Quantum Field Theory I:
Basics in Mathematics and Physics

Eberhard Zeidler

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A Bridge between Mathematicians and Physicists

Springer

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TO THE MEMORY OF JÜRGEN MOSER
(1928–1999)

Preface

Daß ich erkenne, was die Welt im Innersten zusammenhält.[1]
Faust

Concepts without intuition are empty, intuition without concepts is blind.
Immanuel Kant (1724–1804)

The greatest mathematicians like Archimedes, Newton, and Gauss have always been able to combine theory and applications into one.
Felix Klein (1849–1925)

The present comprehensive introduction to the mathematical and physical aspects of quantum field theory consists of the following six volumes:

Volume I: Basics in Mathematics and Physics
Volume II: Quantum Electrodynamics
Volume III: Gauge Theory
Volume IV: Quantum Mathematics
Volume V: The Physics of the Standard Model
Volume VI: Quantum Gravity and String Theory.

Since ancient times, both physicists and mathematicians have tried to understand the forces acting in nature. Nowadays we know that there exist four fundamental forces in nature:

- Newton's gravitational force,
- Maxwell's electromagnetic force,
- the strong force between elementary particles, and
- the weak force between elementary particles (e.g., the force responsible for the radioactive decay of atoms).

In the 20th century, physicists established two basic models, namely,

- the Standard Model in cosmology based on Einstein's theory of general relativity, and
- the Standard Model in elementary particle physics based on gauge theory.

[1] So that I may perceive whatever holds the world together in its inmost folds. The alchemist Georg Faust (1480–1540) is the protagonist of Goethe's drama *Faust* written in 1808.

One of the greatest challenges of the human intellect is the discovery of a unified theory for the four fundamental forces in nature based on first principles in physics and rigorous mathematics. For many years, I have been fascinated by this challenge. When talking about this challenge to colleagues, I have noticed that many of my colleagues in mathematics complain about the fact that it is difficult to understand the thinking of physicists and to follow the pragmatic, but frequently non-rigorous arguments used by physicists. On the other hand, my colleagues in physics complain about the abstract level of the modern mathematical literature and the lack of explicitly formulated connections to physics. This has motivated me to write the present book and the volumes to follow.

> *It is my intention to build a bridge between mathematicians and physicists.*

The main ideas of this treatise are described in the Prologue to this book. The six volumes address a broad audience of readers, including both undergraduate students and graduate students as well as experienced scientists who want to become familiar with the mathematical and physical aspects of the fascinating field of quantum field theory. In some sense, we will start from scratch:

- For students of mathematics, I would like to show that detailed knowledge of the physical background helps to motivate the mathematical subjects and to discover interesting interrelationships between quite different mathematical questions.
- For students of physics, I would like to introduce fairly advanced mathematics which is beyond the usual curriculum in physics.

For historical reasons, there exists a gap between the language of mathematicians and the language of physicists. I want to bridge this gap.[2] I will try to minimize the preliminaries such that undergraduate students after two years of studies should be able to understand the main body of the text. In writing this monograph, it was my goal to follow the advise given by the poet Johann Wolfgang von Goethe (1749–1832):

> *Textbooks should be attractive by showing the beauty of the subject.*

Ariadne's thread. In the author's opinion, the most important prelude to learning a new subject is strong motivation. Experience shows that highly motivated students are willing to take great effort to learn sophisticated subjects.

> *I would like to put the beginning of Ariadne's thread into the hands of the reader.*

[2] On November 7th 1940, there was a famous accident in the U.S.A. which was recorded on film. The Tacoma Narrows Bridge broke down because of unexpected nonlinear resonance effects. I hope that my bridge between mathematicians and physicists is not of Tacoma type.

Remember the following myth. On the Greek island of Crete in ancient times, there lived the monster Minotaur, half human and half bull, in a labyrinth. Every nine years, seven virgins and seven young men had to be sacrificed to the Minotaur. Ariadne, the daughter of King Minos of Crete and Pasiphaë fell in love with one of the seven young men – the Athenian Prince Theseus. To save his life, Ariadne gave Theseus a thread of yarn, and he fixed the beginning of the thread at the entrance of the labyrinth. After a hard fight, Theseus killed the Minotaur, and he escaped from the labyrinth by the help of Ariadne's thread.[3] For hard scientific work, it is nice to have a kind of Ariadne's thread at hand. The six volumes cover a fairly broad spectrum of mathematics and physics. In particular, in the present first volume the reader gets information about

- the physics of the Standard Model of particle physics and
- the magic formulas in quantum field theory,

and we touch the following mathematical subjects:

- finite-dimensional Hilbert spaces and a rigorous approach to the basic ideas of quantum field theory,
- elements of functional differentiation and functional integration,
- elements of probability theory,
- calculus of variations and the principle of critical action,
- harmonic analysis and the Fourier transform, the Laplace transform, and the Mellin transform,
- Green's functions, partial differential equations, and distributions (generalized functions),
- Green's functions, the Fourier method, and functional integrals (path integrals),
- the Lebesgue integral, general measure integrals, and Hilbert spaces,
- elements of functional analysis and perturbation theory,
- the Dirichlet principle as a paradigm for the modern Hilbert space approach to partial differential equations,
- spectral theory and rigorous Dirac calculus,
- analyticity,
- calculus for Grassmann variables,
- many-particle systems and number theory,
- Lie groups and Lie algebras,
- basic ideas of differential and algebraic topology (homology, cohomology, and homotopy; topological quantum numbers and quantum states).

We want to show the reader that many mathematical methods used in quantum field theory can be traced back to classical mathematical problems. In

[3] Unfortunately, Theseus was not grateful to Ariadne. He deserted her on the Island of Naxos, and she became the bride of Dionysus. Richard Strauss composed the opera *Ariadne on Naxos* in 1912.

particular, we will thoroughly study the relation of the procedure of renormalization in physics to the following classical mathematical topics:

- singular perturbations, resonances, and bifurcation in oscillating systems (renormalization in a nutshell on page 628),
- the regularization of divergent infinite series, divergent infinite products, and divergent integrals,
- divergent integrals and distributions (Hadamard's finite part of divergent integrals),
- the passage from a finite number of degrees of freedom to an infinite number of degrees of freedom and the method of counterterms in complex analysis (the Weierstrass theorem and the Mittag–Leffler theorem),
- analytic continuation and the zeta function in number theory,
- Poincaré's asymptotic series and the Ritt theorem in complex analysis,
- the renormalization group and Lie's theory of dynamical systems (one-parameter Lie groups),
- rigorous theory of finite-dimensional functional integrals (path integrals).

The following volumes will provide the reader with important additional material. A summary can be found in the Prologue on pages 11 through 15.

Additional material on the Internet. The interested reader may find additional material on my homepage:

Internet: `www.mis.mpg.de/ezeidler/`

This concerns a carefully structured panorama of important literature in mathematics, physics, history of the sciences and philosophy, along with a comprehensive bibliography. One may also find a comprehensive list of mathematicians, physicists, and philosophers (from ancient until present time) mentioned in the six volumes. My homepage also allows links to the leading centers in elementary particle physics: CERN (Geneva, Switzerland), DESY (Hamburg, Germany), FERMILAB (Batavia, Illinois, U.S.A.), KEK (Tsukuba, Japan), and SLAC (Stanford University, California, U.S.A.). One may also find links to the following Max Planck Institutes in Germany: Astronomy (Heidelberg), Astrophysics (Garching), Complex Systems in Physics (Dresden), Albert Einstein Institute for Gravitational Physics (Golm), Mathematics (Bonn), Nuclear Physics (Heidelberg), Werner Heisenberg Institute for Physics (Munich), and Plasmaphysics (Garching).

Apology. The author apologizes for his imperfect English style. In the preface to his monograph *The Classical Groups*, Princeton University Press, 1946, Hermann Weyl writes the following:

> The gods have imposed upon my writing the yoke of a foreign tongue that was not sung at my cradle.
>
> "Was das heissen will, weiss jeder,
> Der im Traum pferdlos geritten ist,"[4]

[4] Everyone who has dreamt of riding free, without the need of a horse, will know what I mean.

I am tempted to say with the Swiss poet Gottfried Keller (1819–1890). Nobody is more aware than myself of the attendant loss in vigor, ease and lucidity of expression.

Acknowledgements. First of all I would like to thank the Max Planck Society in Germany for founding the Max Planck Institute for Mathematics in the Sciences (MIS) in Leipzig in 1996 and for creating a superb scientific environment here. This treatise would have been impossible without the extensive contacts of the institute to mathematicians and physicists all over the world and without the excellent library of the institute. My special thanks go to the intellectual fathers of the institute, Friedrich Hirzebruch (chairman of the Founder's Committee) and Stefan Hildebrandt in Bonn, Karl-Heinz Hoffmann and Julius Wess in Munich, and the late Jürgen Moser in Zurich who was an external scientific member of the institute. I would like to dedicate this volume to the memory of Jürgen Moser who was a great mathematician and an amiable man. Moreover, I would like to thank Don Zagier (Max Planck Institute for Mathematics in Bonn and Collège de France in Paris), one of the greatest experts in number theory, for the kindness of writing a beautiful section on useful techniques of number theory in physics.

I am very grateful to numerous colleagues in mathematics and physics from all over the world for illuminating discussions. It is not possible to mention the names of all of them, since the list is very long. In particular, I would like to thank the professors from the Institute of Theoretical Physics at Leipzig University, Bodo Geyer, Wolfhard Janke, Gerd Rudolph, Manfred Salmhofer, Klaus Sibold, Armin Uhlmann, and Rainer Verch for nice cooperation. For many stimulating discussions on a broad spectrum of mathematical problems, I would like to thank the co-directors of the MIS, Wolfgang Hackbusch, Jürgen Jost, and Stefan Müller.

For getting information about new research topics, I am very grateful to my former and present collaborators: Günther Berger, Ludmilla Bordag, Friedemann Brandt, Friedemann Brock, Chand Devchand, Bertfried Fauser, Felix Finster, Christian Fleischhack, Jörg Frauendiener, Hans-Peter Gittel, Matthias Günther, Bruce Hunt, Konrad Kaltenbach, Satyanad Kichenassamy, Klaus Kirsten, Christian Klein, Andreas Knauf, Alexander Lange, Rainer Matthes, Johannes Maul†, Erich Miersemann, Mario Paschke, Hoang Xuan Phu, Karin Quasthoff, Olaf Richter†, Alexander Schmidt, Rainer Schumann, Friedemann Schuricht, Peter Senf†, Martin Speight, Jürgen Tolksdorf, Dimitri Vassilevich, Hartmut Wachter, and Raimar Wulkenhaar.

For experienced assistance in preparing this book, I would like to thank Kerstin Fölting (graphics, tables, and a meticulous proof-reading of my entire latex-file together with Rainer Munck), Micaela Krieger–Hauwede (graphics, tables, and layout), and Jeffrey Ovall (checking and improving my English style). For supporting me kindly in various aspects and for helping me to save time, I am also very grateful to my secretary, Regine Lübke, and to the staff of the institute including the librarians directed by Ingo Brüggemann,

the computer group directed by Rainer Kleinrensing, and the administration directed by Dietmar Rudzik. Finally, I would like to thank the staff of the Springer publishing house in Heidelberg, Ruth Allewelt, Angela Schulze-Thomin, Joachim Heinze, and Martin Peters, for a harmonious collaboration.

I hope that the reader of this book enjoys getting a feel for the unity of mathematics and physics by discovering interrelations between apparently completely different subjects.

Leipzig, Fall 2005 *Eberhard Zeidler*

Preface to the Corrected Second Printing

I am very pleased that Springer at Heidelberg is publishing a corrected reprint of Volume I. In this edition, I made minor revisions and updated the references. In particular, the panorama of literature in Chapter 17 was changed substantially. I would like to thank the readers for their words of encouragement and for their useful suggestions. Volume II appeared in 2008, and Volume III is in preparation.

Leipzig, Spring 2009 *Eberhard Zeidler*

Contents

Part III. Heuristic Magic Formulas of Quantum Field Theory

Prologue

We begin with some quotations which exemplify the philosophical underpinnings of this work.

Theoria cum praxi.
Gottfried Wilhelm Leibniz (1646–1716)

> It is very difficult to write mathematics books today. If one does not take pains with the fine points of theorems, explanations, proofs and corollaries, then it won't be a mathematics book; but if one does these things, then the reading of it will be extremely boring.
>
> Johannes Kepler (1571–1630)
> *Astronomia Nova*

> The interaction between physics and mathematics has always played an important role. The physicist who does not have the latest mathematical knowledge available to him is at a distinct disadvantage. The mathematician who shies away from physical applications will most likely miss important insights and motivations.
>
> Marvin Schechter
> *Operator Methods in Quantum Mechanics*[5]

> In 1967 Lenard and I found a proof of the stability of matter. Our proof was so complicated and so unilluminating that it stimulated Lieb and Thirring to find the first decent proof. Why was our proof so bad and why was theirs so good? The reason is simple. Lenard and I began with mathematical tricks and hacked our way through a forest of inequalities without any physical understanding. Lieb and Thirring began with physical understanding and went on to find the appropriate mathematical language to make their understanding rigorous. Our proof was a dead end. Theirs was a gateway to the new world of ideas collected in this book.
>
> Freeman Dyson
> From the Preface to *Elliott Lieb's Selecta*[6]

The state of the art in quantum field theory. One of the intellectual fathers of quantum electrodynamics is Freeman Dyson (born in 1923) who

[5] North-Holland, Amsterdam, 1982.
[6] Stability of Matter: From Atoms to Stars, Springer, New York, 2002.

works at the Institute for Advanced Study in Princeton.[7] He characterizes the state of the art in quantum field theory in the following way:

> All through its history, quantum field theory has had two faces, one looking outward, the other looking inward. The outward face looks at nature and gives us numbers that we can calculate and compare with experiments. The inward face looks at mathematical concepts and searches for a consistent foundation on which to build the theory. The outward face shows us brilliantly successful theory, bringing order to the chaos of particle interactions, predicting experimental results with astonishing precision. The inward face shows us a deep mystery. After seventy years of searching, we have found no consistent mathematical basis for the theory. When we try to impose the rigorous standards of pure mathematics, the theory becomes undefined or inconsistent. From the point of view of a pure mathematician, the theory does not exist. This is the *great unsolved paradox of quantum field theory.*
> To resolve the paradox, during the last twenty years, quantum field theorists have become string-theorists. String theory is a new version of quantum field theory, exploring the mathematical foundations more deeply and entering a new world of multidimensional geometry. String theory also brings gravitation into the picture, and thereby unifies quantum field theory with general relativity. String theory has already led to important advances in pure mathematics. It has not led to any physical predictions that can be tested by experiment. We do not know whether string theory is a true description of nature. All we know is that it is a rich treasure of new mathematics, with an enticing promise of new physics. During the coming century, string theory will be intensively developed, and, if we are lucky, tested by experiment.[8]

Five golden rules. When writing the latex file of this book on my computer, I had in mind the following five quotations. Let me start with the mathematician Hermann Weyl (1885–1955) who became a successor of Hilbert in Göttingen in 1930 and who left Germany in 1933 when the Nazi regime came to power. Together with Albert Einstein (1879–1955) and John von Neumann (1903–1957), Weyl became a member of the newly founded Institute for Advanced Study in Princeton, New Jersey, U.S.A. in 1933. Hermann Weyl wrote in 1938:[9]

> The stringent precision attainable for mathematical thought has led many authors to a mode of writing which must give the reader an impression of being shut up in a brightly illuminated cell where every detail sticks out with the same dazzling clarity, but without relief. I prefer the open landscape under a clear sky with its depth of perspective, where the wealth of sharply defined nearby details gradually fades away towards the horizon.

[7] F. Dyson, Selected Papers of Freeman Dyson with Commentaries, Amer. Math. Soc., Providence, Rhode Island, 1996. We recommend reading this fascinating volume.

[8] In: Quantum Field Theory, A 20th Century Profile. Edited by A. Mitra, Indian National Science Academy and Hindustan Book Agency, 2000 (reprinted with permission).

[9] H. Weyl, The Classical Groups, Princeton University Press, 1938 (reprinted with permission).

For his fundamental contributions to electroweak interaction inside the Standard Model in particle physics, the physicist Steven Weinberg (born 1933) was awarded the Nobel prize in physics in 1979 together with Sheldon Glashow (born 1932) and Abdus Salam (1926–1996). On the occasion of a conference on the interrelations between mathematics and physics in 1986, Weinberg pointed out the following:[10]

> I am not able to learn any mathematics unless I can see some problem I am going to solve with mathematics, and I don't understand how anyone can teach mathematics without having a battery of problems that the student is going to be inspired to want to solve and then see that he or she can use the tools for solving them.

For his theoretical investigations on parity violation under weak interaction, the physicist Cheng Ning Yang (born 1922) was awarded the Nobel prize in physics in 1957 together with Tsung Dao Lee (born 1926). In an interview, Yang remarked:[11]

> In 1983 I gave a talk on physics in Seoul, South Korea. I joked "There exist only two kinds of modern mathematics books: one which you cannot read beyond the first page and one which you cannot read beyond the first sentence. The *Mathematical Intelligencer* later reprinted this joke of mine. But I suspect many mathematicians themselves agree with me."

The interrelations between mathematics and modern physics have been promoted by Sir Michael Atiyah (born 1929) on a very deep level. In 1966, the young Atiyah was awarded the Fields medal. In an interview, Atiyah emphasized the following:[12]

> The more I have learned about physics, the more convinced I am that physics provides, in a sense, the deepest applications of mathematics. The mathematical problems that have been solved, or techniques that have arisen out of physics in the past, have been the lifeblood of mathematics... The really deep questions are still in the physical sciences. For the health of mathematics at its research level, I think it is very important to maintain that link as much as possible.

The development of modern quantum field theory has been strongly influenced by the pioneering ideas of the physicist Richard Feynman (1918–1988). In 1965, for his contributions to the foundation of quantum electrodynamics, Feynman was awarded the Nobel prize in physics together with Julian Schwinger (1918–1994) and Sin-Itiro Tomonaga (1906–1979). In the beginning of the 1960s, Feynman held his famous *Feynman lectures* at the California Institute of Technology in Pasadena. In the preface to the printed version of the lectures, Feynman told his students the following:

> Finally, may I add that the main purpose of my teaching has not been to prepare you for some examination – it was not even to prepare you to

[10] Notices Amer. Math. Soc. **33** (1986), 716–733 (reprinted with permission).
[11] Mathematical Intelligencer **15** (1993), 13–21 (reprinted with permission).
[12] Mathematical Intelligencer **6** (1984), 9–19 (reprinted with permission).

> serve industry or military. I wanted most to give you some appreciation of the wonderful world and the physicist's way of looking at it, which, I believe, is a major part of the true culture of modern times.[13]

The fascination of quantum field theory. As a typical example, let us consider the anomalous magnetic moment of the electron. This is given by the following formula

$$\boxed{\mathbf{M}_e = -\frac{e}{2m_e}\, g_e \mathbf{S}}$$

with the so-called gyromagnetic factor

$$g_e = 2(1 + a)$$

of the electron. Here, m_e is the mass of the electron, $-e$ is the negative electric charge of the electron. The spin vector $\mathbf{S}$ has the length $\hbar/2$, where h denotes Planck's quantum of action, and $\hbar := h/2\pi$. High-precision experiments yield the value

$$a_{\text{exp}} = 0.001\ 159\ 652\ \mathit{188\ 4} \pm 0.000\ 000\ 000\ \mathit{004\ 3}.$$

Quantum electrodynamics is able to predict this result with high accuracy. The theory yields the following value

$$\begin{aligned} a = \frac{\alpha}{2\pi} - 0.328\ 478\ 965 \left(\frac{\alpha}{\pi}\right)^2 &+ (1.175\ 62 \pm 0.000\ 56)\left(\frac{\alpha}{\pi}\right)^3 \\ &- (1.472 \pm 0.152)\left(\frac{\alpha}{\pi}\right)^4 \end{aligned} \tag{0.1}$$

with the electromagnetic fine structure constant

$$\alpha = \frac{1}{137.035\ 989\ \mathit{500} \pm 0.000\ 000\ \mathit{061}}.$$

Explicitly,

$$a = 0.001\ 159\ 652\ \mathit{164} \pm 0.000\ 000\ 000\ \mathit{108}.$$

The error is due to the uncertainty of the electromagnetic fine structure constant α. Observe that 9 digits coincide between the experimental value a_{exp} and the theoretical value a.

The theoretical result (0.1) represents a highlight in modern theoretical physics. The single terms with respect to powers of the fine structure constant α have been obtained by using the method of perturbation theory. In order to represent graphically the single terms appearing in perturbation theory, Richard Feynman (1918–1988) invented the language of Feynman diagrams in about 1945.[14] For example, Fig. 0.1 shows some simple Feynman diagrams

[13] R. Feynman, R. Leighton, and M. Sands, The Feynman Lectures in Physics, Addison-Wesley, Reading, Massachusetts, 1963.

[14] For the history of this approach, see the quotation on page 27.

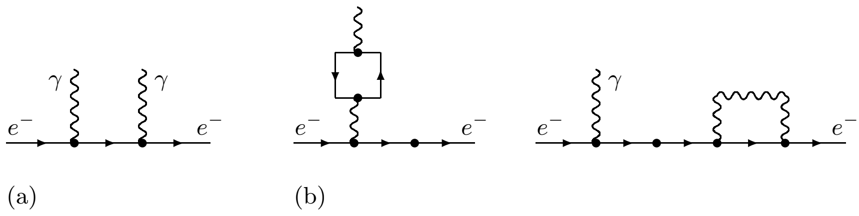

Fig. 0.1. Feynman diagrams

for the Compton scattering between electrons and photons. In higher order of perturbation theory, the Feynman diagrams become more and more complex. In particular, in order to get the α^3-term of (0.1), one has to use 72 Feynman diagrams. The computation of the α^3-term has taken 20 years. The α^4-term from (0.1) is based on 891 Feynman diagrams. The computation has been done mainly by numerical approximation methods. This needed years of supercomputer time.[15] The mathematical situation becomes horrible because of the following fact.

Many of the Feynman diagrams correspond to divergent higher-dimensional integrals called algebraic Feynman integrals.

Physicists invented the ingenious method of renormalization in order to give the apparently meaningless integrals a precise interpretation. Renormalization plays a fundamental role in quantum field theory. Physicists do not expect that the perturbation series (0.1) is part of a convergent power series expansion with respect to the variable α at the origin. Suppose that there would exist such a convergent power series expansion

$$a = \sum_{n=1}^{\infty} a_n \alpha^n, \qquad |\alpha| \le \alpha_0$$

near the origin $\alpha = 0$. This series would then converge for small negative values of α. However, such a negative coupling constant would correspond to a repelling force which destroys the system. This argument is due to Dyson.[16]

Therefore, we do not expect that the series (0.1) *is convergent.*

In Sect. 15.5.2, we will show that each formal power series expansion is indeed the asymptotic expansion of some analytic function in an angular domain, by the famous 1916 Ritt theorem in mathematics.

[15] See M. Veltman, Facts and Mysteries in Elementary Particle Physics, World Scientific, Singapore, 2003; this is a beautiful history of modern elementary particle physics.

[16] F. Dyson, Divergence of perturbation theory in quantum electrodynamics, Phys. Rev. **85** (1952), 631–632.

From the mathematical point of view, the best approach to renormalization was created by Epstein and Glaser in 1973. The Epstein–Glaser theory avoids the use of divergent integrals and their regularization, but relies on the power of the modern theory of distributions (generalized functions).

Physicists have also computed the magnetic moment of the myon. As for the electron, the coincidence between theory and experiment is of fantastic accuracy. Here, the theory takes all of the contributions coming from electromagnetic, weak, strong, and gravitative interaction into account.[17]

> *It is a challenge for the mathematics of the future to completely understand formula* (0.1).

Let us now briefly discuss the content of Volumes I through VI of this monograph.

Volume I. The first volume entitled *Basics in Mathematics and Physics* is structured in the following way.

Part I: Introduction
- Chapter 1: Historical Introduction
- Chapter 2: Phenomenology of the Standard Model in Particle Physics
- Chapter 3: The Challenge of Different Scales in Nature.

Part II: Basic Techniques in Mathematics
- Chapter 4: Analyticity
- Chapter 5: A Glance at Topology
- Chapter 6: Many-Particle Systems
- Chapter 7: Rigorous Finite-Dimensional Magic Formulas of Quantum Field Theory
- Chapter 8: Rigorous Finite-Dimensional Perturbation Theory
- Chapter 9: Calculus for Grassmann Variables
- Chapter 10: Infinite-Dimensional Hilbert Spaces
- Chapter 11: Distributions and Green's Functions
- Chapter 12: Distributions and Quantum Physics.

Part III: Heuristic Magic Formulas of Quantum Field Theory
- Chapter 13: Basic Strategies in Quantum Field Theory
- Chapter 14: The Response Approach
- Chapter 15: The Operator Approach
- Chapter 16: Peculiarities of Gauge Theories
- Chapter 17: A Panorama of the Literature.

Describing the content of Volume I by a parable, we will first enter a mountain railway in order to reach easily and quickly the top of the desired mountain and to admire the beautiful mountain ranges. Later on we will try to climb to the top along the rocks.

[17] See M. Böhm, A. Denner, and H. Joos, Gauge Theories of the Strong and Electroweak Interaction, Teubner, Stuttgart, 2001, p. 80.

In particular, the heuristic magic formulas from Part III should help the reader to understand quickly the language of physicists in quantum field theory. These magic formulas are non-rigorous from the mathematical point of view, but they are extremely useful for computing physical effects.

Modern elementary particle physics is based on the Standard Model in particle physics introduced in the late 1960s and the early 1970s. Before studying thoroughly the Standard Model in the next volumes, we will discuss the phenomenology of this model in the present volume. It is the goal of quantum field theory to compute

- the cross sections of scattering processes in particle accelerators which characterize the behavior of the scattered particles,
- the masses of stable elementary particles (e.g., the proton mass as a bound state of three quarks), and
- the lifetime of unstable elementary particles in particle accelerators.

To this end, physicists use the methods of perturbation theory. Fortunately enough, the computations can be based on only a few basic formulas which we call magic formulas. The magic formulas of quantum theory are extremely useful for describing the experimental data observed in particle accelerators, but they are only valid on a quite formal level.

> *This difficulty is typical for present quantum field theory.*

To help the reader in understanding the formal approach used in physics, we consider the finite-dimensional situation in the key Chapter 7.

> *In the finite-dimensional case, we will rigorously prove all of the magic formulas used by physicists in quantum field theory.*

Furthermore, we relate physics to the following fields of mathematics:

- causality and the analyticity of complex-valued functions,
- many-particle systems, the Casimir effect in quantum field theory, and number theory,
- propagation of physical effects, distributions (generalized functions), and the Green's function,
- rigorous justification of the elegant Dirac calculus,
- duality in physics (time and energy, time and frequency, position and momentum) and harmonic analysis (Fourier series, Fourier transformation, Laplace transformation, Mellin transformation, von Neumann's general operator calculus for self-adjoint operators, Gelfand triplets and generalized eigenfunctions),
- the relation between renormalization, resonances, and bifurcation,
- dynamical systems, Lie groups, and the renormalization group,
- fundamental limits in physics,
- topology in physics (Chern numbers and topological quantum numbers),
- probability, Brownian motion, and the Wiener integral,

- the Feynman path integral,
- Hadamard's integrals and algebraic Feynman integrals.

In fact, this covers a broad range of physical and mathematical subjects.

Volume II. The second volume entitled *Quantum Electrodynamics* consists of the following parts.[18]

Part I: Introduction
- Chapter 1: Mathematical Principles of Modern Natural Philosophy
- Chapter 2: The Basic Strategy of Extracting Finite Information from Infinities – Ariadne's Thread in Renormalization Theory
- Chapter 3: The Power of Combinatorics and Hopf Algebras
- Chapter 4: The Strategy of Equivalence Classes in Mathematics.

Part II: Basic Ideas in Classical Mechanics
- Chapter 5: Geometrical Optics
- Chapter 6: The Principle of Critical Action and the Harmonic Oscillator – Ariadne's Thread in Classical Mechanics.

Part III: Basic Ideas in Quantum Mechanics
- Chapter 7: Quantization of the Harmonic Oscillator – Ariadne's Thread in Quantization
- Chapter 8: Quantum Particles on the Real Line – Ariadne's Thread in Scattering Theory
- Chapter 9: A Glance at General Scattering Theory.

Part IV: Quantum Electrodynamics (QED)
- Chapter 10: Creation and Annihilation Operators
- Chapter 11: The Basic Equations in Quantum Electrodynamics
- Chapter 12: The Free Quantum Fields of Electrons, Positrons, and Photons
- Chapter 13: The Interacting Quantum Field, and the Magic Dyson Series for the S-Matrix
- Chapter 14: The Beauty of Feynman Diagrams in QED
- Chapter 15: Applications to Physical Effects.

Part V: Renormalization
- Chapter 16: The Continuum Limit
- Chapter 17: Radiative Corrections of Lowest Order
- Chapter 18: A Glance at Renormalization to all Orders of Perturbation Theory
- Chapter 19: Perspectives.

The final goal of quantum field theory is the foundation of a rigorous mathematical theory which contains the Standard Model as a special low-energy approximation. At present we are far away from reaching this final goal. From

[18] This volume appeared in 2008.