GENERAL LINEAR METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS

ZDZISŁAW JACKIEWICZ

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GENERAL LINEAR METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS
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ZDZISŁAW JACKIEWICZ
To my wife, Elżbieta
my son, Wojciech Tomasz
and my daughter, Hanna Katarzyna
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Preface

This book is concerned with the theory, construction and implementation of general linear methods for ordinary differential equations. This is a very general class of methods which include the classical methods such as Runge-Kutta, linear multistep, and predictor-corrector methods as special cases. Some theoretical and practical aspects related to general linear methods are discussed in Numerical Methods for Ordinary Differential Equations by J.C. Butcher, in Solving Ordinary Differential Equations I: Nonstiff Problems by E. Hairer, S.P. Nørsett, and G. Wanner, and in Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems by E. Hairer and G. Wanner. However, these monographs cover the entire area of numerical solution of ordinary differential equations and devote only a limited amount of space to the discussion of general linear methods. This monograph is an attempt to present a complete analysis of some classes of general linear methods that have good potential for practical use. These classes include diagonally implicit multistage integration methods, two-step Runge-Kutta methods, and general linear methods with inherent Runge-Kutta stability.

In Chapter 1 we present a short introduction to ordinary differential equations, including existence and uniqueness theory, continuous dependence on the initial data and right-hand side, stability theory, and discussion of stiff differential equations and systems. Chapter 2 is an introduction to general
linear methods. In particular, we discuss preconsistency, consistency, stage-
consistency, zero-stability, convergence, order and stage order conditions, local
discretization error, and linear stability theory, and present examples of meth-
ods that are appropriate for nonstiff or stiff differential systems in sequential
or parallel computing environments. We also discuss briefly algebraic stabil-
ity, the concept of an underlying one-step method, starting procedures, and
codes based on general linear methods.

Chapters 3 to 8 constitute the main part of the book. In Chapters 3 and
4 we deal with the construction and implementation of diagonally implicit
multistage integration methods. In Chapters 5 and 6 the theory and imple-
mentation of two-step Runge-Kutta methods is discussed. In Chapters 7 and
8 we describe the theory and implementation of general linear methods with
inherent Runge-Kutta stability. The topics in these chapters related to the
theory and construction of these methods include the derivation of order and
stage order conditions, representation formulas for the coefficient matrices of
these methods, construction of formulas with desirable accuracy and stabil-
ity properties, and Nordsieck representation of these methods. The topics
in these chapters related to implementation issues include the construction
of appropriate starting procedures, local error estimation for small and large
step sizes, step size and order changing strategies, construction of continuous
interpolants of uniform high order, updating the vector of external approxi-
mations, and the solution of nonlinear systems of equations for stiff systems by
the modified Newton method. We also present many examples of these meth-
ods of all types, mainly of order \( p \) and stage order \( q = p \) or \( q = p - 1 \). Many
implementation issues are illustrated by the results of numerical experiments
with different classes of general linear methods.

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Z. Jackiewicz

Arizona State University
April, 2009
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CHAPTER 1

DIFFERENTIAL EQUATIONS AND SYSTEMS

1.1 THE INITIAL VALUE PROBLEM

Many problems in science and engineering can be modeled by the initial value problem for systems of ordinary differential equations (ODEs), which we write in autonomous form as follows:

\[
\begin{align*}
    u'(t) &= f(u(t)), \quad t \in [t_0, T], \\
    u(t_0) &= u_0.
\end{align*}
\]

Here \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is a given function that usually satisfies some regularity conditions, and \( u_0 \in \mathbb{R}^m \) is a given initial vector. Introducing the notation

\[
    u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad u' = \begin{bmatrix} u'_1 \\ \vdots \\ u'_m \end{bmatrix}, \quad f(u) = \begin{bmatrix} f_1(u_1, \ldots, u_m) \\ \vdots \\ f_m(u_1, \ldots, u_m) \end{bmatrix}, \quad u_0 = \begin{bmatrix} u_{0,1} \\ \vdots \\ u_{0,m} \end{bmatrix}.
\]

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(1.1.1) can be written in the following scalar form:

\[
\begin{align*}
    u'_1 &= f_1(u_1, \ldots, u_m), \quad u_1(t_0) = u_{0,1}, \\
    \vdots & \quad \vdots \\
    u'_m &= f_m(u_1, \ldots, u_m), \quad u_m(t_0) = u_{0,m}, \\
\end{align*}
\]

\(t \in [t_0, T]\), where we have suppressed dependence on the independent variable \(t\) in \(u_i\) and \(u'_i\), \(i = 1, 2, \ldots, m\). Observe that the nonautonomous equation

\[
y'(t) = g(t, y(t)), \quad t \in [t_0, T], \\
y(t_0) = y_0,
\]

(1.1.2)

\(g : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m, y_0 \in \mathbb{R}^m\), can always be reduced to a system of the form (1.1.1) of dimension \(m + 1\) if we define

\[
u = \begin{bmatrix} y \\ t \end{bmatrix}, \quad f(u) = \begin{bmatrix} g(t, y) \\ 1 \end{bmatrix}, \quad u_0 = \begin{bmatrix} y_0 \\ t_0 \end{bmatrix}.
\]

Systems of the form (1.1.1) or (1.1.2) can also arise in practice from the conversion of initial value problem for differential equations of higher order. Consider, for example, an autonomous form of such a problem:

\[
y^{(m)} = g(y, y^{(1)}, \ldots, y^{(m-1)}), \quad t \in [t_0, T], \\
y(t_0) = y_0, \quad y^{(1)}(t_0) = y_0^{(1)}, \ldots, y^{(m-1)}(t_0) = y_0^{(m-1)},
\]

where \(y^{(i)}\) stands for the derivative of order \(i\) and we again suppressed dependence on \(t\). Setting

\[
u_1 = y, \quad u_2 = y^{(1)}, \ldots, u_m = y^{(m-1)},
\]

we obtain

\[
\begin{align*}
    u'_1 &= u_2, \quad u_1(t_0) = y_0, \\
    u'_2 &= u_3, \quad u_2(t_0) = y_0^{(1)}, \\
    \vdots & \quad \vdots \\
    u'_{m-1} &= u_m, \quad u_{m-1}(t_0) = y_0^{(m-2)}, \\
    u'_m &= g(u_1, u_2, \ldots, u_m), \quad u_m(t_0) = y_0^{(m-1)},
\end{align*}
\]
which is equivalent to (1.1.1) with

\[
    f(u) = \begin{bmatrix}
        u_2 \\
        u_3 \\
        \vdots \\
        u_m \\
        g(u_1, u_2, \ldots, u_m)
    \end{bmatrix}, \quad u_0 = \begin{bmatrix}
        y_0 \\
        y_0^{(1)} \\
        \vdots \\
        y_0^{(m-2)} \\
        y_0^{(m-1)}
    \end{bmatrix}
\]

In Section 1.3 we discuss the existence and uniqueness of solutions to (1.1.1) and (1.1.2) under various conditions on the functions \( f \) and \( g \). In this discussion we often assume that these problems are defined not only for \( t \in [t_0, T] \) but on the larger interval \( t \in I \), where \( I = \{ t : |t - t_0| \leq T \} \).

### 1.2 EXAMPLES OF DIFFERENTIAL EQUATIONS AND SYSTEMS

We list in this section several examples of differential equations and differential systems. These problems are used later in our numerical experiments with various algorithms for numerical solution of ODEs. These algorithms are based on some classes of general linear methods discussed herein. All equations in this section are examples of nonstiff equations and systems. Problem of stiffness and stiff differential equations and systems are discussed in Sections 1.7 and 1.8.

**SCALAR** — the scalar problem [143, p. 237]:

\[
    y'(t) = -\text{sign}(t)|1 - |t|| y^2, \quad t \in [-2, 2],
\]

\[
    y(-2) = \frac{2}{3}. \tag{1.2.1}
\]

The solution to this initial value problem has a discontinuity in the first derivative \( y' \) at the point \( t = 0 \) and discontinuities in the second derivative \( y'' \) at \( t = -1 \) and \( t = 1 \).

**BUBBLE** — a model of cavitating bubble [200, 257]:

\[
    \frac{dy_1}{ds} = y_2, \quad \frac{dy_2}{ds} = 5 \exp(-s/s^*) - 1 - 1.5 y_2^2 - \frac{a y_2 + D}{y_1^\gamma} + \frac{1 + D}{y_1^{3\gamma + 1}}, \tag{1.2.2}
\]

\[
    y_1(0) = 1, \quad y_2(0) = 0,
\]

\( t \in [0, T] \). Here \( s^*, a, D, \) and \( \gamma \) are real parameters. As observed by Shampine [257], this problem places great demands on the precision and step size control strategies of numerical algorithms.
AREN — Arenstorf orbit for the restricted three body problem [12, 13, 143]. This is an example from astronomy which describes the movement of two bodies of scaled masses $1 - \mu$ and $\mu$ in a circular rotation in a plane and the movement of a third body of negligible mass (e.g., satellite or spacecraft) in the same plane. The equations of motion are

$$
y''_1 = y_1 + 2y'_2 - (1 - \mu) \frac{y_1 + \mu}{D_1} - \mu \frac{y_1 - 1 + \mu}{D_2},$$
$$y''_2 = y_2 - 2y'_1 - (1 + \mu) \frac{y_2}{D_1} - \mu \frac{y_2}{D_2},$$

(1.2.3)

$t \in [0, T]$, where

$$D_1 = ((y_1 + \mu)^2 + y_2^2)^{3/2}, \quad D_2 = ((y_1 - 1 + \mu)^2 + y_2^2)^{3/2}.$$

This problem with $\mu = 0.012277471$ corresponds to the earth-moon system. The periodic orbits of a satellite or spacecraft moving in such a system were discovered by Arenstorf [12, 13] by theoretical analysis of periodicity conditions and numerical calculations. Such periodic orbits may facilitate low-cost space exploration and are of interest to NASA. The initial conditions for which the solution to (1.2.3) is periodic are, for example,

$$y_1(0) = 0.994, \quad y'_1(0) = 0, \quad y_2(0) = 0, \quad y'_2(0) = -2.001585106379,$$

with the period of motion $T_1$ given by $T_1 = 17.06522$, or

$$y_1(0) = 0.994, \quad y'_1(0) = 0, \quad y_2(0) = 0, \quad y'_2(0) = -2.031732629557,$$

with the period of motion $T_2$ given by $T_2 = 11.1234$. Such orbits are plotted in, for example, [52, Fig. 102(i) and (ii)], and [143, Fig. 0.1].

LRNZ — the Lorenz model [209]. This is a system of three differential equations of the form

$$
y'_1 = -\sigma y_1 + \sigma y_2,$$
$$y'_2 = -y_1 y_3 + r y_1 - y_2,$$
$$y'_3 = y_1 y_2 - b y_3,$$

(1.2.4)

t \in [0, T]$. Here $b$, $\sigma$, and $r$ are positive constants. For example, for $b = 8/3$, $\sigma = 10$, and $r = 28$, this system has aperiodic solutions.

EULR — Euler’s equations of rotation of a rigid body [143]. This is a system of three differential equations given by

$$I_1 y'_1 = (I_2 - I_3)y_2 y_3,$$
$$I_2 y'_2 = (I_3 - I_1)y_3 y_1,$$
$$I_3 y'_3 = (I_1 - I_2)y_1 y_2 + f(t),$$

(1.2.5)
$t \in [0, T]$. Here $y_1$, $y_2$, and $y_3$ are the coordinates of the rotation vector; $I_1$, $I_2$, and $I_3$ are the principal moments of inertia; and the third coordinate has an additional exterior force $f(t)$.

**PLEI** — a celestial mechanics problem “the Pleiades” from [143, p. 245]. The equations of motion are

\[ x_i'' = \sum_{j \neq i} m_j (x_j - x_i) / r_{ij}, \]
\[ y_i'' = \sum_{j \neq i} m_j (y_j - y_i) / r_{ij}, \]  

(1.2.6)

$t \in [0, 3]$, where

\[ r_{ij} = \left((x_i - x_j)^2 + (y_i - y_j)^2\right)^{3/2}, \quad i, j = 1, 2, \ldots, 7. \]

The initial conditions are

\[
\begin{align*}
x_1(0) &= 3, \quad x_2(0) = 3, \quad x_3(0) = -1, \quad x_4(0) = -3, \\
x_5(0) &= 2, \quad x_6(0) = -2, \quad x_7(0) = 2, \\
y_1(0) &= 3, \quad y_2(0) = -3, \quad y_3(0) = 2, \quad y_4(0) = 0, \\
y_5(0) &= 0, \quad y_6(0) = -4, \quad y_7(0) = 4,
\end{align*}
\]

\[ x_i'(0) = y_i'(0) = 0, \text{ for all } i \text{ with the exception of } \]

\[ x_6'(0) = 1.75, \quad x_7'(0) = -1.5, \quad y_4'(0) = -1.25, \quad y_5'(0) = 1. \]

This problem describes the movement of seven stars in the plane with coordinates $x_i$, $y_i$ and masses $m_i = i$, $i = 1, 2, \ldots, 7$. The trajectories of these stars are plotted by Hairer et al. [143, Fig. 10.2a], and speeds $x'_i$ and $y'_i$, $i = 1, 2, \ldots, 7$, [143, Fig. 10.2b].

**ROPE** — the movement of a hanging rope of length 1 under gravitation and the influence of horizontal $F_y(t)$ and vertical $F_x(t)$ forces [143]. As explained by Hairer et al. [143], the discretization of this problem leads to a system of differential equations of second order for the angles $\theta_i = \theta_i(t)$ between the tangents to the rope and the vertical axis at a discrete arc length $s_l$. This system takes the form

\[
\sum_{k=1}^{n} a_{kl} \dot{\theta}_k = - \sum_{k=1}^{n} b_{lk} \dot{\theta}_k^2 - n \left(n + \frac{1}{2} - l\right) \sin(\theta_l) \\
- n^2 \sin(\theta_l) F_x(t) + \begin{cases} 
  n^2 \cos(\theta_l) F_y(t) & \text{if } l \leq 3n/4, \\
  0 & \text{if } l > 3n/4,
\end{cases}
\]  

(1.2.7)
$t \in [0, T], \ l = 1, 2, \ldots, n$, where

$$a_{lk} = g_{lk} \cos(\theta_l - \theta_k), \quad b_{lk} = g_{lk} \sin(\theta_l - \theta_k), \quad g_{lk} = n + \frac{1}{2} - \max\{l, k\}.$$

The horizontal force $F_y(t)$ acting at the point $s = 0.75$ is

$$F_y(t) = \left( \frac{1}{\cosh(4t - 2.5)} \right)^4,$$

and the vertical force $F_x(t)$ acting at the point $s = 1$ is

$$F_x(t) = 0.4.$$

This system will be solved for $n = 40$ with initial conditions

$$\theta_l(0) = \dot{\theta}_l(0) = 0, \quad l = 1, 2, \ldots, n,$$

on the interval $[0, 3.723]$.

Setting

$$A = \begin{bmatrix} a_{lk} \end{bmatrix}, \quad B = \begin{bmatrix} b_{lk} \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_n \end{bmatrix}^T,$$

$$\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \cdots & \dot{\theta}_n \end{bmatrix}^T, \quad \ddot{\theta} = \begin{bmatrix} \ddot{\theta}_1 & \ddot{\theta}_2 & \cdots & \ddot{\theta}_n \end{bmatrix}^T,$$

system (1.2.7) can be written in vector form as

$$A \ddot{\theta} = -B \dot{\theta}^2 + g(t, \theta), \quad t \in [0, T],$$

$$\theta(0) = \dot{\theta}(0) = 0,$$  \hspace{1cm} (1.2.8)

where $\dot{\theta}^2$ denotes componentwise exponentiation and $g(t, \theta)$ is an appropriately defined vector function. The solution of (1.2.8) requires computation of the inverse matrix $A^{-1}$. As explained by Hairer et al. [143, 146] this can be done very efficiently in $O(n)$ operations, due to the special structure of the matrix $A$. It can be verified that

$$A + iB = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) G \text{diag}(e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_n}),$$

where $G = [g_{kl}]$. This matrix has the inverse

$$G^{-1} = \begin{bmatrix}
1 & -1 & & & \\
-1 & 2 & -1 & & \\
& & \ddots & \ddots & \ddots \\
& & & -1 & 2 & -1 \\
& & & & -1 & 3
\end{bmatrix}.$$
and it follows that

\[(A + i B)^{-1} = C + i D\]

\[= \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) G^{-1} \text{diag}(e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_n}),\]

where \(C\) and \(D\) are tridiagonal matrices of the form

\[
C = \begin{bmatrix}
1 & c_{12} & & \\
& 2 & c_{23} & \\
& & \ddots & \ddots & \ddots \\
& & & c_{n-1,n-2} & 2 & c_{n-1,n} \\
& & & & c_{n,n-1} & 3
\end{bmatrix}
\]

and

\[
D = \begin{bmatrix}
0 & s_{12} & & \\
& 0 & s_{23} & \\
& & \ddots & \ddots & \ddots \\
& & & 0 & s_{n-1,n} & \\
& & & & 0 & s_{n,n-1}
\end{bmatrix}
\]

with

\[c_{kl} = -\cos(\theta_k - \theta_l), \quad s_{kl} = -\sin(\theta_k - \theta_l).\]

Since \((A + i B)(C + i D) = I\), we have

\[AC - BD = I, \quad AD + BC = 0\]

and it follows that

\[C = A^{-1} + A^{-1} BD = A^{-1} + A^{-1} BCC^{-1} D = A^{-1} - DC^{-1} D\]

or \(A^{-1} = C + DC^{-1} D\). We also have \(A^{-1} B = -DC^{-1}\) and system (1.2.8) can be written as

\[\ddot{\theta} = DC^{-1} (\dot{\theta}^2 + Dg(t, \theta)) + Cg(t, \theta), \quad t \in [0, T].\]

As observed by Hairer et al. [143] this suggests the following efficient algorithm for computation of the acceleration vector \(\ddot{\theta}\).

1. Compute \(w = \dot{\theta}^2 + Dg(t, \theta)\).
2. Solve the tridiagonal system \(Cu = w\).
3. Compute \(\ddot{\theta} = Du + Cg(t, \theta)\).
BRUS — a reaction-diffusion equation (the Brusselator with diffusion) [143]. This is the system of partial differential equations of the form

$$\begin{align*}
\frac{\partial u}{\partial t} &= 1 + u^2 v - 4.4u + \alpha \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\
\frac{\partial v}{\partial t} &= 3.4u - u^2 v + \alpha \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right),
\end{align*}$$

(1.2.9)

$$0 \leq x \leq 1, \ 0 \leq y \leq 1, \ t \geq 0, \ \alpha = 2 \times 10^{-2},$$

together with the Neumann boundary conditions

$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0,$$

where \( n \) is the normal vector to the boundary of the region \([0,1] \times [0,1]\) and the initial conditions

$$u(x, y, 0) = 0.5 + y, \quad v(x, y, 0) = 1 + 5x.$$

Let \( N > 1 \) be an integer and define the grid in space variables \( x \) and \( y \) by

$$x_i = (i - 1)\Delta x, \quad y_j = (j - 1)\Delta y, \quad i, j = 1, 2, \ldots, N,$$

where \( \Delta x = \Delta y = 1/(N - 1) \). Define also the functions

$$U_{ij}(t) = u(x_i, y_j, t), \quad V_{ij}(t) = v(x_i, y_j, t), \quad i, j = 1, 2, \ldots, N.$$

Discretizing (1.2.9) by the method of lines, where the space derivatives are approximated by finite differences of second order leads to the system of ordinary differential equations

$$\begin{align*}
U'_{ij} &= 1 + U^2_{ij} V_{ij} - 4.4U_{ij} \\
&\quad + \frac{\alpha}{\Delta x^2} \left( U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{ij} \right), \\
V'_{ij} &= 3.4U_{ij} - U^2_{ij} V_{ij} \\
&\quad + \frac{\alpha}{\Delta y^2} \left( V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{ij} \right),
\end{align*}$$

(1.2.10)

t \in [0, T], \ i, j = 1, 2, \ldots, N, \ of \ dimension \ 2N^2. \ The \ boundary \ conditions \ imply \ that

$$U_{0,j} = U_{2,j}, \quad U_{N+1,j} = U_{N-1,j}, \quad U_{i,0} = U_{i,2}, \quad U_{i,N+1} = U_{i,N-1},$$

$$V_{0,j} = V_{2,j}, \quad V_{N+1,j} = V_{N-1,j}, \quad V_{i,0} = V_{i,2}, \quad V_{i,N+1} = V_{i,N-1},$$

and the initial conditions are

$$U_{ij}(0) = 0.5 + y_j, \quad V_{ij}(0) = 1 + 5x_i, \quad i, j = 1, 2, \ldots, N.$$
1.3 EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this section we formulate results regarding the existence and uniqueness of solutions to (1.1.2). We begin with the classical Peano existence theorem for the system (1.1.2), which assumes only that the function $g$ is continuous in some domain. The proof of this result is based on the Arzela-Ascoli theorem about a family of vector-valued functions that is uniformly bounded and equicontinuous. Consider a family $\mathcal{F}$ of vector-valued functions $y = y(t)$ defined on an interval $I = \{ t : |t - t_0| \leq T \}$. Define $\|y\| := \sup\{\|y(t)\| : t \in I\}$, where $\| \cdot \|$ is any norm on $\mathbb{R}^m$. We introduce the following definitions.

**Definition 1.3.1** A family $\mathcal{F}$ of vector-valued functions $y = y(t)$ is said to be uniformly bounded if there exists a constant $M$ such that $\|y\| \leq M$ for every $y \in \mathcal{F}$.

**Definition 1.3.2** A family $\mathcal{F}$ of vector-valued functions $y = y(t)$ is said to be equicontinuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that the condition $|t - s| < \delta$, $t, s \in I$, implies that $\|y(t) - y(s)\| < \epsilon$ for all functions $y \in \mathcal{F}$.

**Theorem 1.3.3** (Arzela-Ascoli; see [212]) Let $y_n(t)$, $n = 1, 2, \ldots$, be a uniformly bounded and equicontinuous sequence of vector functions defined on the interval $I$. Then there exists a subsequence $y_{n_j}(t)$, $j = 1, 2, \ldots$, which is uniformly convergent on $I$.

We are now ready to formulate and prove the classical existence result for system (1.1.2).

**Theorem 1.3.4** (Peano [235]) Assume that the function $g(t, y)$ is continuous in the domain

$$D = \left\{(t, y) : |t - t_0| \leq T, \|y - y_0\| \leq K\right\} \quad (1.3.1)$$

and that there exists a constant $M$ such that $\|g(t, y)\| \leq M$ for $(t, y) \in D$. Then system (1.1.2) has at least one solution $y = y(t)$ defined for

$$|t - t_0| \leq T_1 := \min\{T, K/M\}$$

and passing through the point $(t_0, y_0)$.

**Proof:** It is generally agreed that the original proof of Peano [235] was inadequate, and a satisfactory proof was found many years later (see e.g., Perron [236]). Here, we follow the presentation given by Birkhoff and Rota [21]. We will prove the theorem for the interval $[t_0, t_0 + T_1]$; the proof for the interval $[t_0 - T_1, t_0]$ is analogous. Consider the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} g(s, y(s))ds, \quad (1.3.2)$$
Define the sequence of functions 
\[ y_n = y_n(t), \quad n = 1, 2, \ldots, \]
by the formulas
\[
y_n(t) = \begin{cases} 
  y_0, & t \in [t_0, t_0 + T_1/n], \\
  y_0 + \int_{t_0}^{t-T_1/n} g(s, y_n(s))\,ds, & t \in (t_0 + T_1/n, t_0 + T_1].
\end{cases}
\]

Observe that the right-hand side of the second formula above defines \( y_n(t) \) for \( t \in (t_0 + T_1/n, t_0 + T_1] \) in terms of \( y_n(t) \) already defined for \( t \in [t_0, t_0 + T_1/n] \). This sequence is well defined since
\[
\left\| y_n(t) - y_0 \right\| \leq \int_{t_0}^{t-T_1/n} \left\| g(s, y_n(s)) \right\| \,ds \\
\leq M \left( t - t_0 - \frac{T_1}{n} \right) \leq M(t - t_0) \leq MT_1 \leq K
\]
and \( y_n(t) \) are clearly continuous on \([t_0, t_0 + T_1] \). We have \( \|y_n(t)\| \leq \|y_0\| + MT_1 \), which shows that the sequence \( y_n(t) \) is uniformly bounded. We also have
\[
\left\| y_n(t_2) - y_n(t_1) \right\| \leq \int_{t_1 - T_1/n}^{t_2 - T_1/n} \left\| g(s, y_n(s)) \right\| \,ds \leq M|t_2 - t_1|
\]
which shows that \( y_n(t) \) is also equicontinuous. Hence, it follows from the Arzela-Ascoli theorem, Theorem 1.3.3, that there exists a subsequence \( y_{n_j}(t) \), \( j = 1, 2, \ldots \), which is uniformly convergent to a continuous function \( \overline{y}(t) \); that is,
\[
\lim_{j \to \infty} y_{n_j} = \overline{y}.
\]

This subsequence satisfies the integral equation, which we write in the form
\[
y_{n_j}(t) = \int_{t_0}^{t} g(s, y_{n_j}(s))\,ds - \int_{t-T_1/n_j}^{t} g(s, y_{n_j}(s))\,ds.
\]
We have
\[
\lim_{j \to \infty} \int_{t_0}^{t} g(s, y_{n_j}(s))\,ds = \int_{t_0}^{t} g(s, \overline{y}(s))\,ds
\]
since the function \( g(t, y) \) is uniformly continuous. We also have
\[
\left\| \int_{t-T_1/n_j}^{t} g(s, y_{n_j}(s))\,ds \right\| \leq M \frac{T_1}{n_j} \to 0 \quad \text{as} \quad j \to \infty.
\]
Hence, passing to the limit as \( j \to \infty \) in the integral equation for \( y_{n_j} \), we obtain
\[
\overline{y}(t) = y_0 + \int_{t_0}^{t} g(s, \overline{y}(s))\,ds,
\]
which proves that \( g(t) \) satisfies the integral equation (1.3.2); hence it also satisfies (1.1.2) for \( t \in [t_0, t_0 + T_1] \).

A solution whose existence is guaranteed by the Peano theorem, Theorem 1.3.4, is not necessarily unique. A simple example that illustrates this is given by a scalar initial value problem

\[
y' = 3y^{2/3}, \quad y(0) = 0,
\]

where \( D = \{(t, y) : |t| \leq 1, |y| \leq 1\} \). Here the function \( g(t, y) = y^{2/3} \) is continuous on \( D \), but the problem has solutions \( y_1(t) = 0 \) and \( y_2(t) = t^3 \).

Assume that a function \( g(t, y) \) is defined in some region \( R \subset \mathbb{R} \times \mathbb{R}^m \). To formulate uniqueness results for (1.1.2), we usually assume that the function \( g(t, y) \) is not only continuous but satisfies some additional regularity properties. We introduce the following definitions.

**Definition 1.3.5** A function \( g(t, y) \) satisfies a Lipschitz condition in \( R \) with a Lipschitz constant \( L \) if

\[
||g(t, y_1) - g(t, y_2)|| \leq L||y_1 - y_2||
\]

for all \((t, y_1), (t, y_2) \in R\), where \( \cdot \) is any norm in \( \mathbb{R}^m \).

**Definition 1.3.6** A function \( g(t, y) \) satisfies a one-sided Lipschitz condition in \( R \) with a one-sided Lipschitz constant \( \nu \) if

\[
(g(t, y_1) - g(t, y_2))^T(y_1 - y_2) \leq \nu||y_1 - y_2||^2
\]

for all \((t, y_1), (t, y_2) \in R\). Here \( \cdot \) is the Euclidean norm in \( \mathbb{R}^m \); that is, \( ||u|| := \sqrt{u^Tu} \) for \( u \in \mathbb{R}^m \).

One-sided Lipschitz condition (1.3.4) plays an important role in the analysis of numerical methods for stiff systems of ODEs (compare [109, 146]). Assume that the function \( g(t, y) \) satisfies Lipschitz condition (1.3.3) in the Euclidean norm \( || \cdot || \) with a constant \( L \). Then using the Schwartz inequality and (1.3.3), we obtain

\[
(g(t, y_1) - g(t, y_2))^T(y_1 - y_2) \leq ||g(t, y_1) - g(t, y_2)||||y_1 - y_2|| \leq L||y_1 - y_2||^2
\]

and it follows that \( g(t, y) \) also satisfies one-sided Lipschitz condition (1.3.4) with the same constant \( L \). However, as observed, for example, by Dekker and Verwer [109], the reverse is not true. A counterexample is provided by any monotonically nonincreasing function \( g : \mathbb{R} \to \mathbb{R} \) which has, for some value of \( \bar{y} \in \mathbb{R} \), an infinite slope. We then have

\[
(g(y_1) - g(y_2))(y_1 - y_2) \leq 0
\]

for all \( y_1, y_2 \in \mathbb{R} \) and it follows that \( g(y) \) satisfies (1.3.4) with \( \nu = 0 \). However, this function does not satisfy (1.3.3) in any neighborhood of \( \bar{y} \), where the slope is infinite.
Next we formulate a local existence and uniqueness theorem. We also show that the solution to (1.1.2) can be obtained as a limit of a uniformly convergent sequence of continuous functions starting with an arbitrary initial function that satisfies the appropriate initial condition.

**Theorem 1.3.7** Assume that the function \( g(t, y) \) is continuous and satisfies a Lipschitz condition (1.3.3) in the domain \( D \) defined by (1.3.1). Set

\[
M = \max \left\{ \|g(t, y)\| : (t, y) \in D \right\}.
\]

Then (1.1.2) has a unique solution defined on the interval

\[
|t - t_0| \leq T_1 := \min\{T, K/M\}
\]

passing through \((t_0, y_0)\).

**Proof:** First consider the interval \([t_0, t_0 + T_1]\); the proof for the interval \([t_0 - T_1, t_0]\) is analogous. Define the integral operator

\[
z(t) = \phi(y(t)) := y_0 + \int_{t_0}^{t} g(s, y(s))\,ds, \quad (1.3.5)
\]

\( t \in [t_0, t_0 + T_1]. \) Put \( Y = \{y \in \mathbb{R}^m : \|y - y_0\| \leq K\} \) and denote by \( C([t_0, t_0 + T_1], Y) \) the space of continuous functions from \([t_0, t_0 + T_1]\) into \( Y \) with a uniform norm. Observe that if \( y \in C([t_0, t_0 + T_1], Y) \), then

\[
\|z(t) - y_0\| \leq \int_{t_0}^{t} \|g(s, y(s))\|\,ds \leq MT_1 \leq MK/M = K,
\]

and it follows that the operator \( \phi \) takes the functions from \( C([t_0, t_0 + T_1], Y) \) into \( C([t_0, t_0 + T_1], Y) \):

\[
\phi : C([t_0, t_0 + T_1], Y) \to C([t_0, t_0 + T_1], Y).
\]

Define the sequence of functions \( y_n(t) \in C([t_0, t_0 + T_1], Y) \) by the formula

\[
y_{n+1}(t) = \phi(y_n(t)) = y_0 + \int_{t_0}^{t} g(s, y_n(s))\,ds, \quad (1.3.6)
\]

\( n = 0, 1, \ldots, \) where \( y_0(t) \equiv y_0, \ t \in [t_0, t_0 + T_1]. \) Then we have the bound

\[
\|y_n(t) - y_{n-1}(t)\| \leq \frac{ML^{n-1}(t - t_0)^n}{n!} \quad (1.3.7)
\]

We prove (1.3.7) by induction with respect to \( n. \) Since

\[
\|y_1(t) - y_0(t)\| \leq \int_{t_0}^{t} \|g(s, y_0(s))\|\,ds \leq M(t - t_0),
\]