Precedence-Type Tests and Applications

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Precedence-Type Tests
and Applications
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Precedence-Type Tests and Applications

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To

My Late Father and Mother,
R. Narayanaswamy Iyer and N. Lakshmi,
for their love, support, and encouragement!

(NB)

My Parents,
Cheong Leung Ng and Kit Ching Wong,
for their love and affection!

(HKTN)
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Preface

Nonparametric statistics are intuitive and easily understood and inferential procedures based on ranks and runs are often heuristically simple to follow and implement. One such family of test procedures are the so-called precedence-type tests. These tests, which are quite useful in life-testing situations to make quick and reliable decisions early in the experiment, are also time- and cost-efficient as they are based on only a few early failures (instead of failures of all units in the life-test). However, the development of precedence-type tests requires care and usage of a wide range of statistical techniques. This volume provides a thorough and comprehensive overview of various theoretical as well as applied developments on a variety of problems in which precedence-type test procedures may be applied effectively.

This volume comprises 10 chapters, and may be broadly classified into four parts—Part A, comprising Chapter 3, deals with the original precedence test and some properties of precedence and related test procedures; Part B, comprising Chapters 4-6, deals with some alternatives to precedence test such as maximal precedence, weighted forms of precedence and maximal precedence, and Wilcoxon-type rank-sum precedence tests and their properties; Part C, comprising Chapter 7, deals with the extension of precedence, maximal precedence, and Wilcoxon-type rank-sum precedence tests to the situation when the sample arising from the life-testing experiment is progressively Type-II censored, and their properties; and Part D, comprising Chapters 8-10, deals with precedence-type tests in multisample situations and selection problems. Throughout the volume, several tables have been presented so as to facilitate the use of these tests in practical problems, and also some examples have been included in order to illustrate all the precedence-type procedures.

The length of this volume as well as the extensive bibliography at the end of the volume (with a good number of publications being in the last 25 years or so) provides ample testimony to the growth and continued interest in this
topic of research. Even though we have discussed a number of variations of
the precedence test and also different applications of these test procedures,
there is clearly a lot more potential to develop new precedence-type tests as
well as to apply them to diverse inferential problems. It is our sincere hope
that this volume would enable and encourage this to happen.

In a volume of this nature and size, inevitably there will be omission of
some results that should have been included. Any such omission is accidental
and by no means due to personal nonscientific antipathy.

We encourage readers to comment on the contents of this volume and
thank them in advance for informing us of any errors, omissions, or misrep-
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Chapter 1

Introduction

1.1 PROBLEMS OF INTEREST

The comparison of the quality of products from different manufacturing processes or the effectiveness of different treatments for an illness is a commonly encountered problem in practice. For example, a manufacturer of a product may wish to compare a new manufacturing process with the existing process. If there is significant statistical evidence that the new process results in better product (meaning, more reliable or with longer lifetime), then the manufacturer may wish to abandon the existing process and implement the new process into production. Another example is when a medical researcher wishes to compare a new treatment with a control. In this case, patients may be assigned randomly to treatment and control groups, and their remission times (or lifetimes) are recorded. Based on these data, the researcher will be primarily interested in determining whether the treatment is effective.

The development of efficient statistical procedures for these problems is, therefore, of great interest and importance.

1.2 SPECIAL CONSIDERATIONS

In the examples described above, we may have some special considerations. For example, in the medical experiment, the treatment may be toxic and harmful to the patients; therefore, based on ethical grounds, the researcher may wish to terminate the experiment as soon as there is evidence to draw a reliable conclusion, one way or the other. In the other example involving
quality or reliability of products, the manufacturer may want (1) to make quick and reliable decisions early in the life-testing experiment, and (2) to minimize the number of failures of units from the new process since their cost of production may be relatively high so that the units that had not failed could be used for some other testing purposes.

For these reasons, our main goal is to make decisions during the early stage of the experiment, not having observed many failures.

1.3 SPECIAL FORM OF TESTING

As we are concerned with the comparison of the lifetime distributions of units from the new process with those from the standard process, and because we would expect more failures to occur from the standard process than from the new process during the early stage of the experiment, we would naturally like to utilize this to collect data in this special form and then carry out a test suitably based on these data.

For this purpose, we assume throughout this book that sample units from the processes are placed simultaneously on a life-testing experiment and that failures are observed as they arise in a naturally time-increasing manner.

At this point, there are clearly two ways to proceed: one is to assume specific lifetime distributions for the samples and carry out the test under a parametric setup, and the other is to collect data in a nonparametric manner (for example, only the ranks of failure times rather than the failure times themselves) and carry out the test under a nonparametric setup. We have chosen the latter because we will have only very few failures (and so assumption of a family of lifetime distributions for data in hand may be difficult to justify or verify) and the decisions made will be somewhat robust (as compared to those from efficient tests based on some specific family of lifetime distributions).

1.4 PRECEDENCE TESTS

From the life-testing experiment described in the previous section, one form of (nonparametric) data that we could collect easily is the number of failures from the standard process that preceded the first failure from the new process, the number of failures from the standard process that occurred between the first and second failures from the new process, and so on.
If the experimenter had decided to allow only a certain number (say, \( r \)) of failures from the new process (for reasons stated earlier), then the life-test would be terminated as soon as this \( r \)th failure occurred from the new process. We would then have the data on the numbers of failures from the standard process only until this particular failure time.

This form of life-test is called a \textit{precedence test} and any test statistic based on these “precedences” is called a \textit{precedence-type statistic}; see, for example, Nelson (1963, 1986) and Ng and Balakrishnan (2006). Of course, the simplest precedence-type statistic is the number of failures from the standard process that preceded the \( r \)th failure from the new process; as a matter of fact, this is what Nelson (1963) has called as a \textit{precedence statistic}. One may then change the functional form (instead of just the sum of the numbers of failures from the standard process) and come up with different precedence-type statistics, each with its own special features and properties. Furthermore, the idea of precedence-type statistics can also be extended to some other statistical problems. These, indeed, form the bases for all the developments in this book!

\section*{1.5 DEVELOPMENTS}

For the problems described in Section 1.1, with the goals as stated in Section 1.2, many different precedence-type test procedures are developed in this book and their properties are evaluated.

First, in Chapter 2, we present briefly the basic concepts and results that are essential for the developments in the subsequent chapters. After describing the form and nature of data arising from a life-testing experiment, we introduce order statistics and present some important formulas and results concerning order statistics. We then explain the concept of censoring and different forms of censored data that could arise from a life-test. We pay special attention to progressive censoring and present some formulas and results concerning the progressively censored order statistics. Some useful lifetime distributions are described next, and these distributions are used throughout this book for evaluating the performance of all the test procedures. Since the test procedures are developed under a stochastically ordered alternative, it will be quite natural to compare their performance with the well-known Wilcoxon’s rank-sum statistic for complete samples; so, we present a brief description of Wilcoxon’s rank-sum test and also explain how a randomized
test could be developed if one wishes to have a test that attains exactly a
prespecified level of significance.

Next, in Chapter 3, we introduce the concept of precedence testing and
present the precedence test statistic. We derive the exact null distribution
of this test statistic by combinatorial method and also by means of an order
statistics approach. We evaluate the power properties of this test through the
exact power function under the Lehmann alternative as well as through the
simulated power under the location-shift alternative. We then discuss various
properties of the precedence test and some other related nonparametric tests,
and present finally some examples to illustrate the use of precedence tests.

Since the precedence test developed in Chapter 3 suffers from a masking
effect, we introduce in Chapter 4 the maximal precedence test statistic. We
derive the exact null distribution of this test statistic by means of an order
statistics approach. We evaluate the power properties of this test through the
exact power function under the Lehmann alternative as well as through the
simulated power under the location-shift alternative. We then make some
comparisons of this test with the precedence test and show that the maximal
precedence test, unlike the precedence test, does not suffer from the masking
effect. Finally, we present some examples to illustrate the use of maximal
precedence tests.

In Chapter 5, we introduce the concept of weighted precedence and
weighted maximal precedence tests. We derive the exact null distributions of
these test statistics. We evaluate their power properties through their exact
power functions under the Lehmann alternative as well as through their sim-
ulated power values under the location-shift alternative. Finally, we present
some examples to illustrate the use of weighted precedence and weighted
maximal precedence tests.

In Chapter 6, we introduce three Wilcoxon-type rank-sum precedence
test statistics—the minimal, maximal, and expected rank-sum statistics. We
derive the exact null distributions of these three test statistics. Since the
large-sample normal approximation for the null distributions is not satisfac-
tory in the case of small or moderate sample sizes, we present an Edgeworth
approximation to the significance probabilities. We evaluate the power prop-
erties of these three tests through their exact power functions under the
Lehmann alternative as well as through their simulated power values un-
der the location-shift alternative. We then make some comparisons of these
three tests with the precedence test, the maximal precedence test, and the
Wilcoxon's rank-sum test based on complete samples. Finally, we present