MECHANICS OF OPTIMAL STRUCTURAL DESIGN

Minimum Weight Structures

David W. A. Rees
School of Engineering and Design, Brunel University, Uxbridge, UK

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This edition first published 2009

Registered office
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

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Library of Congress Cataloguing-in-Publication Data

Rees, D. W. A. (David W. A.), 1947-
Mechanics of optimal structural design : minimum weight structures / David Rees.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-470-74623-3 (cloth)
1. Lightweight construction. I. Title.
TA663.R43 2009
624.1′7–dc22

2009025913

A catalogue record for this book is available from the British Library.
ISBN: 978-0-470-74623-3

Typeset in 10/12 Times by Laserwords Private Limited, Chennai, India.
Printed and bound in Singapore by Markono

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Preface

Design is the process of finding a solution to a problem, with all the constraints and requirements that it presents. Accordingly, there would normally be more than one design solution, depending upon the requirements perceived from the application. Specifically, an optimum design will narrow the solutions to one or two depending upon the criteria adopted. For example, the criteria may relate to structural form, the material characteristics, ease of manufacture and assembly, a demand for standard parts, or a specified mechanical performance involving aerodynamics, heat transfer, wear resistance, etc. In principle, the designer would wish to optimise some merit factor common to all of these. In practice, an optimisation can be achieved when related items are considered together but more usually specific requirements are optimised in isolation, as with attaining minimal waste in machining and maximal heat transfer in material selection. In this book we are concerned with minimising the weight of load bearing structures through making informed choices upon their structural forms. In fact, we find that this problem can be treated in terms of its building blocks. This amounts to optimising all the key elements of a structure – struts, ties, beams, shear panels, plates, etc. – in terms of their strength and weight contributions.

An almost infinite number of structural forms can bear load. Whilst a strut, a beam, a braced frame, a tripod would usually be adequate, the choice rests with which is best for the particular application. In the fields of aerospace, transport and structural engineering the reduction in weight has become a key factor with their associated environmental issues. Quite apart from the obvious reduction in material cost, the improvement in performance from the reduction in weight becomes an overriding factor when designing to reduce emissions in road vehicles and aircraft. In this context the usual reference is made to a strength-to-weight ratio, which implies a maximising of strength for a given weight. In fact, it is the weight that is to be minimised with respect to a required strength and so the term weight-to-strength is a more precise description of the approach to be adopted here.

Though the structures considered in this book are many and varied, they share the common theme of optimising their weight to strength. The question as to whether an optimised design exists for thin-walled tubes, Ts, Is and channel sections is answered by applying additional design criteria involving both global and local buckling. Generally, the solution is based upon similar loading stresses from all sources being reached simultaneously within each design criterion adopted. For example, in a thin-walled open strut, with an I or T cross-section, Euler's global stress for flexural buckling in the length and the local buckling stress for the sections' flanges are made to coincide with the limiting design stress. Struts as compression members, both wide and narrow, appear typically as columns, pillars and walls (Chapters 1 and 2). Beams and bar sections under uniform bending moments and torques are treated in Chapters 3 and 4. Various types of strut and beam sections are covered. Solid sections in standard shapes are usually more amenable to an optimum design, but tubular and thin-walled sections obviously present less weight at the outset. Thin sections present wall thickness as a further variable to an optimum design and an additional, local buckling failure mode within their walls.

Transverse shear and torsion in thin-walled tubes are given special consideration (Chapters 5 and 6) because of their ability to sustain a constant shear flow irrespective of the wall thickness, unlike an open section. However, a pure flexure is only possible when shear forces act at the shear centre of a tube. With shear forces displaced from the shear centre, an example of combined loading upon a structure arises through bending, torsion and shear. The effect is similar to when a torque is not applied at the centre of twist, this giving rise to a similar loading combination (Chapter 6). Here we might simplify the problem by idealising a section in which webs carry shear force and booms carry bending moments. The interested reader can find the detail of these more subtle aspects of optimum design within Chapter 6 and 7. Methods of reducing weight while preserving the shear resistance of webs are considered in Chapter 8. Corrugations, Z and top-hat stiffeners are effective in offsetting the risk of buckling in thin webs over long lengths and depths. A method of optimising the design of pin-jointed frame assemblies is given in Chapter 9. Whilst the structures considered consist of relatively few bars, they are sufficient to show how the approach may be extended to plane and three-dimensional frames with many bars.

Beams which feature as lateral carriers for vertical loading are allowed in Chapters 10 and 11 to vary in cross-section. In particular, the beam supports are altered from resting upon simple knife edges or rollers to having one or both ends built in. The influence of both shear force and bending moment upon an optimum beam design has a significant influence upon weight reductions compared to standard designs in uniform section. Savings of over 50% are possible by allowing the section to vary whilst maintaining the load bearing integrity. In the case of combined bending and shear in beams, their effects can be separated because at the surface, where the bending stress is at its maximum, the shear stress is zero. Conversely, at the neutral axis, where the bending stress is zero, the shear stress is at its maximum. So in a given cross-section we need only alternate from their respective maximum stress positions as either bending or shear governs the design of the section depth. However, we might seek an ideal optimum design in which all points in the depth of a section are stressed equally, say at a given equivalent (von Mises) stress. Chapter 11 shows that all the common solid beam sections are inefficient, even with the aforementioned weight savings. The ideal section is found numerically, so that when its depth is allowed to change with the varying force and moment, the greatest possible weight saving is achieved.

In Chapter 12 further examples of combined loading in structures present themselves in various ways: tension-torsion in shafts, bending and torsion of beams, compression and bending of struts and combinations of concentrated and distributed loadings in beams. Limbs are separated within free-body diagrams within which the design stress is limited by a yield criterion before the appropriate loading criteria are applied. Here the von Mises yield criterion is used to account for combined loading and also in converting material yield stress from the shear to the tension mode.

The fully encastré beam (Chapter 13), being a statically indeterminate structure, is optimised for minimum weight as it is known that these provide a stiffer, lighter structure than simply-supported beams bearing similar loading. It is shown that encastré beams can operate at minimum weight with a variable profile in circular and square cross-sections for central point loading and uniformly distributed loading acting separately and in combination. Both bending and shear effects are combined within a true optimum design. In applying the limiting stresses from bending and shear, regions of the length appear over which the beam section is optimised, depending upon which design criterion dominates.

In all that has been mentioned these designs are essentially elastic since the limiting stress is normally set at the yield or at a proof stress involving less than 1% plastic strain. Some allowance for a greater degree of plasticity is allowed in the buckling of plates by adopting a tangent modulus in the critical stress formula. Gross plasticity is broached in the final two Chapters 14 and 15 where an optimum design at the ultimate strength of a material is given. This is made possible by the application of energy balance equations to a collapsing structure. A collapsing structure is taken to behave as a rigid mechanism following the development of plastic hinges at the ultimate stress level of its elements. Given that there may be many ways for a combination of these hinges to form possible collapse mechanisms, the optimum plastic design is based upon collapse under the least loading. In achieving an efficient design, we can alter the section geometry to make possible the coincidence of a number of mechanisms under the applied loading. Graphical and dynamic programming techniques are shown to assist with this procedure in Chapter 15.

There are a number of books on the subject of optimum structural design (see the Further Reading section of Chapter 9) but none present the material in quite the same way as the reader will find here. While the basic method of weight to strength optimisation remains common to all structures, it has been embellished with all the necessary mechanics to allow for this. The following topics feature among the supporting material which require an understanding through their optimum design applications: Euler strut buckling, torsion of non-circular sections, buckling of plates in compression and shear, shear flow in thin sections, the construction of shear force and bending moment diagrams, properties of areas, the principle of virtual work and plastic collapse.

The author would like to express his gratitude to J. J. Richardson for his invitation some years ago to attend his inspiring short course, ‘An Introduction to Optimum Structural Design’, within the Department of Mechanical, Aeronautical and Production Engineering at Kingston Polytechnic (now Kingston University). The course content is reflected within earlier chapters of this book and underpins the author’s developments of all integrated new material. As a student receiving new material there appeared, within this carefully structured and thought-provoking course, a clear need for further work to be done in certain areas. Faced with that challenge, the author has produced this book to inform optimum design practice and perhaps inspire the reader in turn to advance the subject further. The challenge is extended to all readers to make a contribution to weight saving in engineering applications, especially to those environmentally sensitive designs involving the planet’s dwindling resources and fragile ecological system.

Glossary of Terms

The following are among the most common terms that appear on the subject of mechanics in the optimum design of structures. The terms are centred upon the common theme of optimising the weight to strength of a structure. This involves applying design criteria to all failure modes, both global and local. The optimum solution is based upon similar loading stresses from each design criterion adopted being reached simultaneously.

Design constraints: Many factors impose limits on the design variables. For example, restrictions may be placed upon material, geometry, available space, use of non standard parts, and fabrication methods.

Design variables: These refer to the control of the shape and proportion of a structure as imposed by the designer.

Failure criteria: These refer to the limits placed upon the structural response to the applied loading. Commonly, this amounts to limiting stress to that which will produce the onset of plastic yielding, local necking, unstable buckling and brittle fracture. Optimisation requires that, where such modes co-exist within a structure, their critical stresses are attained simultaneously.

Loading: This refers to all types of external actions, including tension, compression, bending, shear and torsion, all of which must be specified in magnitude and direction in terms of the internal stress that they induce.

Material factor (or material efficiency) M or m: An independent variable appearing as an argument of the objective function expression. The analyses that follow reveal that the material efficiency appears consistently as a ratio between particular material properties such as density, elastic modulus and yield stress, typically $M = \sqrt{E}/\rho$ for struts, $m = \rho/\sigma_y^{2/3}$ for beams, thereby revealing how material choice influences the objective function.

Objective function: Generally, a mathematical function of a number of independent structural variables arises from imposing any criterion that is to be maximised or minimised. Here the objective function g in equation (1) refers to an expression that will serve to minimise the weight for a given strength. Often this is more easily achieved by expressing the weight/strength ratio as an equivalent ratio R between two properties. This ratio may either be minimised or maximised, depending upon whether it is directly proportional to or inversely proportional to the weight/strength ratio. The objective function itself may be split into the product of three independent variables – the shape factor F, the material factor M and the structural index S – to allow their separate influences upon weight to be examined. Thus, we may express these dependencies in the objective function equation

$$R = g(F, M, S) \tag{1}$$

Hence the essential pre-requisite for an optimisation procedure is to derive the arguments F, M and S in equation (1). Despite there being a large number of sections available as struts, beams and shear webs, the procedure for the optimisation of R is constant for each type of loading. For example, in the struts considered in Chapters 1 and 2, the independent variables in equation (1) are found to be separable as

$$R = F \times M \times S^n \tag{2}$$

where the exponent n is fractional. Thus, for a given M and S, a maximum in F gives a maximum in R, this being a common route to minimising the weight of struts.

Problem definition: This refers to a clear analysis of all factors that impose upon the eventual design, including the loading, geometry and material. An optimum design will depend entirely upon the criteria set within its definition.

Shape efficiency factor F: This is used with struts, where it is inversely proportional to weight. For standard sections, e.g. square, circular and tubular, F appears as a numerical coefficient in which a maximum is sought. Here F-values usually lie in the range $0 < F < 1$ (see Table 1.4, p. 25), indicating that the effect of F upon R is less influential than M and S in equation (1). For non-uniform, thin-walled sections, F depends upon the section geometry, so allowing their dimensions to be optimised.

Shape factor f: This is used with beams and shear webs, where it is proportional to weight. For standard sections, e.g. square, circular and tubular, f appears as a numerical coefficient in which a minimum is sought. Here $f > 1$ (see Table 3.1, p. 88). For open and closed sections, with non-uniform thin walls, f depends upon the section geometry, so allowing their dimensions to be optimised.

Structural index S or s: This refers to the loading expressed in a form to facilitate the minimum weight to strength analysis of a structure. The unit of the index S corresponds to that of stress, i.e MPa or N/m^2. Thus, for struts and plates of length L under a compression (P or w), the index is P/L^2 and w/L, while for beams in bending, the index M/D^3 refers to a circular section of diameter D. Here the loading symbols are: a concentrated force P, a force/unit width w and an applied moment M. In equation (1) the index S, as an argument of the objective function g, is found to bear the non-linear relation to R for struts and beams given in equation (2). The index S that optimises R is then sought by mathematical means, i.e. differentiation for the function's turning points or by alternative graphical methods. For many structures a non-dimensional structural index $s = S/\sigma_y$, where σ_y is the yield stress, may be used across five decades (i.e. $10^{-3} \leq s \leq 10^2$) to show the optimum weight dependence (below).

Transitions in s between beam section designs based upon the bending moment M and shear force F are denoted by s_{FM} and s_{MF}.

Weight parameter W or n: Weight W is the quantity to be minimised in all structures. Weight arises from the mass of their structural elements: beams, bars plates, tubes, etc. A non-dimensional weight parameter n is used with s (above) to show graphically the optimum weight range of most structural geometries for a practical range of working loads. The normalised weight parameter is defined as $n = (1/\rho)W/L^3$, where ρ is the material density (see equation (3) below). In a structure with varying section the s versus n relation is found from the integration of an optimum section profile (see below).

Key Symbols

In addition to the symbols reserved for the glossary terms above, other general symbols are employed to denote the following. The list is not intended to be exhaustive as various specialist terms and their symbols are defined more meaningfully where they first appear in the text.

Applied loading: This is a generic term denoting the relatively few external actions that induce stress within a structure. These are: tension and compression P, bending moment M, shear force F, torque T, pressure and distributed loading w. The problem is complicated when two or more loads act in combination.

Buckling coefficient: Appearing within both global and local buckling formulae, K connects the critical stress to an aspect ratio, e.g. the thickness/width ratio for a thin plate. Moreover, the dependence of K upon different edge fixings – hinged, fixed, etc. – is represented typically in design data sheets (derived from the listed published sources). Buckling is an especially important criterion of failure to consider under shear and compression of thin-walled sections. In slender struts, for example, a local buckling of webs and flanges within the section is taken concurrently with flexural buckling of the length.

Dimensionless parameters: Non-dimensional load, weight and length parameters *s, n, q* are especially useful for the analysis of beams (see Glossary). Thus the optimum weight is modified to $n = (1/\rho)W/L^3$ and the structural index is written as $s = (1/\sigma_y)F/L^2$. Here, when finding the beam contour, it is found convenient to normalise with length both the half depth ($d/2L$) and the length position ($q = z/L$). Correspondingly, these might appear within a non-dimensional form of the objective function (2), as follows:

$$\frac{W}{L^3} = f\left(\frac{\rho}{{\sigma_y}^n}\right)\left(\frac{F}{L^2}\right)^n, \quad \Rightarrow \quad \frac{1}{\rho}\frac{W}{L^3} = f\left(\frac{1}{\sigma_y}\frac{F}{L^2}\right)^n, \quad \Rightarrow \quad n = fs^n \qquad (3)$$

Plots of n versus s appear for four standard cases of beam loadings in which their sections are circular, square and rectangular in their optimised forms. By allowing s to scan 5 decades within these plots all manner of applications are contained, both within short studs and rivets placed under shear and in the extended lengths required of transversely loaded beams with simple and encastré supports.

Elastic modulus: The elastic modulus in tension E and the shear modulus G appear with various subscripts: S and T to denote secant and tangent moduli. These are employed where some measure of plasticity is admitted to the design. The tangent modulus $E_T = \mathrm{d}\sigma/\mathrm{d}\varepsilon$ is the gradient to the stress–strain curve at a point in the plastic region corresponding to a given offset plastic strain. Alternatively, the ratio between the coordinates for this point define a secant modulus $E_S = \sigma/\varepsilon$. For either definition, a suitable empirical description of the stress–strain curve is required. For this the Ramberg–Osgood equation is used in which elastic and plastic strains are additive (see Appendix B, p. 531).

Properties of areas: Any section area A has four important properties that appear throughout this book. They are $(\overline{x}, \overline{y})$ denoting the coordinates of the centroid position, i the first moment of area, I the second moment of area and $k(= \sqrt{I/A})$ the radius of gyration of the section. Both i and I refer to axes passing through the centroid. Subscripts (x, y) are used for general centroidal axes and (u, v) for principal axes.

Section geometry: This appears within symbols b, d, t and p, denoting breadth, depth, thickness and pitch, respectively. These may appear with subscripts, say, t_w and t_f, denoting the thickness of the web and the flange of a T-section. Having minimised weight, the section geometry is said to be optimised. That is, a unique combination of dimensions will appear to ensure that all the design criteria are met. The preferred design is one which ensures that failures from all potential sources occur together rather than a design based upon one or other failure mode.

Strain and displacement: Corresponding to direct and shear stress (below), the direct and shear strains are denoted as ε and γ. Direct strain arises from a dimensional change, i.e. a lengthening or a shortening (say, x) arising from tension and compression, respectively. Shear strain refers to the angular distortion (in radians), arising from shear force and torsion, between two initially perpendicular directions. Here a shear distortion or a twist (θ) occurs between the unstrained reference directions. Beam curvature R is involved where surface tension and compression occur together from bending. In-line displacements (δ or Δ) appear beneath the loads applied to pin-jointed structures.

Stress: This is the measure of a material's resistance to the applied loading, whether it be a direct stress σ or a shear stress τ. A safe design is ensured by placing a limit upon stress, and an optimum structural design is achieved where this limit (the design stress) is reached simultaneously from its various sources: flexure, compression, buckling, torsion, etc. The stress measure known as *shear flow* q refers to the product τt reserved for the stress measure in thin-walled sections under torsion and flexural shear.

Weight: The two weight measures, W and n, depend upon section geometry (see Glossary above), length L and material density ρ. For varying cross-sections, the weight integral depends upon position z in the length. Where, for example, an optimum design proposes that diameter d varies with length z, we have the weight integral

$$W = \frac{\rho\pi}{4}\int_{z=0}^{z=L} d^2\,\mathrm{d}z \quad \Rightarrow \quad n = \frac{1}{\rho}\frac{W}{L^3} = \rho\pi\int_{q=0}^{q=1}\left(\frac{d}{2L}\right)^2 \mathrm{d}q \qquad (4)$$

The corresponding normalised weight parameter n in equation (4) employs $q = z/L$ and $d/2L$. The integration of equation (4) leads to the form of equation (3), in which it appears that n depends upon both s and q.

1

Compression of Slender Struts

1.1 Introduction

The stress σ within a long slender strut of uniform cross-section is affected by the magnitude of the load applied P and its length L. It will be shown that the weight of the strut is a minimum when the stress is a maximum. It is therefore necessary to investigate how σ varies with both P and L and the shape of the cross-section, each being under the control of the designer. The general approach is to seek an objective function in which the strut's weight W is expressed as the product of the strut's volume and density ρ. This gives

$$W = \rho AL = \rho \left(\frac{P}{\sigma}\right) L = \frac{PL}{\sigma/\rho} \tag{1.1}$$

where the elastic stress σ in the strut prior to its buckling is equated to the axial load per unit area, i.e. $\sigma = P/A$, and L is the 'pinned' strut length. Hence, to minimise W it follows from equation (1.1) that σ/ρ, the equivalent objective function, is to be maximised. The following failure criteria provide the various limiting stress measures upon which the strut's minimum weight is to be based.

1.2 Failure Criteria

The failure criteria for a strut would need to be expressed in terms of the section's shape, whether this be solid, hollow, thin-walled, tubular, etc.

1.2.1 Flexural Buckling

Euler's theory [1] expresses the critical elastic buckling load P_c for a pinned-end strut as

$$P_c = \frac{\pi^2 EI}{L^2} \tag{1.2a}$$

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where $I = Ak^2$ is the least second moment of area, which depends upon the radius of gyration of the section. Hence, the critical flexural buckling stress σ_F may be expressed as

$$\sigma_F = \frac{\pi^2 E}{(L/k)^2} \tag{1.2b}$$

In shorter, stockier struts, where buckling is elastic-plastic, the tangent modulus E_T may replace the elastic modulus E in equations (1.2a,b).

1.2.2 Local Buckling

A local buckling failure refers specifically to struts with thin walls in their cross-sections. Typically, this mode of failure appears as an indentation of diamond shape upon the surface or in a bowing of the section walls [2]. Local buckling does not arise in struts with solid sections. For buckling of the flat plates (i.e. the walls) within thin-walled tubular sections, the local buckling stress takes the form [3]

$$\sigma_L = K_L\ E_T \left(\frac{t}{d}\right) \tag{1.3}$$

where K_L is a buckling coefficient that depends upon the plates aspect ratio and the support provided to its edges (see equation (D.1a,b) in Appendix D).

1.2.3 Working Stress

In the absence of buckling, the axial, compressive, working stress σ_W is found simply by dividing the applied load P by the section area A:

$$\sigma_W = \frac{P}{A} \tag{1.4}$$

1.2.4 Limiting Stress

The stress in equations (1.2b), (1.3) and (1.4) would normally be limited to the yield stress where buckling is elastic. In the case of plastic buckling the limiting stress is raised to correspond to a given offset (plastic) strain, i.e. the 0.1% proof stress. Let σ_y be the limiting yield or proof stress of the strut material appropriate to its buckling behaviour. Then, its relation to the applied stress (σ_F, σ_L and σ_W above) from those sources in equations (1.2b)–(1.4), is simply

$$\sigma \leq \sigma_y \tag{1.5}$$

1.2.5 Objective Function

An optimum section size is found by equating (1.2b), (1.3) (where appropriate) and (1.4), where they all have been limited by equation (1.5). Finally, all appropriate failure criteria in § 1.2.1–1.2.4 are combined within the objective function to minimise the weight. We

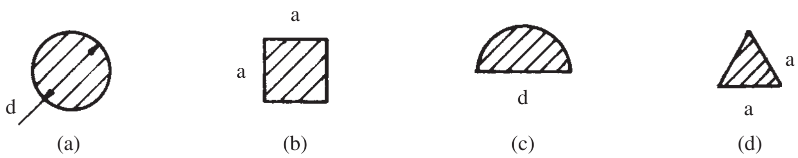


Figure 1.1 Standard, solid strut cross-sections

shall demonstrate this design procedure, firstly, with the more common solid cross-sections in Figure 1.1.

1.3 Solid Cross-Sections

The four solid cross-section shown in Figure 1.1 are the most likely contenders for strut cross-sections as these are available in long bars of extruded stock.

1.3.1 Circular Section, Diameter d (see Figure 1.1a)

With $I = \pi d^4/64$, then $k^2 = d^2/16$ and the buckling failure criterion (1.2a), becomes

$$\sigma_F = \frac{\pi^2 E d^2}{16L^2} \tag{1.6}$$

When equation (1.6) is combined with the axial stress formula (1.4) with $A = \pi d^2/4$, so that $\sigma_F = \sigma_W$, this gives the optimum diameter:

$$d_{\text{opt}} = \frac{(64P)^{1/4} L^{1/2}}{\pi^{3/4} E^{1/4}} = 1.199 \left(\frac{PL^2}{E}\right)^{1/4} \tag{1.7a}$$

It will be seen that all solid sections will conform to an equation of similar form for an optimum section dimension (here the diameter) as:

$$d_{\text{opt}} = C \left(\frac{PL^2}{E}\right)^{1/4} \tag{1.7b}$$

where C is a *shape coefficient*. Substituting equation (1.7a) into equation (1.6) and dividing by ρ leads to the equivalent objective function for a solid, circular-section strut:

$$\left(\frac{\sigma}{\rho}\right)_{\text{opt}} = 0.886 \left(\frac{E^{1/2}}{\rho}\right) \left(\frac{P}{L^2}\right)^{1/2} \tag{1.8a}$$

Here, a *shape efficiency factor* $F = 0.886$ appears. The *material efficiency factor* is $M = \sqrt{E}/\rho$ and the *structural index* $S = P/L^2$ is raised to the fractional power ½. Hence, we may write the quantity to be maximised, the *objective function* R, more generally as

$$R = \left(\frac{\sigma}{\rho}\right)_{\text{opt}} = F \times M \times S^n \tag{1.8b}$$

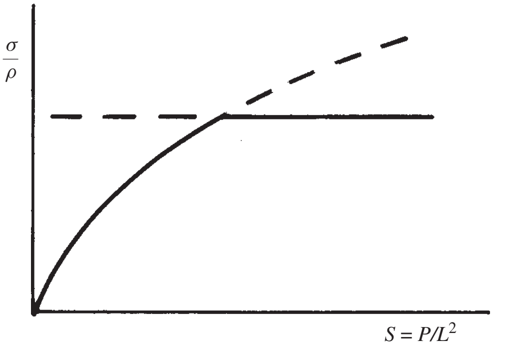


Figure 1.2 Objective function plot from equation (1.8a) showing limiting stress cut-off

where, for a circular cross-section, $n = {}^1\!/_2$. If we wish to employ a tangent modulus E_T this will reduce M by the ratio $\sqrt{(E_T/E)}$. We can derive from equation (1.8a) the plot given in Figure 1.2 with limiting stress cut-offs at an appropriate yield, proof or ultimate stress level.

1.3.2 Solid Square Bar a × a (see Figure 1.1b)

With $I = a^4/12$ and $A = a^2$, then $k^2 = a^2/12$ and the buckling failure criterion, equation (1.2b), becomes

$$\sigma_F = \frac{\pi^2 E a^2}{12L^2} \tag{1.9}$$

Equate (1.9) to the axial stress formula (1.4), i.e. $\sigma_W = \sigma_F$, and on setting $A = a^2$, this gives the square side length as

$$a_{\text{opt}} = \frac{(12P)^{1/4} L^{1/2}}{\pi^{1/2} E^{1/4}} = 1.050 \left(\frac{PL^2}{E}\right)^{1/4} \tag{1.10}$$

Substituting equation (1.10) into equation (1.9) and dividing by ρ leads to the objective function required:

$$\left(\frac{\sigma}{\rho}\right)_{\text{opt}} = 0.907 \left(\frac{E^{1/2}}{\rho}\right)\left(\frac{P}{L^2}\right)^{1/2} \tag{1.11}$$

Equation (1.11) is similar in form to the circular section's objective function, equation (1.8a). Note here that the greater value of the shape efficiency factor $F = 0.907$ indicates that more of the material in the square section is fully stressed. Figure 1.3 presents equation (1.11) graphically for four materials whose properties and relevant ratios appear in Table 1.1 (see also Appendix A). The figure shows working stress ranges cut off by the limiting stress. The latter is taken to be the 0.1% proof stress for the metallic materials and the ultimate stress for Douglas fir and the glass fibre-reinforced composite (GFRC).

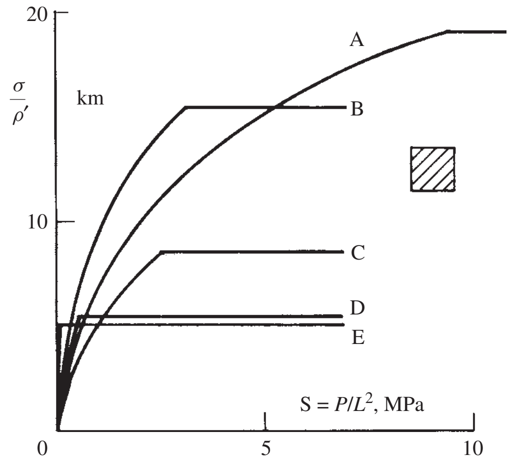


Figure 1.3 Objective function versus structural index for struts with solid square sections (key: A, Ti alloy; B, Al alloy; C, steel; D, GFRC; E, Douglas fir (see Table 1.1))

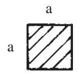
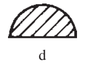
Within the range of the index $S = P/L^2$ shown, the aluminium alloy (L65) appears to optimise the stress most efficiently at a given S. Notably, in extending the range of S threefold, titanium alloy (DTD 5053) allows higher objective functions to be reached. The high grade bolt steel (S96) is a poor performer on a weight/strength basis. GFRC lies between the aluminium and titanium alloys but is cut off at a much lower value, $\sigma_{ult}/\rho = 5.7$ km. Douglas fir has a similar cut-off at 5.1 km and, within a very restricted range of structural indices ($P/L^2 < 0.2$), provides the greatest objective function of all the materials considered within this figure. In this case both E and σ_{ult} are properties of fir taken parallel to the grain.

A similar analysis may be applied to a strut of any solid cross-section. Table 1.2 summarises the results obtained here for the circular and square sections together with those that apply to semi-circular and equilateral triangular sections (see Figure 1.1c,d). The comparison between the four sections may be made simply through the shape coefficient C and the shape factor F, appearing in equations (1.7b) and (1.8b). Of these four sections an equilateral triangle appears to bear the greatest stress. The implication of this is that when $(\sigma/\rho)_{opt}$ is to be maximised, in order to minimise the strut's weight, the triangular

Table 1.1 Properties of common structural materials

Material\Property → ↓ Unit →	E GPa	ρ kg/m^3	$E^{1/2}/\rho$ m^2/N$^{1/2}$	$E^{2/3}/\rho$ (m^5/N)$^{1/3}$	$E^{3/5}/\rho$ (m^9/N^2)$^{1/5}$
A Ti alloy (DTD 5053)	118	4540	7.22	494.6	9.2
B Al alloy (L65)	75	2790	9.83	634.6	119.8
C Steel (S96)	207	7800	5.79	441.7	78
D GFRC	20	1800	8	41.73	85.84
E Douglas fir	11	497	21.51	1014.47	217.18

Table 1.2 Shape coefficients for slender struts of solid section

Cross-section				
C	1.199	1.050	1.960	1.539
F	0.886	0.907	0.663	0.975

section strut would provide the lowest weight for supporting a predetermined compressive load in a given material.

1.4 Thin-Walled, Tubular Sections

The objective function is again $R = \sigma/\rho$, but two cross-section variables arise in tubular sections: a mean section dimension d and the wall thickness t (see Figure 1.4a). Appendix D shows that failure criteria must now include local buckling in addition to flexural buckling. In deriving the objective function R, the usual procedure is to establish the failure criteria first. Then, by ensuring that the critical stresses by these criteria are attained simultaneously, the geometry of the tubular section is optimised, from which the usual form for R will follow.

1.4.1 Thin-Walled Circular Tube

Local, inelastic buckling in thin-walled circular tubes (Figure 1.4a) of moderate length under compression has been reported in [4]. Mostly, buckling appeared in the two-lobe failure mode (Figure 1.4b), even though four lobes are generally assumed for an elastic failure [2].

(i) Flexural Buckling
With $I = \pi d^3 t/8$ and $A \approx \pi dt$, then $k^2 = I/A \approx d^2/8$. Hence the buckling stress in equation (1.2b) becomes

$$\sigma_F = \frac{\pi^2 E_T d^2}{8L^2} \tag{1.12}$$

(ii) Limiting Stress
The working, compressive stress σ_W in the strut is given as

$$\sigma_W = \frac{P}{A} = \frac{P}{\pi dt} \tag{1.13}$$

which is limited to σ_y, i.e. $\sigma_W \leq \sigma_y$. (Here we take: $\sigma_W = \sigma_y$)

(iii) Local Buckling
A collapse of the wall surface occurs (see Figure 1.4b) when local depressions appear under an axial stress:

$$\sigma_L = K_L E_T \left(\frac{t}{d}\right) \tag{1.14}$$