Ernst Zermelo

Heinz-Dieter Ebbinghaus
In Cooperation with Volker Peckhaus

Ernst Zermelo

An Approach to His Life and Work

With 42 Illustrations

Springer

Heinz-Dieter Ebbinghaus
Mathematisches Institut
Abteilung für Mathematische Logik
Universität Freiburg
Eckerstraße 1
79104 Freiburg, Germany
E-mail: hde@math.uni-freiburg.de

Volker Peckhaus
Kulturwissenschaftliche Fakultät
Fach Philosophie
Universität Paderborn
Warburger Straße 100
33098 Paderborn, Germany
E-mail: volker.peckhaus@upb.de

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To the memory of
Gertrud Zermelo (1902–2003)

Preface

Ernst Zermelo is best-known for the explicit statement of the axiom of choice and his axiomatization of set theory. The axiom of choice led to a methodological enrichment of mathematics, the axiomatization was the starting point of post-Cantorian set theory. His achievements, however, did not unfold in an undisputed way. They became the object of serious criticism sparked, in particular, by the inconstructive character of the axiom of choice, making it one of the most debated principles in the history of mathematics. Zermelo defended his point of view with clear insights and discerning arguments, but also with polemical formulations and sometimes hurtful sharpness. The controversial attitude shining through here has become a dominating facet of his image. Further controversies such as those with Ludwig Boltzmann about the foundations of the kinetic theory of heat and with Kurt Gödel and Thoralf Skolem about the finitary character of mathematical reasoning support this view.

Even though these features represent essential constituents of Zermelo's research and character, they fall short of providing a conclusive description. Neither is Zermelo's major scientific work limited to set theory, nor his personality to controversial traits. His scientific interests included applied mathematics and purely technical questions. His dissertation, for example, promoted the Weierstraßian direction in the calculus of variations, he wrote the first paper in what is now called the theory of games, and created the pivotal method in the theory of rating systems. The complexity of his personality shows in his striving for truth and objectivity, and in the determination with which he stood up for his convictions. Well-educated in and open-minded about philosophy, the classics, and literature, he had the ability of encountering others in a stimulating way.

Due to serious illness, which hindered and finally ended his academic career, and due to growing isolation from the dominating directions in the foundations of mathematics, he became caught in a feeling of being denied due scientific recognition, and controversy seemed to gain the upper hand. Those close to him, however, enjoyed his other sides.

The present biography attempts to shed light on all facets of Zermelo's life and achievements. In doing so, quotations from various sources play a major role. Personal and scientific aspects are kept separate as far as coherence allows, in order to enable the reader to follow the one or the other of these threads. The discussion of Zermelo's scientific work does not require detailed knowledge of the field in question. Rather than aiming at an in-depth technical analysis of his papers, the presentation is intended to explore motivations, aims, acceptance, and influence. Selected proofs and information gleaned from drafts, unpublished notes, and letters add to the analysis.

The main text is followed by a *curriculum vitae* which summarizes the main events of Zermelo's life, now in a more schematical manner. It thus provides some kind of chronological index.

All facts presented are documented by appropriate sources. Whenever possible, English versions of German texts follow a published translation. In some special cases such as axioms, the original German version is given in the main text, as well. Original German versions which have not been published previously and whose wording may be of some importance or interest, are compiled in the appendix. In particular, the appendix contains all unpublished quotations from Zermelo himself, supplemented by samples of his literary activity.

There is no claim that the biography offers a complete picture. Rather, the description of Zermelo's life and work should be considered as an approach to a rich and multifaceted personality, which may invite the reader to take an even closer look.

When Zermelo's late wife Gertrud learnt about this project, she gave all possible support. The information and the documents which she provided with warm-hearted interest contributed in essential ways to the picture of Zermelo presented here. With deep gratitude, the biography is dedicated to her memory.

Freiburg, September 2006 Heinz-Dieter Ebbinghaus

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Editorial Remarks

Archives and *Nachlässe* which are frequently quoted are given in an abbreviated form as follows:

- ASD: Archiv der sozialen Demokratie, Bad Godesberg, *Nachlass* Nelson;
- CAH: Center for American History, The University of Texas at Austin, Dehn (Max) Papers, 3.2/86-4/Box 1;
- DMA: Deutsches Museum, Archiv;
- GSA: Geheimes Staatsarchiv Preußischer Kulturbesitz Berlin;
- SAZ: Staatsarchiv des Kantons Zürich;
- SUB: Niedersächsische Staats- und Universitätsbibliothek Göttingen;
- UAF: Universitätsarchiv Freiburg;
- UAG: Universitätsarchiv Göttingen;
- UAW: University Archives Wrocław;
- UBA: University of Buffalo Archives, Marvin Farber, Papers on Philosophy and Phenomenology, 1920s–1980s, 22/5F/768.

The abbreviation "MLF" refers to the Abteilung für Mathematische Logik, Universität Freiburg. The corresponding documents are currently being incorporated into the Zermelo *Nachlass* held in the Universitätsarchiv Freiburg under signature C 129.

Photographs reprinted without reference to a source are contained in the Zermelo photo collection held in the Abteilung für Mathematische Logik at Freiburg University. This photo collection will also be integrated into the Zermelo *Nachlass*.

References of type OV n.xy, which follow an English translation of a German text, refer to the original German version given under the same reference in the Appendix (Chapter 7). The numbers n = 1, 2, 3, 4 correspond to Chapters 1 (Berlin), 2 (Göttingen), 3 (Zurich), 4 (Freiburg), respectively.

Contents

1

Berlin *1871–1897*

1.1 Family and Youth

Ernst Friedrich Ferdinand Zermelo was born in Berlin on 27 July 1871.[1] It was the year when Georg Cantor wrote his papers on trigonometric series.[2] As Zermelo expressed it about 60 years later, these papers led with inherent necessity to the conception of transfinite ordinal numbers; they hence could be considered the birthplace of Cantorian set theory, i. e. the theory whose transformation into an axiomatic form would be an essential part of Zermelo's scientific achievements.

There exist several hypotheses about the origin of the name "Zermelo." They all seem to originate with Zermelo himself. According to the most well-known ([Rei70], 98), Zermelo is said to have replied to newcomers in Göttingen when asked about his strange name, that it resulted from the word "Walzermelodie" ("waltz melody") by deleting the first and the last syllable. In conversations with his wife Gertrud, Zermelo suggested the more seriously meant possibility that it was derived from "Termeulen," a form of the name "Zurmühlen" ("at the mill") current in the lower Rhine and Friesian areas. The name most probably originates from the variant "Tormählen"[3] which evolved in northeastern Germany, the region the Zermelo family comes from, during the 16th century; the Low German "T" might have been transformed into the High German "Z" in the course of one of the frequently observed hypercorrections.[4] With the death of Zermelo's widow Gertrud in December 2003 the name "Zermelo" disappeared from the German telephone listings. She seems to have been its last bearer.

Zermelo had five sisters, the one-year older Anna and the younger ones Elisabeth, Margarete, Lena, and Marie. His parents had married in Delitzsch,

[1] Details about Zermelo's family are taken from documents held in MLF.
[2] [Can71a], [Can71b], [Can72].
[3] The "ä" is pronounced like the "ea" in "bear."
[4] [Kun98], 159, and oral communication with Konrad Kunze.

Saxony, on 18 May 1869. His mother, Maria Auguste née Zieger, was born there on 11 February 1847 as the only child of the surgeon Dr. Ottomar Hugo Zieger and his wife Auguste née Meißner. Both her parents died early, her mother just one week after giving birth to her, her father one year later from tuberculosis. She then grew up in the wider family, which included the natural historian Christian Gottfried Ehrenberg (1795–1876) who was to become known for his works on microorganisms and who accompanied Alexander von Humboldt on his Russian expedition of 1829.[5] Like her parents, Maria Auguste suffered from poor health. Exhausted from the strain of multiple pregnancies, she died on 3 June 1878, soon after the birth of Marie. After her death a maid, Olga Pahlke, took care of the household.

Lena and Marie died at a young age, in 1906 and 1908, respectively. Not only they, but also the other children inherited their mother's delicate health; they all fell ill with tuberculosis. Neither of the three surviving sisters married. The letters which Zermelo wrote to them attest to a close relationship between the siblings, and care and concern on Zermelo's side.

Zermelo's father, Theodor Zermelo, was born on 9 June 1834 in the East Prussian town of Tilsit (today Sovjetsk, Russia) on the river Memel (Neman). Theodor's father Ferdinand Zermelo, a bookbinder who had graduated from the college of arts and crafts in Königsberg (today Kaliningrad, Russia), took over a shop for books, arts, and stationery in Tilsit in 1839. Documentation about him and his wife Bertha née Haberland is sparse: a carefully written diary about a trip to Paris and London, which Ferdinand undertook in the summer of 1851, and newspaper reports and obituary notices on the occasion of his death showing that he was an integral part and a supporter of cultural life in Tilsit and engaged in the town's social institutions. He served as a councillor and was a co-founder of the Tilsit Schiller Society and the Tilsit Art Society.

After finishing his secondary school education in Tilsit, Theodor Zermelo studied mainly history and geography, first in Königsberg and then in Berlin. Having attained his doctorate in 1856 with a dissertation in history, he completed his studies with the exams *pro facultate docendi,* i. e., his state exams for teaching at secondary schools, which allowed him to teach history and geography as main subjects and Greek and Latin as subsidiary ones; a later supplementary examination also permitted him to teach French. He refrained from a further examination in the natural sciences, but completed an additional diploma in mathematics. In its evaluation of 8 March 1857, the Königlich-Wissenschaftliche Prüfungskommission (Royal Academic Examination Board) found him to have "an adequate knowledge of the elements of geometry and algebra for teaching mathematics in the lower grades." It further observes: "As concerns his philosophical knowledge, he has a solid grasp of the basic notions of logic and general grammar (allgemeine Grammatik) and

[5] In a *curriculum vitae* written after 1933 (UAF, C 129/273) Zermelo counts not only Ehrenberg, but also the physicist Arnold Sommerfeld among his relatives.

shows an assured competence in their application." Theodor Zermelo became a teacher at the Friedrichswerder Gewerbeschule in Berlin, a secondary school that was vocationally oriented and did not have Latin and Greek as subjects like the traditional *Gymnasium.* In 1886 he was appointed *Gymnasialprofessor.*


Zermelo's parents Maria Auguste and Theodor Zermelo

Altogether, Ernst Zermelo grew up in a family with an academic background, but not necessarily with an orientation towards mathematics and the sciences. His sisters Anna and Elisabeth became painters, his sister Margarete a teacher of history and modern languages. Too little information is available about Zermelo's childhood and youth to come to reliable conclusions concerning his education. As his mother died when he was seven years old, her influence was limited to his early childhood. Zermelo described her to his wife Gertrud as a delicate and sensitive woman. She confided many of her thoughts to a diary.

The influence of his father is more obvious. As a teacher, Ferdinand Zermelo appears to have been popular and highly regarded. In a letter which former pupils wrote to congratulate him on his nomination as a professor it says:

> We all are delighted about this event, since each of us has had the good fortune to have been taught by you just a few years ago. We sincerely regret that we now have to do without. We all owe you a great part of our education, and therefore we wish you with all our heart a long and happy life.

Zermelo also respected his father and held him in high esteem. This is illustrated by a letter in which he thanks him for the presents he received on his 17th birthday:[6]

[6]Letter of 28 July 1888; MLF. A letter which Zermelo wrote to his father in April (copy in MLF) shows that Theodor Zermelo was not living together with his

Left: Zermelo as a baby with his mother
Right: Zermelo with his parents and his sisters Anna (right) and Elisabeth

> Dear father,
> Your affectionate congratulations on my birthday yesterday and your presents, which are as generous as ever, caused me great pleasure. Above all I would like to express my heartfelt thanks to you. Your very kind offer of a trip which I certainly do not particularly deserve was a complete surprise to me. I can only hope that your plan will indeed be realised, not only in my own interest, but especially in yours, as it is to depend precisely on your state of health. It goes without saying that I will gladly obey your "conditions." [. . .]
> Your obedient son
> Ernst. (OV 1.01)

His father's state of health to which Zermelo alludes, got worse and led to his death half a year later, on 24 January 1889. Zermelo and his sisters had become orphans. Thanks to the family's assets, their livelihood was secured for the time being.

Zermelo had a kind heart, as is attested by some of his contemporaries.[7] However, he was also prone to sharp and polemical reactions and did not refrain from trenchant irony when he was convinced of his opinions, both in

children at that time; perhaps he was staying in hospital because of his serious illness. Earlier letters (MLF) document that the Zermelo siblings often spent their holidays without their father.

[7]So Wilhelm Süss in a letter to Zermelo of 27 July 1951; MLF.

scientific and in political matters. He did so even if he had to suffer from the consequences, as with his criticism of the Nazis. Whereas the first trait might be attributed to the influence of his mother, his determination to speak his mind may be due to the example of his father. In 1875, Theodor Zermelo published a widely acknowledged[8] paper ([Zert75]) on the historian and philologist August Ludwig von Schlözer (1735–1809) of Göttingen, the most influential anti-absolutist German publicist of the early age of enlightenment.[9] It was his aim to raise awareness of those values expressed by Schlözer's plea for free speech, sincerity, tolerance, and humanity on which, in his opinion, the newly founded German Reich should be built. There are indications that the views of his father as reflected in this treatise and the example of von Schlözer had their effect on Zermelo. In the copy of the treatise, which he owned, many passages are underlined, among them those that concern the determined defence of one's own convictions. In a 1935 letter to his sisters, which describes his attitude towards the Nazis (4.11.3), he argues in the spirit of his father. Moreover, he used to talk about von Schlözer with his wife Gertrud, thereby arousing her interest in von Schlözer's daughter Dorothea (1770–1825), known as "the loveliest incarnation of enlightenment." In 1787, at the age of 17, she was the first woman to receive a Ph. D. at the University of Göttingen.[10]

Zermelo shared with his father an interest in poetry. As a young teacher, Theodor Zermelo had compiled a large selection of his own translations of poems from England, the US, France, and Italy under the title "Aus der Fremde" ("From Foreign Lands"). Zermelo was later to translate parts of the Homeric epics, and he enjoyed commenting on daily events in the form of poems. In 1885, at the age of thirteen, he made "a metrical translation from the first book of Virgil's Aeneid" as a birthday present for his father (MLF).

1.2 School and University Studies

In 1880 Ernst Zermelo entered the Luisenstädtisches Gymnasium in Berlin, situated in Brandenburgstraße (today Lobeckstraße).[11] Its name refers to the neighbourhood Luisenstadt honouring the Prussian Queen Luise Auguste Wilhelmine Amalie (1776–1810), the wife of King Friedrich Wilhelm III (1770–1840). The Luisenstädtisches Gymnasium had been opened on 10 October 1864, after the Berlin authorities had decided to establish new secondary schools and to extend existing ones in a reaction to the large increase in the population of Berlin.[12] Teaching started in the autumn of 1864 with 86 stu-

[8] Cf. *Brockhaus' Konversationslexikon*, 14th edition, 14th Volume (1895), 525.

[9] Cf., e. g., [Pet03].

[10] Cf. [Ker88] or [Kue76].

[11] The certificates relating to Zermelo's education at school and university are held in MLF.

[12] *Erster Jahresbericht über das Luisenstädtische Gymnasium in Berlin*, Berlin 1865, 33–34.

Zermelo as a schoolboy with his sisters Anna (right) and Elisabeth

dents. When Zermelo entered the school it had 587 students with 50 students in each entry class.[13] After the turn of the century the number of students in Luisenstadt decreased steadily. The school was eventually closed at its first site and transferred further to the west. The building was then destroyed in World War II.

Shortly after the death of his father, Zermelo finished school. His school leaving certificate (*Zeugnis der Reife*), dated 1 March 1889, certifies good results in religious education, German language, Latin, Greek, history, geography, mathematics, and physics. His performance in French was adequate. The comment in mathematics says that he followed the instructions with good understanding and that he has reliable knowledge and the ability to use it for skilfully solving problems. In physics it is testified that he is well-acquainted with a number of phenomena and laws. Under the heading "Behaviour and Diligence" it is remarked that he followed the lessons with reflection, but

[13] *Sechzehnter Jahresbericht über das Luisenstädtische Gymnasium in Berlin*, Berlin 1881, 7.

that he occasionally showed a certain passivity as a result of physical fatigue, a clear indicator that Zermelo already suffered from poor health during his school days. The certificate furthermore informs us that he was exempted from oral examinations and that he was now going to take up university studies in mathematics and physics.

In the summer semester 1889 Zermelo began his course of studies, which he did mainly at the Friedrich Wilhelm University (now Humboldt University) in Berlin, but also for one semester each at Halle-Wittenberg and Freiburg, and finally, after his Berlin examinations, at Göttingen.

When Zermelo's father died in 1889, he left behind six children who were still minors, now under the guardianship of Amtsgerichtsrat (judge of a county court) Muellner. On 24 December 1891 Muellner testified that the father's estate was needed to provide for the younger siblings, thereby helping Zermelo get a grant to finish his university studies. Zermelo was awarded grants from the Moses Mendelssohn Foundation (probably 1891/92) and the Magnus Foundation (1892/93).[14] Both foundations belonged to the Friedrich Wilhelm University, set up to support gifted students. The first was named after Moses Mendelssohn (1729–1786), the Berlin philosopher of German enlightenment, the second after the physicist Gustav Magnus (1802–1870) who is regarded as the progenitor of Berlin physics. The Magnus Foundation was the most important one in the second half of the 19th century. Gustav Magnus' widow had donated 60000 Marks to the University. The interest allowed the support of two excellent students of mathematics or the sciences with an annual grant of 1200 Marks, a significant sum at that time. Karl Weierstraß himself had outlined and defended its statutes ([Bier88], 126). As stipulated, Zermelo had to pass special examinations in order to prove that he had the level of knowledge expected at that stage of his studies. Two testimonials in mathematics, one for the Mendelssohn Foundation signed by Lazarus Fuchs and one for the Magnus Foundation signed by Hermann Amandus Schwarz, can be found among his papers.

In Berlin Zermelo enrolled for philosophy, studying mathematics, above all with Johannes Knoblauch, a student of Karl Weierstraß, and with Fuchs. Furthermore, he attended courses on experimental physics and heard a course on experimental psychology by Hermann Ebbinghaus. In the winter semester 1890/91 he changed to Halle-Wittenberg where he enrolled for mathematics and physics. There he attended Georg Cantor's courses on elliptical functions and number theory, Albert Wangerin's courses on differential equations and on spherical astronomy, but also a course given by Edmund Husserl on the philosophy of mathematics at the time when Husserl's *Philosophie der Arithmetik* ([Hus91a]) was about to be published. Zermelo also took part in the course on logic given by Benno Erdmann, one of the leading philosophical (psychologistic) logicians of that time. The next semester saw him at the University of Freiburg studying mathematical physics with Emil Warburg, analytical

[14] Cf. [Bier88], 126.

geometry and the method of least squares with Jakob Lüroth, experimental psychology with Hugo Münsterberg, and history of philosophy with Alois Riehl. Furthermore he attended a seminar on Heinrich von Kleist.

Zermelo then returned to Berlin. Between the winter semester 1891/92 and the summer semester 1894 he attended several courses by Max Planck on theoretical physics, among them a course on the theory of heat in the winter semester 1893/94. He also attended a course on the principle of the conservation of energy by Wilhelm Wien (summer semester 1893). In mathematics he took part in courses on differential equations by Fuchs, algebraic geometry by Georg Ferdinand Frobenius, and non-Euclidean geometry by Knoblauch. The calculus of variations, the central topic of Zermelo's later work, was taught by Hermann Amandus Schwarz in the summer semester 1892. In philosophy he attended "Philosophische Übungen" by the neo-Kantian Friedrich Paulsen and Wilhelm Dilthey's course on the history of philosophy. He also took a course on psychology, again by Hermann Ebbinghaus (winter semester 1893/94).

Zermelo's Ph. D. thesis *Untersuchungen zur Variations-Rechnung* (*Investigations On the Calculus of Variations*) (cf. 1.3) was suggested and guided by Hermann Amandus Schwarz. Schwarz, who had started his studies in chemistry, was brought over to mathematics by Weierstraß and was to become his most eminent student. In particular, he aimed at "keeping a firm hold on and handing down the state of mathematical exactness which Weierstraß had reached" ([Ham23], 9). In 1892, then a professor in Göttingen, he was appointed Weierstraß' successor in Berlin. Zermelo became his first Ph. D. student there.

On 23 March 1894 Zermelo, then 22 years old, applied to begin the Ph. D. procedure.[15] Schwarz and the second referee Fuchs delivered their reports in July 1894. The oral examination took place on 6 October 1894. It included the defence of three theses that could be proposed by the candidate. Zermelo had made the following choice ([Zer94], 98):

I. In the calculus of variations one has to attach importance to an exact definition of maximum or minimum more than has been done up to now.[16]
II. It is not justified to confront physics with the task of reducing all phenomena in nature to the mechanics of atoms.[17]
III. Measuring can be understood as the everywhere applicable means to distinguish and to compare continuously changing qualities.[18]

[15] Details follow the files of Zermelo's Ph. D. procedure in the archives of Humboldt University Berlin and are quoted from [Thie06], 298–303.

[16] "In der Variations-Rechnung ist auf eine genaue Definition des Maximums oder Minimums grösserer Wert als bisher zu legen."

[17] "Mit Unrecht wird der Physik die Aufgabe gestellt, alle Naturerscheinungen auf Mechanik der Atome zurückzuführen."

[18] "Die Messung ist aufzufassen als das überall anwendbare Hilfsmittel, stetig veränderliche Qualitäten zu unterscheiden und zu vergleichen."

The latter two theses, in particular the second one, are clearly aimed against early atomism in physics and the mechanical explanations coming with it. They will gain in substance within the next two years and lead to a serious debate with Ludwig Boltzmann about the scope of statistical mechanics (cf. 1.4).

After having received his doctorate, Zermelo got a position as an assistant to Max Planck at the Berlin Institute for Theoretical Physics from December 1894 to September 1897.[19] During this time he edited a German translation ([Gla97]) of Richard Tetley Glazebrook's elementary textbook on light ([Gla94]). He also prepared for his exams *pro facultate docendi*, which he passed successfully on 2 February 1897. In philosophy he wrote an essay entitled "Welche Bedeutung hat das Prinzip der Erhaltung der Energie für die Frage nach dem Verhältnis von Leib und Seele?" ("What is the Significance of the Principle of the Conservation of Energy for the Question of the Relation Between Body and Mind?").[20] According to the reports Zermelo was well-informed about the history of philosophy and systematical subjects. He was examined in religious education (showing excellent knowledge of the Holy Bible and church history) and in German language and literature. As to his exams in mathematics the report says that although he was not always aware of the methods in each domain, it was nevertheless evident that he had acquired an excellent mathematical education. He passed the physics part excellently. The exams in geography showed that he was excellent in respect to theoretical explanations, but that he was not that affected by "studying reality." Expressing its conviction that he would be able to close these gaps in the future, the committee allowed him to teach geography in higher classes. Finally it was certified that he had the knowledge for teaching mineralogy in higher classes and chemistry in the middle classes.

Already in the summer of 1896 Zermelo applied for a position as assistant at the Deutsche Seewarte in Hamburg, the central institution for maritime meteorology of the German Reich. The application was supported by Hermann Amandus Schwarz who reported on 20 July 1896 that he knew Zermelo from mathematics courses at Berlin University, from a private course ("Privatissimum"), and as supervisor of his dissertation. He expressed his conviction that Zermelo had excellent skills for investigations in theoretical mathematics. Planck confirmed on 7 July 1896 (MLF) that he was extraordinarily satisfied with Zermelo's achievements where he "made use of his special mathematical talent in an utmost conscentious manner" (OV 1.02).

Finally, however, Zermelo continued his academic career, taking up further studies at the University of Göttingen. He enrolled for mathematics on 4 November 1897. In the winter semester 1897/98 he heard David Hilbert's course on irrational numbers and attended the mathematical seminar on differential equations in mechanics directed by Hilbert and Felix Klein. He took

[19] Assessment from Max Planck of 3 November 1897; MLF.
[20] Draft in shorthand in UAF, C 129/270.

exercises in physics with Eduard Riecke and heard thermodynamics with Oskar Emil Meyer. The next semester he took part in a course on set theory given by Arthur Schoenflies and in Felix Klein's mathematical seminar.

Looking back to Zermelo's university studies, one may emphasize the following points:

He acquired a solid and broad knowledge in both mathematics and physics, his main subjects. His advanced studies directed him to his early specialities of research, to mathematical physics and thermodynamics, and to the calculus of variations, the topic of his dissertation.

He attended courses by Georg Cantor, but no courses on set theory, neither in Halle[21] nor at other places, at least until the summer semester 1898 when Schoenflies gave his course in Göttingen. Furthermore, he had no instruction in mathematical logic at a time, however, when German universities offered this subject only in Jena with Gottlob Frege and in Karlsruhe with Ernst Schröder. He got his logical training from Benno Erdmann, who defined logic as "the general, formal and normative science of the methodological preconditions of scientific reasoning" ([Erd92], 25).

Given his teachers in philosophy, Riehl, Paulsen, Dilthey, and Erdmann, it can be assumed that he had a broad overall knowledge of philosophical theories. In Halle he even got acquainted with Husserl's phenomenological philosophy of mathematics *in statu nascendi.* His interest in experimental psychology as taught by Hermann Ebbinghaus and Hugo Münsterberg is quite evident.

1.3 Ph. D. Thesis and the Calculus of Variations

We recall that Zermelo's Ph. D. thesis *Untersuchungen zur Variations-Rechnung* (*Investigations on the Calculus of Variations*) was guided by Hermann Amandus Schwarz. It may have been inspired by Schwarz's lectures in the summer semester of 1892. Schwarz proposed to generalize methods and results in the calculus of variations, which Weierstraß had obtained for derivations of first order, to higher derivations.[22]

The calculus of variations treats problems of the following kind: Given a functional J from the set M into the set of reals, what are the elements of M for which J has an extremal value? The transition from J to $-J$ shows that it suffices to treat either minima or maxima. In a classical example, a special form of the so-called *isoperimetric problem,* M is the set of closed curves in

[21] Cantor lectured on set theory only in the summer semester 1885 in a course entitled "Zahlentheorie, als Einleitung in die Theorie der Ordnungstypen" ("Number Theory, as an Introduction to the Theory of Order Types") (information by Rüdiger Thiele; cf. also [Gra70], 81, or [PurI87], 104).

[22] Also here we follow the files of Zermelo's Ph. D. procedure in the archives of Humboldt University Berlin as quoted in [Thie06], 298–303.

the plane of a fixed given perimeter, and J maps a curve in M to the area it encloses. The problem asks for the curves enclosing an area of maximal size. In Zermelo's Ph. D. thesis M is a set of curves and J the formation of integrals along curves in M for a given integrand which may now be of a more general kind. In Zermelo's own words ([Zer94], 24):

> [Given] an integral
>
> $$J = \int_{t_1}^{t_2} F\, dt,$$
>
> [...] our task is the following: If
>
> $$F\left(x^{(\mu)}, y^{(\mu)}\right) = F\left(x, x', .. x^{(n)};\ y, y', .. y^{(n)}\right)$$
>
> is a function which is analytical in all arguments and which has the character of an entire function in the domain under consideration and obeys the integrability conditions developed in the first section, we search in the totality A of all curves
>
> $$x = \varphi(t), \quad y = \psi(t)$$
>
> which satisfy certain conditions, a special curve a for which the value of the definite integral
>
> $$J = \int_{t_1}^{t_2} F\left(x^{(\mu)}, y^{(\mu)}\right) \quad \left(x^{(\mu)} = \frac{d^\mu x}{dt^\mu},\ y^{(\mu)} = \frac{d^\mu y}{dt^\mu}\right)$$
>
> along the curve between certain boundaries is smaller than for all neighbouring curves $\bar{a}$ of the same totality A.

The integrability conditions to which Zermelo refers ensure that the value of the integral does not depend on the parametrization of the curve in question.

The case $n = 1$ had been solved by Weierstraß and treated in his courses, in particular in the course given in the summer semester 1879.[23] The latter one had been written up under the auspices of the Berlin Mathematical Society. Edmund Husserl had also taken part in this project.[24]

Zermelo studied the lecture notes in the summer of 1892, when he attended Schwarz's course on the calculus of variations ([Zer94], 1). Less than two years later he succeeded in solving the task Schwarz had set. In his report of 5 July 1884 Schwarz describes the subject as "very difficult;" he is convinced that Zermelo provided the very best solution and predicts a lasting influence of the methods Zermelo had developed and of the results he had obtained:

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[23] A comprehensive version of Weierstraß' lectures has been edited as [Wei27]; cf. also [Thie06], 183–243, or [Gol80], Ch. 5.

[24] Husserl (1859–1938) studied mathematics, first in Leipzig (1876–1878) and then in Berlin (1878–1880) mainly with Weierstraß. His Ph. D. thesis *Beiträge zur Theorie der Variationsrechnung* (*Contributions to the Theory of the Calculus of Variations*) ([Hus82]) is written in the spirit of Weierstraß; cf. [Thie06], 293–295.

> According to my judgement the author succeeds in generalizing the main investigations of Mr. Weierstraß [...] in the correct manner. In my opinion he thus obtained a valuable completion of our present knowledge in this part of the calculus of variations. Unless I am very much mistaken, all future researchers in this difficult area will have to take up the results of this work and the way they are deduced. (OV 1.03)

He marked the thesis with the highest degree possible, *diligentia et acuminis specimen egregium*. Co-referee Fuchs shared his evaluation.

The dissertation starts by exhibiting the integrability conditions mentioned above. Taking them as "a task of interest in itself" ([Zer94], 14), Zermelo develops them in a more general framework than needed for the later applications. In the second part he provides a careful definition of minimum (ibid., 25-29), quite in accordance with the first thesis he had chosen for the oral examination. The notion of minimum results from the conditions which he imposes on the totalities A of (parametrizations of) curves and on the relation "$\overline{a}$ is a neighbouring curve of a."[25] In the final parts he carries out the Weierstraßian programme in the framework thus created.[26]

The dissertation ends (p. 96) with "the first clear formulation and proof of the important envelope theorem,"[27] according to Gilbert A. Bliss ([Bli46], 24) "one of the most interesting and most beautiful theorems in the domain of geometrical analysis." The theorem generalizes the following proposition[28] about geodetic lines to one-parameter families of extremals, likewise providing a criterion for the non-existence of a minimum of the corresponding variational problem among the extremals under consideration:[29]

> If $\mathcal{F}$ is a one-parameter family of geodetic lines issuing from a point P of a surface and E an envelope of $\mathcal{F}$, and if, moreover, $G_1 \in \mathcal{F}$ and $G_2 \in \mathcal{F}$ are tangent to E in P_1 and P_2, respectively, and P_1 precedes P_2 on E, then
>
> $$\text{length of } PP_2 = \text{length of } PP_1 + \text{length of } P_1P_2,$$
>
> where, for example, PP_1 denotes the arc of G_1 between P and P_1, and P_1P_2 denotes the arc on E between P_1 and P_2.

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[25] Partly, these conditions generalize those of Weierstraß for the case "n = 1" in a natural way, demanding, for instance, that for curves in A the nth derivation exists and is continuous. Over and above that, the curves in A have to have not only the same boundary values, but also the same so-called "boundary osculation invariants," among them as a simple example the torsion.

Analogously, the definition of neighbourhood does not only refer to a natural distance between a and $\overline{a}$, but also to quantities corresponding to the osculation invariants, among them again the torsion.

[26] For technical details see [Thie06], 298–303.

[27] So Hilbert in an assessment of Zermelo of 16 January 1910; SUB, Cod. Ms. D. Hilbert 492, fols. 4/1-2. – For a sketch of the proof see [Gol80], 340–341.

[28] Attributed to Jean Gaston Darboux; cf. [Bolz04], 166, or [Gol80], 339.

[29] Namely, by providing a proof of the necessity of the so-called Jacobi condition; cf., for example, [Bli46], Sect. 10.

In the "triangle" formed by P, P_1, and P_2, the arc P_1P_2 can be replaced by a shorter curve (the envelope is not geodetic). Hence, there is a curve connecting P with P_2 that is shorter than PP_2, i.e., the (geodetic) arc PP_2 does not provide a curve of minimal length connecting P with P_2.

Later, Adolf Kneser provided generalizations of Zermelo's theorem, discussing also the existence of an envelope – a desideratum Zermelo had left open.[30] Zermelo is, however, more general than Kneser in respect to the variational integrand, allowing derivations of arbitrary order. Goldstine speaks of the Zermelo-Kneser results as of the "Zermelo-Kneser envelope theorem," giving credit to Zermelo for the "most elegant argument" ([Gol80], 340).

Various voices confirm the significance of Zermelo's dissertation for the development of the Weierstraßian direction in the calculus of variations.

Adolf Kneser's 1900 monograph on the calculus of variations recommends the dissertation as giving valuable information about the Weierstraßian methods and the case of higher derivatives. In the preface Kneser says ([Knes00], IV):

> The relationship between my work and the investigations of Weierstraß requires a special remark. By his creative activity Weierstraß opened new ways in the calculus of variations. As is well-known, his research is not available in a systematic representation; as the most productive sources I used the Ph. D. thesis of Zermelo and a paper of Kobb[31] [...]. In a modified and generalized form they contain all essential ideas of Weierstraß as far as they are related to our topic.

Oskar Bolza (1857–1942), one of the most influential proponents of the calculus of variations, also gives due respect to Zermelo: his epochal monograph ([Bolz09]) contains numerous quotations of Zermelo's work.[32]

Constantin Carathéodory, in his similarly influential monograph on the subject, comments that "in the beginning Weierstraß' method was known to only a few people and opened to a larger public by Zermelo's dissertation" ([Car35], 388).

The quality of the dissertation played a major role when Zermelo was considered for university positions after his *Habilitation.* In 1903 the search committee for an extraordinary professorship[33] of mathematics at the University of Breslau (now Wrocław, Poland) remarks that "[Zermelo's] doctoral dissertation, when it appeared, was taken note of considerably more than

[30] [Knes1898], 27; [Knes00], §15. Cf. [Gol80], Sect. 7.5, or [Bolz09], §43, for details. For a further generalization of the envelope theorem obtained by Caratheodory in 1923, cf. [Car35], 292–293 and 398.

[31] The two parts of the paper are [Kob92a], [Kob92b].

[32] The monograph is a substantial extension of Bolza's *Lectures in the Calculus of Variations* ([Bolz04]). The latter is rooted in a series of eight talks about the history and recent developments of the subject which he had given at the 1901 meeting of the American Mathematical Society in Ithaca, N. Y.

[33] Corresponding to the position of an associate professor.

usually happens to such writings."[34] In 1909 Zermelo was shortlisted for the succession of Eduard von Weber at the University of Würzburg. The report of the Philosophical Faculty states that "in the calculus of variations his work had a really epoch making influence."[35] In 1913, when Zermelo was chosen for the first place in a list of three for a full professorship in mathematics at the Technical University of Breslau, the report to the minister characterizes his thesis as an investigation "basic (grundlegend) to the development of the calculus of variations."[36]

Zermelo's interest in the calculus of variations never weakened. He gave lecture courses (for example, in the summer semester 1910 at the University of Zurich and in the winter semester 1928/29 at the University of Freiburg) and continued publishing papers in the field.

In the first of these papers ([Zer02c]), he provides an intuitive description of some extensions of the problem of shortest lines on a surface, namely for the case of bounded steepness with or without bounded torsion, illustrating them with railroads and roads, respectively, in the mountains.[37] About 40 years later he will choose this topic as one of the chapters of a planned book *Mathematische Miniaturen* (*Mathematical Miniatures*) under the title "Straßenbau im Gebirge" ("Building Roads In the Mountains").

In the next paper ([Zer03]) Zermelo gives two simplified proofs of a result of Paul du Bois-Reymond ([DuB79b]) which says that, given n and an analytical F, any function y for which $y^{(n)}$ exists and is continuous and which yields an extremum of $\int_a^b F(x, y, \ldots, y^{(n)})dx$, possesses derivatives of arbitrarily high order;[38] the theorem shows that the Lagrangian method for solving the related variational problem which uses the existence of $y^{(2n)}$ and its continuity, does not exclude solutions for which $y^{(n)}$ exists and is continuous. Constantin Caratheodory characterizes Zermelo's proofs as "unsurpassable with respect to simplicity, shortness, and classical elegance."[39]

Roughly, Zermelo's proceeds as follows: Using partial integration according to Lagrange together with a suitable transformation of the first variation according to du Bois-Reymond, he obtains the first variation of the given problem in the form

$$\int_a^b \Lambda(x)\eta^{(n)}(x)dx,$$

where the variation η is supposed to possess a continuous n-th derivation and to satisfy

$$\eta^{(i)}(a) = \eta^{(i)}(b) = 0 \text{ for } 0 \le i \le n.$$

[34] UAW, signature F73, p. 114.

[35] Archives of the University of Würzburg, UWü ZV PA Emil Hilb (No. 88).

[36] UAW, signature TH 156, p. 14.

[37] Technically, the paper is concerned with so-called discontinuous solutions; cf. [Car35], 400.

[38] Zermelo adds a variant of the second proof which he attributes to Erhard Schmidt.

[39] [Car10], 222; [Car54], Vol. V, 305.

The main simplification of du Bois-Reymond's proof consists in showing that Λ is a polynomial in x of a degree $\leq n-1$.[40] In particular, Λ is arbitrarily often differentiable. It is easy to show that this property is transferred to the solutions y.

Together with Hans Hahn, one of the initiators of linear functional analysis, Zermelo writes a continuation of Adolf Kneser's contribution ([Knes04]) on the calculus of variations for the *Encyklopädie der Mathematischen Wissenschaften,* giving a clear exposition of the Weierstraßian method ([HahZ04]).

Finally, he formulates and solves "Zermelo's navigation problem" concerning optimal routes of airships under changing winds.[41]

1.4 The Boltzmann Controversy

We recall the second thesis that Zermelo defended in the oral examination of his Ph. D. procedure: "It is not justified to confront physics with the task of reducing all phenomena in nature to the mechanics of atoms." Probably he had met the questions framing his thesis in Max Planck's course on the theory of heat in the winter semester 1893/94 and had come to acknowledge Planck's critical attitude against early atomism and the mechanical explanations of natural phenomena accompanying it. During his time as an assistant to Planck (1894–1897) his criticism consolidated, now clearly aiming also against statistical mechanics, in particular against Ludwig Boltzmann's statistical explanation of the second law of thermodynamics. Having a strong mathematical argument and backed by Planck, he published it as aiming directly against Boltzmann ([Zer96a]), thus provoking a serious controversy which took place in two rounds in the *Annalen der Physik und Chemie* in 1896/97. Despite being interspersed with personal sharpness, it led to re-considering foundational questions in physics at a time when probability was showing up, competing with the paradigm of causality.[42]

1.4.1 The Situation

Around 1895 the atomistic point of view, while widely accepted in chemistry, was still under debate in physics. Looking back to this situation, Max Planck remembers ([Pla14], 87):

[40] For $n = 1$ this result had already been obtained by du Bois-Reymond himself in 1879 ([DuB79a]) and, with a different proof, by Hilbert in 1899 (cf. [Bolz04], §6). – It is here that Zermelo provides two arguments.

[41] [Zer30c], [Zer31a]; cf. 4.2.2.

[42] For a description of the debates on the kinetic theory of gas, also in the context of 19th century cultural development, cf., for example, [Bru66] (containing reprints or English translations of the most important contributions) and [Bru78]. For an analysis of the controversy cf. also [Beh69], [JunM86], 213–217, [Kai88], 125–127, [Uff07], 983–992.

> To many a cautious researcher the huge jump from the visible, directly controllable region into the invisible area, from macrocosm into microcosm, seemed to be too daring.

Discussions mainly crystallized in the theory of heat. The so-called energeticists such as Ernst Mach and Wilhelm Ostwald regarded energy as the most fundamental physical entity and the basic physical principles of heat theory as autonomous phenomenological laws that were not in need of further explanation. Among these principles we find the first law of thermodynamics[43] stating the conservation of energy and the second law of thermodynamics[44] concerning the spontaneous transition of physical systems into some state of equilibrium. The latter may be illustrated by a system A consisting of a container filled with some kind of gas and being *closed*, i. e. entirely isolated from its surroundings. As long as temperature or pressure vary inside A, there is a balancing out towards homogeneity. If the universe is considered as a closed system, the equilibrium state or *heat death* it is approaching can be characterized by a total absence of processes that come with some amount of differentiation. The parameter measuring the homogeneity, the so-called *entropy*, can be thought to represent the amount of heat energy that is no longer freely available.

From the early atomistic point of view and the mechanical conceptions accompanying it, physical systems consist of microscopic particles called *atoms* which obey the laws of mechanics. For an "atomist," the principles governing heat theory may no longer be viewed as unquestionable phenomena, but are reducible to the mechanical behaviour of the particles that constitute the system under consideration. In particular, the heat content of the system is identified with the kinetic energy of its particles. Determining its behaviour means determining the behaviour of all its constituents. However, as a rule, the number of atoms definitely excludes the possibility of calculating exactly how each of them will behave. To overcome this dilemma, atomists used statistical methods to describe the expected behaviour at least approximately. For justification they argued that the phenomena of thermodynamics, which we observe in nature, result from the global and "statistical" view, the only view we have at our disposal of the behaviour of the invisible parts lying at their root. Josiah Willard Gibbs, who shaped the mathematical theory of statistical mechanics, describes the new approach in the preface of his classical book ([Gib02], vii–viii):

> The usual point of view in the study of mechanics is that where the attention is mainly directed to the changes which take place in the course of time in a given system. The principal problem is the determination of the condition of the system with respect to configuration and velocities at any required time, when its condition in these respects has been given for some one time, and

[43] Henceforth also called "First Law."
[44] Henceforth also called "Second Law."

> the fundamental equations are those which express the changes continually taking place in the system. [...]
>
> The laws of thermodynamics, as empirically determined, express the approximate and probable behavior of systems of a great number of particles, or, more precisely, they express the laws of mechanics for such systems as they appear to beings who have not the fineness of perception to enable them to appreciate quantities of the order of magnitude of those which relate to single particles, and who cannot repeat their experiments often enough to obtain any but the most probable results.

According to the last part of this quotation, the statistical view includes a change of paradigm: It replaces causality by probability. In Boltzmann's presentation the probability $W(s)$ of a system A to be in state s is measured by the relative number (with respect to some suitable measure) of the configurations of A which macroscopically represent s. According to this interpretation the Second Law now says that physical systems tend toward states of maximal probability.

Coming back to the system A defined above, let us assume that the container is divided into two parts P_1 and P_2 of equal size with no wall in between. Furthermore, let s be a state where all atoms are in part P_1, and let s' be a state where the gas is homogeneously distributed over both parts. Then the probability $W(s')$ is big and $W(s)$ is low compared to $W(s')$. Therefore, if A is in a state macroscopically represented by s, it will tend toward a state macroscopically represented by s'; in other words, the atoms will tend to distribute homogeneously over the whole container. The entropy $S(s)$ of a state s may now be interpreted as to measure the probability $W(s)$. In fact, it is identified with a suitable multiple of its logarithm:

$$S = k \log W,$$

where k is the so-called Boltzmann constant. The Second Law now provides systems with a tendency towards states of increasing entropy, thereby imposing a direction on the physical processes concerned. Similarly, the distribution of the velocities of their constituents will converge to a final equilibrium distribution, the so-called Maxwell distribution.

1.4.2 The First Round

It was, in particular, Ludwig Boltzmann (1844–1906) who developed the kinetic theory of gas according to the principles of statistical mechanics, thereby providing the statistical interpretation of the Second Law sketched above.[45] In the mid 1890s, attacked already by energeticists and other non-atomists, he

[45] The sketch does not take into consideration the development of the respective notions in Boltzmann's work itself and the way they differ from those in [Gib02]. For respective details cf. [Uff04], [Uff07], or [Kac59], Ch. III.

Ludwig Boltzmann in 1898
Courtesy of Österreichische Nationalbibliothek

found himself confronted with a serious mathematical counterargument written up by Zermelo in December 1895 in the paper entitled "Ueber einen Satz der Dynamik und die mechanische Wärmetheorie" ("On a Theorem of Dynamics and the Mechanical Theory of Heat") ([Zer96a]), the *Wiederkehreinwand* or *recurrence objection*. For an account we return to the system A described above. Its microscopic description will only depend on the spatial coordinates of its atoms together with their velocities. Under reasonable assumptions which Boltzmann had taken for granted, the function describing these data in dependence of time falls under the so-called *recurrence theorem* proved by Poincaré in 1890:[46] System A will infinitely often approach its initial state. Hence, one may argue, also the entropy, depending only on states, will infinitely often approach the initial entropy. It thus cannot constantly increase. Moreover, the velocities of the atoms of A will not tend to a final distribution. To be fully exact, one has to assume that the initial state of A is different from some exceptional states for which Poincaré's argument does not work. These states, however, are "surrounded" by non-exceptional ones.[47] So there really seems to be a contradiction.

[46] [Poi90], esp. Section 8, "Usage des invariants intégraux," 67–72.
[47] In precise terms the exceptional states form a set of measure zero.