately, the editing of this book was saddened by the passing away of A.C. Eringen on December 06, 2010, at the age of 88, after more than sixty years of devotion to engineering science, physics and applied mathematics.

The Colloquium was financially and materially supported by the Engineering UFR of the Université Pierre et Marie Curie (UPMC), the STII Directorate of the French Centre National de la Recherche Scientifique, and the Institut Jean Le Rond d’Alembert, UPMC–Paris Universitas and UMR 7190 of CNRS. Members of the MPIA Team of this Institute helped much in the local organization. Ms Simona Otarasanu is to be thanked for her efficient treatment of many questions. Without the expertise of Ms Janine Indeau, the present volume would not exist.

Paris  Gérard A. Maugin
Delft  Andrei V. Metrikine
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Part I
On the Cosserat’s Works
Chapter 1
Generalized Continuum Mechanics:
What Do We Mean by That?

Gérard A. Maugin

Dedicated to A.C. Eringen

Abstract Discursive historical perspective on the developments and ramifications of generalized continuum mechanics from its inception by the Cosserat brothers (Théorie des corps déformables. Hermann, 1909) with their seminal work of 1909 to the most current developments and applications is presented. The point of view adopted is that generalization occurs through the successive abandonment of the basic working hypotheses of standard continuum mechanics of Cauchy, that is, the introduction of a rigidly rotating microstructure and couple stresses (Cosserat continua or micropolar bodies, nonsymmetric stresses), the introduction of a truly deformable microstructure (micromorphic bodies), “weak” nonlocalization with gradient theories and the notion of hyperstresses, and the introduction of characteristic lengths, “strong nonlocalization” with space functional constitutive equations and the loss of the Cauchy notion of stress, and finally giving up the Euclidean and even Riemannian material background. This evolution is paved by landmark papers and timely scientific gatherings (e.g., Freudenstadt, 1967; Udine, 1970, Warsaw, 1977).

Preliminary note: Over 40 years, the author has benefited from direct studies under, and lectures from, P. Germain, A.C. Eringen, E.S. Suhubi, R.D. Mindlin, W. Nowacki, V. Sokolowski, S. Stojanovic, from contacts with J.L. Ericksen, C.A. Truesdell and D.G.B. Edelen, from friendship with C.B. Kafadar, J.M. Lee, D. Rogula, H.F. Tiersten, J. Jaric, P.M. Naghdi, I.A. Kunin, L.I. Sedov, V.L. Berdichevskii, E. Kröner, and most of the authors in the present volume as co-workers or friends, all active contributors to the present subject matter. He apologizes to all these people who certainly do not receive here the fully deserved recognition for their contribution to the field.

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1.1 Introduction

The following question is naturally raised with the venue of EUROMECH 510 in Paris in May 2009. What do we understand by generalized continuum mechanics? Note already some ambiguity since the last expression can be alternately phrased as “generalized (continuum mechanics)” or “(generalized continuum) mechanics”. We do not pursue this semantic matter. We simply acknowledge the fact that with the publication of the book of the Cosserat brothers in 1909 a true “generalized continuum mechanics” developed, first slowly and rather episodically and then with a real acceleration. That a new era was borne at the time in the field of continuum mechanics is not obvious if we remember that the Cosserats’ theory was published as a supplement to the French translation (by them for some, I suppose, alimentary purpose) of Chowolson’s Russian Encyclopedia of mathematics (the translation was done from the German edition). Another valid subtitle of the present contribution could be “From the classical to the less classical”. But what is classical? Then the “less classical” or “generalized” will be defined by successively discarding the working hypotheses of the classical case, simple as the latter may be.

1.2 From Cauchy and the Nineteenth Century

Here we consider as a classical standard the basic model considered by engineers in solid mechanics and the theory of structures. This essentially is the theory of continua set forth by A.L. Cauchy in the early nineteenth century for isotropic homogeneous elastic solids in small strains. The theory of continua respecting Cauchy’s axioms and simple working hypotheses is such that the following holds true:

1. Cauchy’s postulate. The traction exerted on a facet cut in the solid depends on the geometry of that facet only at the first order (the local unit normal); it will be linear in that normal. From this the notion of a stress tensor follows, the so-called stress being the only “internal force” in the theory.

2. It is understood that both physical space (of Newton) and material manifold (the set of material particles constituting the body) are Euclidean and connected, whence the notion of displacement is well defined.

3. Working hypothesis (i). There are no applied couples in both volume and surface.

4. Working hypothesis (ii). There exists no “microstructure” described by additional internal degrees of freedom.

According to Points 3 and 4, the Cauchy stress tensor is symmetric. This results from the application of the balance of angular momentum. Isotropy, homogeneity, and small strains are further hypotheses but they are not so central to our argument. Then generalizations of various degrees consist in relaxing more or less these different points above, hence the notion of generalized continuum. This notion of generalization depends also on the culture and physical insight of the scientists. For instance, the following generalizations are “weak” ones:
• “Generalized” Hooke’s law (linear, homogeneous, but anisotropic medium);
• Hooke–Duhamel law in thermoelasticity;
• Linear homogeneous piezoelectricity in obviously anisotropic media (no center of symmetry).

These are “weak” generalizations because they do not alter the main mathematical properties of the system. Of course, thermoelasticity and linear piezoelectricity require adding new independent variables (e.g., temperature $\theta$ or scalar electric potential $\varphi$). In some sense, the problem becomes four-dimensional for the basic field (elastic displacement and temperature in one case, elastic displacement and electric potential in the other). The latter holds in this mere simplicity under the hypothesis of weak electric fields, from which there follows the neglect of the so-called ponderomotive forces and couples, e.g., the couple $P \times E$ when electric field and polarization are not necessarily aligned; see Eringen and Maugin [24]. Such theories, just like standard elasticity, do not involve a length scale. But classical linear inhomogeneous elasticity presents a higher degree of generalization because a characteristic length intervenes necessarily.

From here on, we envisage three true (in our view) generalizations.

### 1.2.1 The Cauchy Stress Tensor Becomes Nonsymmetric for Various Reasons

This may be due to

(i) The existence of body couples (e.g., in electromagnetism: $P \times E$ or/and $M \times H$; the case of intense EM fields or linearization about intense bias fields);
(ii) The existence of surface couples (the introduction of “internal forces” of a new type of the so-called couple stresses); the medium possesses internal degrees of freedom that modify the balance of angular momentum;
(iii) The existence of internal degrees of freedom of a nonmechanical nature in origin, e.g., polarization inertia in ferroelectrics, intrinsic spin in ferromagnetics (see Maugin’s book [57]);
(iv) The existence of internal degrees of freedom of “mechanical” nature.

This is where the Cosserats’ model comes into the picture.

The first example in this class pertains to a rigid microstructure (three additional degrees of freedom corresponding to an additional rotation at each material point, independently of the vorticity). Examples of media of this type go back to the early search for a continuum having the capability to transmit transverse waves (as compared to acoustics in a pure fluid), i.e., in relation to optics. The works of McCullagh [64] and Lord Kelvin must be singled out (cf. Whittaker [85]). Pierre Duhem [9] proposes to introduce a triad of three rigidly connected directors (unit vectors) to represent this rotation. In modern physics, there are other tools for this, including Euler’s angles (not very convenient), quaternions and spinors. It is indeed
the Cosserats, among other studies in elasticity, who really introduced internal degrees of freedom of the rotational type (these are micropolar continua in the sense of Eringen) and the dual concept of couple stress. Hellinger [36], in a brilliant essay, recognized at once the new potentialities offered by this generalization but did not elaborate on these. A modern rebirth of the field had to await works in France by crystallographs (Laval [45–47]; Le Corre [49]), in Russia by Aero and Kuvshinskii [1], and Palmov [71], in Germany by Schaeffer [77], Günther [34], Neuber [67], and in Italy by Grioli [33] and Capriz—see Capriz’s book of 1989 [3]. But the best formulations are those obtained by considering a field of orthogonal transformations (rotations) and not the directors themselves, see Eringen [19–21], Kafadar and Eringen [37], Nowacki [70], although we note some obvious success of the “director” representation, e.g., in liquid crystals (Ericksen [17]; Leslie [52]) and the kinematics of the deformation of slender bodies (Ericksen, Truesdell, Naghdi)—in this volume see the contribution of Lhuillier. But there was in the mid 1960s a complete revival of continuum mechanics (cf. Truesdell and Noll [82]) which, by paying more attention to the basics, favored the simultaneous formulation of many more or less equivalent theories of generalized continua in the line of thought of the Cosserats (works by Mindlin and Tiersten [66], Mindlin and Eshel [65], Green and Rivlin [32], and Green and Naghdi [31], Toupin [80, 81], Truesdell and Toupin [83], and Eringen and Suhubi [25, 26], etc.).

More precisely, in the case of a deformable microstructure at each material point, the vector triad of directors of Duhem–Cosserats becomes deformable and the additional degree of freedom at each point, or micro-deformation, is akin to a general linear transformation (nine degrees of freedom). These are micromorphic continua in Eringen’s classification. A particular case is that of continua with microstretch. A truly new notion here is that of the existence of a conservation law of micro-inertia (Eringen [18], Stokes [79]). In the present volume, this is illustrated by several contributions. A striking example is due to Drouot and Maugin [8] dealing with fluid solutions of macromolecules, while Pouget and Maugin [73] have provided a fine example of truly micromorphic solids with the case of piezoelectric powders treated as continua.

Remark 1.1. Historical moments in the development of this avenue of generalization have been the IUTAM symposium organized by E. Kröner in Freudenstadt in 1967 (see Kröner [41]) and the CISM Udine summer course of 1970 (Mindlin, Eringen, Nowacki, Stojanovic, Sokolowski, Maugin, Jaric, Micunovic, etc. were present).

Remark 1.2. Strong scientific initial motivations for the studies of generalized media at the time (1960–1970s) were (i) the expected elimination of field singularities in many problems with standard continuum mechanics, (ii) the continuum description of real existing materials such as granular materials, suspensions, blood flow, etc. But further progress was hindered by a notorious lack of knowledge of new (and too numerous) material coefficients despite trials of estimating such coefficients, e.g., by Gauthier and Jashman [28] at the Colorado School of Mines by building artificially microstructured solids.

Remark 1.3. Very few French works were concluded in the 1960–1970s if we note the exceptional work of Duvaut [10, 11] on finite strains after a short stay in the
USA, the variational principle for micromorphic bodies by Maugin [53] from the USA, and those on micropolar fluids by C. Hartmann [35] under the influence of R. Berker (who had been the teacher of Eringen in Istanbul).

*Remark 1.4.* The intervening of a rotating microstructure allows for the introduction of wave modes of rotation of the “optical” type with an obvious application to many solid crystals (e.g., crystals equipped with a polar group such as NaNO$_2$, cf. Pouget and Maugin [74]).

### 1.2.2 The Loss of Validity of the Cauchy Postulate

Then the geometry of a cut intervenes at a higher order than one (variation of the unit normal, role of the curvature, edges, apices and thus capillarity effects). We may consider two different cases referred to as the *weakly nonlocal theory* and the *strongly nonlocal theory* (distinction introduced by the author at the Warsaw meeting of 1977, cf. Maugin [55]). Only the first type does correspond to the exact definition concerning a cut and the geometry of the cut surface. This is better referred to as *gradient theories of the $n$th order*; it is understood that the standard Cauchy theory is, in fact, a *theory of the first gradient* (by this we mean the first gradient of the displacement or the theory involving just the strain and no gradient of it in the constitutive equations).

#### 1.2.2.1 Gradient Theories

Now, to tell the truth, gradient theories abound in physics, starting practically with all continuum theories in the nineteenth century. Thus, Maxwell’s electromagnetism is a first-gradient theory (of the electromagnetic potentials); the Korteweg [39] theory of fluids is a theory of the first gradient of density (equivalent to a second-gradient theory of displacement in elasticity); Einstein’s [12] (also [13]) theory of gravitation (general relativity, 1916) is a second-gradient theory of the metric of curved space–time, and Le Roux [50] (also [51]) seems to be the first public exhibition of a second-gradient theory of (displacement) elasticity in small strains (using a variational formulation). There was a renewal of such theories in the 1960s with the works of Casal [4] on capillarity, and of Toupin [80], Mindlin and Tiersten [66], Mindlin and Eshel [65], and Grioli [33] in elasticity.

However, it is with a neat formulation basing on the *principle of virtual power* that some order was imposed in these formulations with an unambiguous deduction of the (sometimes tedious) boundary conditions and a clear introduction of the notion of *internal forces* of higher order, i.e., *hyperstresses* of various orders (see, Germain [29, 30], Maugin [56]). Phenomenological theories involving gradients of other physical fields than displacement or density, coupled to deformation, were envisaged consistently by the author in his Princeton PhD thesis (Maugin [54]) dealing with typical ferroïc electromagnetic materials. This is justified by a microscopic
approach, i.e., the continuum approximation of a crystal lattice with medium-range interactions; with distributed magnetic spins or permanent electric dipoles. This also applies to the pure mechanical case (see, for instance, the Boussinesq paradigm in Christov et al. [5]).

Very interesting features of these models are:

**F1.** Inevitable introduction of characteristic lengths;

**F2.** Appearance of the so-called capillarity effects (surface tension) due to the explicit intervening of curvature of surfaces;

**F3.** Correlative boundary layers effects;

**F4.** Dispersion of waves with a possible competition and balance between nonlinearity and dispersion, and the existence of solitonic structures (see Maugin [60], Maugin and Christov [63]);

**F5.** Intimate relationship with the Ginzburg–Landau theory of phase transitions and, for fluids, van der Waals’ theory.

Truly sophisticated examples of the application of these theories are found in

(i) The coupling of a gradient theory (of the carrier fluid) and consideration of a microstructure in the study of the inhomogeneous diffusion of microstructures in polymeric solutions (Drouot and Maugin [8]);

(ii) The elimination of singularities in the study of structural defects (dislocations, disclinations) in elasticity combining higher-order gradients and polar microstructure (cf. Lazar and Maugin [48]).

Most recent works consider the application of the notion of gradient theory in elastoplasticity for nonuniform plastic strain fields (works by Aifantis, Fleck, Hutchinson, and many others)—but see the thermodynamical formulation in Maugin [58]. In the present volume, this trend is exemplified by the first-hand synthesis contribution of E.C. Aifantis.

Insofar as general mathematical principles at the basis of the notion of gradient theory are concerned, we note the fundamental works of Noll and Virga [69] and Dell’Isola and Seppecher [7], the latter with a remarkable economy of thought.

### 1.2.2.2 Strongly Nonlocal Theory (Spatial Functionals)

Initial concepts in this framework were established by Kröner and Datta [42], Kunin [43, 44], Rogula [76], Eringen and Edelen [23]. As a matter of fact, the Cauchy construct does not apply anymore. In principle, only the case of infinite bodies should be considered as any cut would destroy the prevailing long-range ordering. Constitutive equations become integral expressions over space, perhaps with a more or less rapid attenuation with distance of the spatial kernel. This, of course, inherits from the action-at-a-distance dear to the Newtonians, while adapting the disguise of a continuous framework. This view is justified by the approximation of an infinite crystal lattice; the relevant kernels can be justified through this discrete approach. Of course, this raises the matter of solving integro-differential equations instead of
PDEs. What about boundary conditions that are in essence foreign to this representation of matter-matter interaction? There remains a possibility of the existence of a “weak-nonlocal” limit by the approximation by gradient models.

The historical moment in the recognition of the usefulness of strongly nonlocal theories was the EUROMECH colloquium on nonlocality organized by D. Rogula in Warsaw (cf. Maugin [55]). A now standard reference is Eringen’s book [22], also Kunin [44]. A recent much publicized application of the concept of nonlocality is that to damage by Pijaudier–Cabot and Bazant [72].

Note in conclusion to this point that any field theory can be generalized to a nonlocal one while saving the notions of linearity and anisotropy; but loosing the usual notion of flux. Also, it is of interest to pay attention to the works of Lazar and Maugin [48] for a comparison of field singularities in the neighborhood of structural defects in different “generalized” theories of elasticity (micropolar, gradient-like, strongly nonlocal or combining these). In this respect, see Lazar’s contribution in this volume.

1.2.3 Loss of the Euclidean Nature of the Material Manifold

Indeed, the basic relevant problem emerges as follows. How can we represent geometrically the fields of structural defects (such as dislocations associated with a loss of continuity of the elastic displacement, or disclinations associated with such a loss for rotations). A similar question is raised for vacancies and point defects. One possible answer stems from the consideration of a non-Euclidean material manifold, e.g., a manifold without curvature but with an affine connection, or an Einstein–Cartan space with both torsion and curvature, etc. With this, one enters a true “geometrization” of continuum mechanics of which conceptual difficulties compare favorably with those met in modern theories of gravitation. Pioneers in the field in the years 1950–1970 were K. Kondo [38] in Japan, E. Kröner [40] in Germany, Bilby in the UK, Stojanovic [78] in what was then Yugoslavia, W. Noll [68] and C.C. Wang [84] in the USA. Modern developments are due to, among others, M. Epstein and the author [14, 15], M. Elzanowski and S. Preston (see the theory of material inhomogeneities by Maugin [59]). Main properties of this type of approach are (i) the relationship to the multiple decomposition of finite strains (Bilby, Kroener, Lee) and (ii) the generalization of theories such as the theory of volumetric growth (Epstein and Maugin [16]) or the theory of phase transitions within the general theory of local structural rearrangements (local evolution of reference; see Maugin [62], examining Kröner’s inheritance and also the fact that true material inhomogeneities (dependence of material properties on the material point) are then seen as pseudo-plastic effects [61]. All local structural rearrangements and other physical effects (e.g., related to the diffusion of a dissipative process) are reciprocally seen as pseudo material inhomogeneities (Maugin [62]). An original geometric solution is presented in the book of Rakotomanana [75] which offers a representation of a material manifold that is everywhere dislocated. Introduction of the notion
of fractal sets opens new horizons (cf. Ostoja-Starzewski’s contribution in this volume). An antiquated forerunner work of all this may be guessed in Burton [2], but only with obvious good will by a perspicacious reader.

1.3 Conclusion

Since the seminal work of the Cosserats, three more or less successful paths have been taken towards the generalization of continuum mechanics. These were recalled above. They are also fully illustrated in the various contributions that follow. An essential difference between the bygone times of the pioneers and now is that artificial materials can be man-made that are indeed generalized continua. In addition, mathematical methods have been developed (homogenization techniques) that allow one to show that generalized continua are deduced as macroscopic continuum limits of some structured materials. This is illustrated by the book of Forest [27].

References


