Transient Electronics

Pulsed Circuit Technology

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Preface

The analysis of the transient response of electrical and electronic circuits to any transient input signal is a rather more difficult subject than the analysis of the AC response of such circuits when excited by sinusoidal signal sources. Fortunately the use of the Laplace transform method, developed separately by Oliver Heaviside in his operational calculus, has proved to be a very powerful tool, more so than many people realise, for carrying out this type of analysis. The method effectively transforms a difficult problem, based on the classical solution of linear differential equations, into a much simpler one that even first year undergraduates can tackle. However, in a few cases, the classical method can prove to be more efficient particularly in the transient analysis of circuits containing electrical components whose values change with time.

The book therefore starts with a detailed chapter on the Laplace transform method together with a introductory section on the use of the classical method. The chapter gives an insight into the origins of the Laplace transform method so that interested readers can get some sort of understanding of the way in which differential equations, that may be difficult to solve directly, can be converted into simple algebraic equations that are much easier to solve. The use of the method is illustrated by many worked examples and it is recommended that those new to the subject work through the examples to achieve competence in its application. The Laplace transform method is used heavily throughout the later stages of the book as the standard method for analysing the transient response of the many components and circuits described. This chapter is then followed up by a second chapter on transmission lines which is written in such a way as to explain how the Laplace transform method may also be applied to the transient analysis of the response of transmission lines to transient signals. These two chapters then provide the foundation for the rest of the book which is devoted to specific electrical components, circuits and circuit techniques which are used to generate and transform short electrical pulses with pulse duration's down to a few hundreds of picoseconds.

Chapters 3 and 4 are then devoted to the subject of pulse forming using transmission lines (chapter 3) and line simulating *LC* ladder networks (chapter 4). The transient response of conventional wound transformers is described in chapter 5 which is then followed by a chapter on the more recently developed family of transmission line transformers. Chapter 7 deals with the design of pulse generators which are based on the discharge of energy either stored in capacitors or inductors and includes a detailed description of the Marx generator, perhaps the most important generator used in pulsed power systems. Finally chapter 8 introduces the exciting new field of nonlinear pulse generators. The use of nonlinear components has led to the development of a whole new family of pulse generating circuits whose performance, particularly in terms of speed, can far out exceed that of circuits which are restricted to the use of linear components.

It is hoped that this book will provide readers with a comprehensive guide to the most important pulse generating circuits and components that have been reported so far. It can be quite difficult to find information on many pulsed circuit techniques as much of the work has been published in the form of internal research reports, often at defence research establishments, or in books and papers that can only be discovered in the world's largest engineering and physics libraries. For this reason there are extensive lists of references at the end of each chapter so that more detailed information on particular circuits and components can be found relatively easily.

There are very few books that have been written specifically on transient electronics and pulsed electrical circuits. Most notable are the books by Glasoe and Lebacqz, *Pulse Generators* and Lewis and Wells, *Millimicrosecond Pulse Techniques*. Both books are now very old having been written around 50 years ago. A later book by Zepler and Nichols, *Transients in Electronic Engineering* is also worth noting as it also deals specifically with the transient analysis of electrical circuits although it does not include chapters on pulse generating circuits and components that are to be found in this book.

This book, therefore, is designed to give an up-to-date approach to the subjects of transient electronics and pulse generator circuits and is, in part, based on the vast amount of research work carried out over the last 50 years or so in the field of pulsed power technology. This work has been primarily directed towards the development of pulsed electrical circuits capable of generating short electrical pulses at very high power levels for applications mostly in experimental physics and defence. Much of the source material comes from the Proceedings of the International IEEE Pulsed Power Conferences (started in 1977), the Proceedings of the IEEE Power Modulator Symposia (started in 1950) and the IEE Pulsed Power Colloquia (started by the author in 1991).

The book is written so that it should be of use both to undergraduates in electrical and electronic engineering (chapters 1 and 2) and, in particular, to all researchers in pulsed power technology. It should also be of value to engineers who need to know about transient analysis and pulse generation, such as aerospace engineers (lightning and EMP protection), the defence community (electric guns, flash X-rays, etc.), radar engineers (pulsed and impulse radars), and computer engineers (computer protection from transient signals etc.).

In writing this preface, I find it rather amusing to think that I am again writing about the characteristics of a variety of electrical circuits when the first piece of work I wrote on the subject was a project on electrical circuits written at the age of 11 at Whitehorse Road Primary School, Croydon. Some 40 years on I am still devoted to the subject and the book is written as a result of over 30 years research activity in pulsed power. For an experimentalist in physics or electrical engineering, pulsed power technology is arguably the most exciting (quite literally!) field to work in. The world's biggest lasers, plasma experiments, electric guns, particle accelerators are critically dependent on pulsed power technology and simply would not exist without the pioneering research carried out in the field.

Over the years of my research career I have met and worked with many fine physicists and engineers world-wide whose friendship and generosity have made this book possible. However, before thanking those who have helped me to produce this book, I would like to start by expressing my deep gratitude to Jim Holbrook whose lecture course on Network Analysis and Synthesis, that I undertook at Southampton University in 1971, is probably the most valuable course an electrical or electronic engineer could take. His book entitled *Laplace Transforms for Electronic Engineers* clearly illustrates his profound understanding of the Laplace transform method and his ability to communicate this understanding in the most digestible way. A "must buy" for all electrical engineers!

I should also like to thank my colleagues and friends at the former EEV Co. Ltd. (now Marconi Applied Technologies) and, in particular, Peter Maggs, Chris Neale, Colin Pirrie and the late Hugh Menown, for their support and sponsorship over many years. Without their generosity and patronage it would have been impossible to carry out much of the work described in this book. I was also fortunate enough to spend time in the late 70s with the late Charlie Martin and his group at AWE, Aldermaston. Charlie, regarded by many as the Father of pulsed power technology, generously devoted much of his time to the training of new workers in pulsed power technology. His highly individual and unconventional approach to research was very stimulating and a very interesting and amusing account of his career is to be found in the book edited by Martin, Guenther and Kristiansen entitled *J. C. Martin on Pulsed Power.* I should also like to acknowledge the contribution made to my own research by the many postgraduate students and post doctoral research workers who have been part of my research group over the years. Much of this research appears in this book and I should like to thank, in particular, Colin Wilson, Miles Turner, Andy Erickson, Greg Branch, Martin Brown and Osvaldo Rossi for their contribution to the work that is written up in this book. Finally I should like to thank Joanna Ashbourn, John Allen, Nigel Seddon and Peter Choi for agreeing to proof read this book and for their valuable comments.

Paul W. Smith

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1

Mathematical Techniques for Pulse and Transient Circuit Analysis

1.1 INTRODUCTION

The analysis of the transient response of pulsed circuits requires a comprehensive knowledge and understanding of the mathematical methods that can be used. In this chapter the most important mathematical tools are explained. It is not the purpose of this chapter to give a complete description of the techniques of electrical circuit analysis, as there are plenty of good texts available on this topic [1, 2, 3]. It will be assumed, however, that the reader is competent in the basic techniques of network analysis, i.e. the application of Kirchhoff's laws to circuits, the laws of Thévenin and Norton and the principle of superposition. Also desirable is a working knowledge of the technique of signal flow graphs and the associated use of Mason's reduction formula [4] as this can often reduce the labour involved in analysing the transient behaviour of multi-component circuits. It will also be assumed that the reader has an adequate background in mathematical techniques, and in particular is familiar with complex number theory, the solution of integro-differential equations, Fourier analysis and series, and basic matrix and determinental methods. Again there are many texts which cover such topics, but the books by Stephenson [5], Wylie and Barrett [15] and Jeffrey [14] may prove to be among the most useful.

This chapter is largely concerned with the Laplace transform method as developed by Oliver Heaviside. The technique is of prime importance to any electrical engineer concerned with the transient behaviour of electrical circuits [6]. The method has an elegant simplicity for this type of analysis and a very wide range of application. Its application to the transient analysis of circuits involving transmission lines is of particular importance and will be dealt with in the next chapter. Although the growing use of circuit analysis programmes such as PSPICE [7] and MICROCAP [8] can provide a convenient and useful way of analysing the transient behaviour of pulsed circuits, a solid grounding in the application of the Laplace technique to such circuits must be regarded as essential.

1.2 THE CLASSICAL METHOD

Before introducing the Laplace transform method, it is instructive to analyse the transient behaviour of a relatively simple circuit using an integro-differential equation set up using

Kirchhoff's laws. This so called Classical Method [9] can easily become very cumbersome, and in some cases gives equations that are impossible to solve especially if the analysis results in two or more simultaneous integro-differential equations. If a homogeneous solution can be found, arbitrary constants must at some stage be introduced, whose values can only be deduced at the end of the analysis by reference to the initial circuit conditions.

The basis of the method is to use the current voltage relationships for resistors, capacitors and inductors in either algebraic, differential or integral form. These are respectively

$$i(t) = \frac{v(t)}{R} \qquad \text{or} \qquad v(t) = Ri(t) \tag{1.1}$$

$$i(t) = C\frac{dv(t)}{dt} \qquad \text{or} \qquad v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau + V(0) \tag{1.2}$$

$$i(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau + I(0) \qquad \text{or} \qquad v(t) = L\frac{di(t)}{dt} \tag{1.3}$$

where the constants of integration $V(0)$ and $I(0)$ represent the voltage to which the capacitor is charged and the current flowing in the inductor, respectively, at time $t=0$.

Example 1.1

An arbitrary voltage source is switched on to the series combination of a resistor, an inductor and a capacitor as shown in Figure 1.1. Derive expressions for the voltage on the capacitor and the current flowing in the circuit as functions of time if the arbitrary source is chosen to be a DC source of potential V volts. How would the analysis proceed if the source was other than DC?

Applying Kirchhoff's voltage law and using Equations (1.1)–(1.3)

$$v(t) = L\frac{di(t)}{dt} + i(t)R + \frac{1}{C}\int_0^t i(\tau)\,d\tau + V(0) \tag{1.4}$$

Since the capacitor is initially uncharged and the voltage source is DC, this simplifies to

$$V = L\frac{di(t)}{dt} + i(t)R + \frac{1}{C}\int_0^t i(\tau)\,d\tau \tag{1.5}$$

Differentiating with respect to time gives

$$0 = L\frac{d^2i(t)}{dt^2} + \frac{di(t)}{dt}R + \frac{i(t)}{C} \tag{1.6}$$

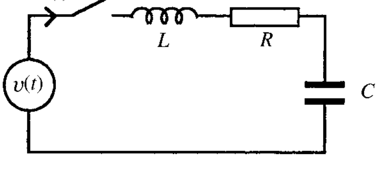

Figure 1.1 Circuit for Example 1.1

Since this is a linear second-order homogeneous differential equation with constant coefficients, a solution of the form

$$i(t) = e^{mt} \tag{1.7}$$

is attempted and gives, on substitution,

$$\left(Lm^2 + Rm + \frac{1}{C}\right)e^{mt} = 0 \tag{1.8}$$

Equation (1.7) must be a solution to Equation (1.6) when m is a root of

$$Lm^2 + Rm + \frac{1}{C} = 0 \tag{1.9}$$

The roots are given by

$$m_1, m_2 = \frac{-R \pm \sqrt{R^2 - 4\frac{L}{C}}}{2L} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\alpha \pm j\omega_d \tag{1.10}$$

where

$$\alpha = \frac{R}{2L}, \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \text{and} \quad \omega_d^2 = \omega_0^2 - \alpha^2 \tag{1.11}$$

ω_0 and ω_d are the natural and damped frequencies of the circuit and it is assumed that $\omega_0^2 > \alpha^2$. Thus the general solution to Equation (1.6) is

$$i(t) = A_1 e^{m_1 t} + A_2 e^{m_2 t} \tag{1.12}$$

To evaluate the constants A_1 and A_2 it is noted that just before closure of the switch in the circuit there was no current flowing, therefore at $t = 0_-$, $i(t) = 0$ and $A_1 = -A_2$. Also, at a time just after the switch has been closed, the impedance of the inductor will be very large compared to that of the resistor and capacitor. Therefore

$$V = L\left(\frac{di(t)}{dt}\right)_{t=0_+} \tag{1.13}$$

where the notations 0_- and 0_+ refer to times that occur at an infinitesimally short time before and after the switch is closed at $t=0$, respectively.

Applying this relationship to Equation (1.12), after differentiation and substitution for m_1 and m_2 from Equation (1.10), gives

$$\frac{V}{L} = A_1(-\alpha + j\omega_d) + A_2(-\alpha - j\omega_d) = 2j\omega_d A_1 \tag{1.14}$$

Thus Equation (1.12) becomes

$$i(t) = \frac{V}{2j\omega_d L}\left(e^{-(\alpha - j\omega_d)t} - e^{-(\alpha + j\omega_d)t}\right) = \frac{V}{\omega_d L} e^{-\alpha t} \sin \omega_d t \tag{1.15}$$

This solution corresponds to the case where the circuit is said to be underdamped and the solution is partly oscillatory due to the sine term. In the case where $\omega_0^2 < \alpha^2$ the roots of Equation (1.9) are both real and may be written in the form

$$m_1, m_2 = -\alpha \pm \beta \qquad \text{where} \quad \beta^2 = \alpha^2 - \omega_0^2 \tag{1.16}$$

In this case the solution to Equation (1.6) can easily be found to be

$$i(t) = \frac{V}{\beta L} e^{-\alpha t} \sinh \beta t \tag{1.17}$$

This represents the overdamped case and the solution for $i(t)$ is now no longer oscillatory. Finally, in the case where $\omega_0^2 = \alpha^2$, the roots of Equation (1.9) are both equal to $-\alpha$. Since the two solutions are no longer independent a different general solution must be used, which takes the form

$$i(t) = (A_1 + A_2 t)e^{-\alpha t} \tag{1.18}$$

Applying the initial conditions, as above, gives values for the constants A_1 and A_2 given by

$$A_1 = 0 \quad \text{and} \quad A_2 = \frac{V}{L} \tag{1.19}$$

The solution to Equation (1.6) is now

$$i(t) = \frac{V}{L} t e^{-\alpha t} \tag{1.20}$$

This is the critically damped solution. Typical plots for $i(t)$ for the three damping conditions are shown in Figure 1.2 shown below.

The voltage on the capacitor $v_c(t)$ can be found by using Equation (1.2) since expressions for the current in the circuit have been derived. For example in the case of an underdamped circuit

$$v_c(t) = \frac{1}{C} \int_0^t \left(\frac{V}{\omega_d L} e^{-\alpha\tau} \sin \omega_{\mathrm{d}} \tau \right) d\tau \tag{1.21}$$

assuming the capacitor was initially uncharged and $v_c(0) = 0$. Carrying out the integration gives

$$v_c(t) = V \left[1 - e^{-\alpha t} \left(\cos \omega_d t + \frac{\alpha}{\omega_d} \sin \omega_d t \right) \right] \tag{1.22}$$

In the situation where the voltage source, which is switched on to the circuit, is other than a DC source the problem then becomes one of finding the solution to Equation (1.5) where

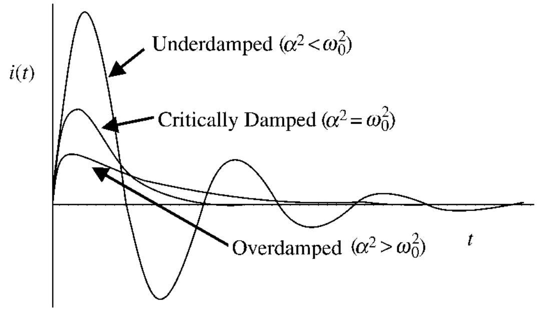

Figure 1.2 Typical current waveforms for an underdamped, critically damped and overdamped series LCR circuit

the source voltage V is replaced by an expression for $v(t)$. After differentiation of the equation, a second-order equation results which is not equal to zero but the derivative of $v(t)$ with respect to time. The general solution of this equation is then the sum of the general solution of the reduced equation (which is the same as Equation (1.6)) i.e. the complementary function and the particular integral of the complete equation. For example if the applied voltage $v(t)$ took the form of a ramp given by

$$v(t) = kt \tag{1.23}$$

then the new form of Equation (1.5) would be

$$v(t) = L\frac{di(t)}{dt} + i(t)R + \frac{1}{C}\int_0^t i(\tau)\,d\tau = kt \tag{1.24}$$

After differentiation this becomes

$$L\frac{d^2i(t)}{dt^2} + \frac{di(t)}{dt}R + \frac{i(t)}{C} = k \tag{1.25}$$

The trial form of the particular integral takes the simple form

$$i(t)_{PI} = A \tag{1.26}$$

Substituting Equation (1.26) into Equation (1.25) gives a value for the constant A of

$$A = kC \tag{1.27}$$

Thus the general solution to Equation (1.25) is given by

$$i(t) = kC + A_3e^{m_1t} + A_4e^{m_2t} \tag{1.28}$$

where the new constants A_3 and A_4 must be evaluated from the initial circuit conditions as before. Since the current in the circuit is initially zero before the switch is closed, then using Equation (1.28) gives

$$kC + A_3 + A_4 = 0 \tag{1.29}$$

As before, by considering the relative sizes of the impedances in the circuit just after the switch has been closed, this results in an equation similar to Equation (1.13)

$$(kt)_{t=0_+} = L\left(\frac{di(t)}{dt}\right)_{t=0_+} \cong 0 \tag{1.30}$$

Applying this relationship to Equation (1.28) gives

$$A_3(-\alpha + j\omega) + A_4(-\alpha - j\omega) = 0 \tag{1.31}$$

From Equations (1.29) and (1.31) the constants A_3 and A_4 are found to be

$$A_3 = \frac{-kC(\alpha + j\omega)}{2j\omega}$$
$$A_4 = \frac{-kC(-\alpha + j\omega)}{2j\omega} \tag{1.32}$$

Thus by substituting these relationships into the general solution Equation (1.28) and rearranging, an expression for the underdamped current $i(t)$ in the circuit can be derived as

$$i(t) = kC\left[1 - e^{-\alpha t}\left(\frac{\alpha}{\omega}\sin\omega t + \cos\omega t\right)\right] \tag{1.33}$$

Clearly, if a less simple function than a ramp for the applied voltage source $v(t)$ had been chosen, the derivation of an expression for the current in the circuit would have been much more laborious, as would have been the case if the initial conditions in the circuit were not zero, i.e. there was an initial current flowing and the capacitor was partially charged prior to switch closure.

Example 1.2

A capacitor charged to a potential V is discharged into a resistor $R(t)$ whose value changes with time. Determine an expression for the current that flows in the circuit $i(t)$ when the capacitor is discharged.

Applying Kirchhoff's voltage law to this simple circuit and using Equations (1.1) and (1.2) gives

$$i(t)R(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau + V = 0 \tag{1.34}$$

Differentiating this expression with respect to time gives

$$\frac{di(t)}{dt}R(t) + \frac{dR(t)}{dt}i(t) + \frac{i(t)}{C} = 0$$

$$\text{or} \qquad \frac{di(t)}{dt} + \frac{\dfrac{dR(t)}{dt}i(t) + \dfrac{i(t)}{C}}{R(t)} = 0 \tag{1.35}$$

This equation has an integrating factor $\phi(t)$ [5] which is given by

$$\phi(t) = \exp\left(\int \frac{\dfrac{dR(t)}{dt}}{R(t)}\,dt + \int \frac{1}{CR(t)}\,dt\right)$$

$$= R(t)\exp(A)\exp\left(\int \frac{1}{CR(t)}\,dt\right), \quad A \text{ constant} \tag{1.36}$$

Hence the current flowing in the circuit $i(t)$ is given by

$$i(t) = \frac{1}{R(t)}B\,\exp\left(-\int \frac{1}{CR(t)}\,dt\right) \tag{1.37}$$

Since the current initially flowing in the circuit $i(0)$ depends on the initial value of $R(t)$, i.e. $R(0)$, then the combined constants of integration B can be found to be

$$B = i(0)R(0) = V \tag{1.38}$$

Hence $i(t)$ is given by

$$i(t) = \frac{V}{R(t)} \exp\left(-\int \frac{1}{CR(t)}\, dt\right) \tag{1.39}$$

Clearly the full solution to this type of problem will require $R(t)$ to be defined. It is worth noting that although the Laplace transform method to be described later in this chapter is generally a far easier method for solving time-dependent circuit analysis problems, in this type of problem the integrating factor method turns out to give the easiest solution.

1.3 THE COMPLEX FREQUENCY METHOD

From the above section it is clear that the use of differential equations to find both the natural response of a network and its response to a forcing function (the forced response) can be quite lengthy and labour-intensive. Fortunately there are other much easier methods by which such responses can be derived. The most powerful of these is the use of Laplace transforms, but before introducing the method it is instructive to examine the complex frequency method which helps to explain the operation of the Laplace method, and also introduces the very important concept of complex frequency.

Referring back to Example 1.1, it was seen that the underdamped response of a series LCR circuit to DC voltage source took the form

$$i(t) = \frac{V}{\omega_d L} e^{-\alpha t} \sin \omega_d t \tag{1.40}$$

i.e. an exponentially decaying sinusoid which is a signal type that is very commonly encountered in the solution of a variety of engineering problems. A voltage signal of this type can also be written in a generalised form as

$$v(t) = V_p e^{\sigma t} \cos(\omega t + \theta) \tag{1.41}$$

where V_p is the peak amplitude of the signal and θ is a phase constant. In Figure 1.3 sketches of the signal are shown for the cases in which ω and/or σ are zero and the polarity of σ is changed. From the figure it can be seen that the representation of a generalised voltage signal in the form of Equation (1.41) is both useful and versatile. By the use of Euler's relationship Equation (1.41) can be written in the form

$$v(t) = \mathscr{Re}(V_p e^{\sigma t} e^{j(\omega t + \theta)}) \tag{1.42}$$

or its complex conjugate. Putting

$$\mathbf{s} = \sigma + j\omega \tag{1.43}$$

Equation (1.42) can be written as

$$v(t) = \mathscr{Re}\left(V_p e^{j\theta} e^{\mathbf{s}t}\right) = \mathscr{Re}(\mathbf{V} e^{\mathbf{s}t}) \tag{1.44}$$

where

$$\mathbf{V} = V_p e^{j\theta} \tag{1.45}$$

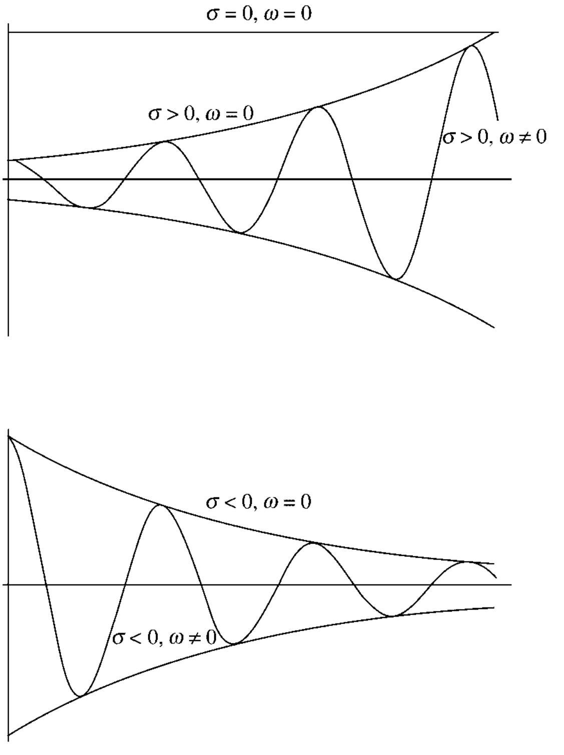

Figure 1.3 Sketches of the generalised voltage signal given by Equation (1.41) with $\theta = 0$

If such a signal is applied as the forcing function to the circuit in Example 1.1, the response of the current in the circuit must take the form

$$i(t) = \mathscr{Re}\left(I_p e^{j\phi} e^{st}\right) = \mathscr{Re}(\mathbf{I}e^{st}), \qquad \text{with} \quad \mathbf{I} = I_p e^{j\phi} \tag{1.46}$$

since there are no non-linear elements in the circuit and the natural frequency and decay rate in the circuit remain the same. The angle $(\phi - \theta)$ represents the difference in phase angle between the applied voltage and the current flowing in the circuit. If the $\mathscr{Re}$ is suppressed, remembering that it must be put back to find the circuit response in the time domain, and the expressions for voltage and current in the circuit (Equations (1.44) and (1.46)) are substituted into Equation (1.4) with $V(0) = 0$, the relationship between current and voltage becomes

$$\mathbf{V}e^{st} = \mathbf{s}L\,\mathbf{I}\,e^{st} + R\mathrm{I}e^{st} + \frac{1}{\mathbf{s}C}\mathrm{I}e^{st} \tag{1.47}$$

or

$$\mathbf{V} = \mathbf{s}L\mathbf{I} + R\mathbf{I} + \frac{1}{\mathbf{s}C}\mathbf{I} \tag{1.48}$$

or

$$\frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}(\mathbf{s}) = \mathbf{s}L + R + \frac{1}{\mathbf{s}C} \tag{1.49}$$

This equation does not contain the time variable t but only the complex frequency parameter **s** and takes a much simpler form than the integro-differential Equation (1.4) from which it was derived. Finding the current in the circuit as a function of time is relatively simple if **V** is replaced using Equation (1.44), **s** is defined, and it is remembered that the real part of **I** is required to give the solution, i.e.

$$i(t) = \mathscr{Re}\left\{\frac{\mathbf{V}}{\mathbf{s}L + R + \dfrac{1}{\mathbf{s}C}} e^{st}\right\} \tag{1.50}$$

It should be also be remembered, however, that this method of solving integro-differential equations of similar form to Equation (1.4) will only work if the forcing function or stimulus applied to a circuit can be put in complex exponential form.

Example 1.3

A voltage signal $v(t) = 10e^{-5t}\cos(10t + 45°)V$ is applied to the series combination of a $10\,\Omega$ resistor and a 5 H inductor. Derive an expression for the current flowing in the circuit as a function of time.

Expressing the applied voltage as the real part of a complex exponential function gives

$$v(t) = \mathscr{Re}(\mathbf{V}e^{st})$$

$$\text{where} \qquad \mathbf{V} = 10\angle 45° \quad \text{and} \quad \mathbf{s} = -5 + j10 \tag{1.51}$$

An expression for the voltage and current in the circuit as a differential equation can be derived, as outlined earlier, using the relations given by Equations (1.1) and (1.3), which is

$$v(t) = 10i(t) + 5\frac{di(t)}{dt} \tag{1.52}$$

Applying the complex frequency method to this equation gives

$$10\angle 45° e^{st} = 10\mathbf{I}e^{st} + 5\mathbf{sI}e^{st} \tag{1.53}$$

Hence

$$\mathbf{I} = \frac{10\angle 45°}{10 + 5\mathbf{s}} = \frac{10\angle 45°}{-15 + j50} = 0.19\angle -28.3° A \tag{1.54}$$

Thus the required expression for the current in the circuit is given by

$$i(t) = 0.19e^{-5t}\cos(10t - 28.3°)\mathrm{A} \tag{1.55}$$

1.4 THE LAPLACE TRANSFORM METHOD

It became clear, in the last section, that the solution of the integro-differential equations, which result from the analysis of the response of a simple circuit or network to a stimulus or forcing function, could be considerably simplified if the forcing function could be expressed in a complex exponential form such as that given by Equation (1.44). Using the method, the integro-differential equations were transformed into simple algebraic equations involving the complex frequency variable **s**. However the method would appear to be confined to forcing functions which can be expressed in complex exponential form which is rather restrictive.

To get over this problem it is possible to express any periodic forcing function, that is likely to be applied to a network, in the form of a Fourier series of sine and cosine terms, or a Fourier series of complex exponential terms using the techniques of Fourier Analysis [6]. Since sine and cosine terms can be expressed as either the imaginary or real part of a complex exponential, it becomes clear that the analysis of the behaviour of a given circuit to any forcing function can be achieved by transforming the function into a Fourier series of either type and then analysing the response of the circuit to each term in the series. Provided the circuit is linear, one can apply the principle of superposition and sum the responses to each of the individual terms to get the complete circuit response. This method will of course be very laborious and involve many separate analyses to get an accurate result. In the case of a non-periodic forcing function the Fourier integral

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{1.56}$$

can be used to transform the forcing function $f(t)$ into its frequency spectrum $F(\omega)$ and this can then be used to determine a given response. Unfortunately applying the Fourier integral to a number of forcing functions that are commonly in use in circuit analysis fails, because for a function to be Fourier transformable requires

$$\int_{-\infty}^{\infty} |f(t)|\,dt \leq \infty \tag{1.57}$$

which then excludes many functions such as periodic functions or functions involving steps. This problem can be resolved by multiplying such non-convergent functions by a convergence factor CF of the form

$$\mathrm{CF} = e^{-ct} \tag{1.58}$$

where the value of c is chosen to make the function convergent [6]. Thus the Fourier integral in Equation (1.56) becomes

$$F(c,\omega) = \int_{0}^{\infty} f(t)e^{-ct}e^{-j\omega t}\,dt \tag{1.59}$$

where the lower limit of the integral has been changed to reflect the fact that in most, if not all, cases the response of a circuit to a forcing function applied at time $t = 0$ is required. Strictly, as has been mentioned before, the lower limit should be written as 0_- to take into account that $f(t)$ may be an impulse or a step function. Clearly c is not a constant and will vary according to the function that is required to become convergent. Thus if we let $c = \sigma$ and the exponents in Equation (1.59) are combined, the equation can be written as

$$F(\sigma + j\omega) = \mathbf{F}(\mathbf{s}) = \int_{0_-}^{\infty} f(t)e^{-\mathbf{s}t}dt \tag{1.60}$$

which is the equation for the direct Laplace transform. As will be seen later, this integral will enable virtually all of the circuit stimuli or forcing functions in the transient analysis of electrical circuits to be dealt with. The inverse transform which allows circuit responses in the complex frequency domain to be converted back to the time domain is given by

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} \mathbf{F}(\mathbf{s})e^{\mathbf{s}t}\,ds \tag{1.61}$$

where σ_1 is a real positive quantity which is greater than the convergence variable σ. This integral is often not easy to perform as it involves contour integration in the complex plane, but as will be explained later much simpler methods exist to find the inverse Laplace transform of a given $\mathbf{F}(\mathbf{s})$.

1.4.1 Application of the Laplace Transform Method

Although the way in which the Laplace transform was introduced may appear a little complicated, in practice it greatly simplifies the problem of analysing the transient response of a circuit or network to the sudden application of an electrical signal (forcing function). Indeed it is arguably the most important technique that an Electrical Engineer must master to be able to cope with such analyses. However its range of application is much broader than at first might be thought, in that the reduction of a given circuit response in the complex frequency domain, by setting either $\sigma = 0$ and $j\omega = 0$, or $\sigma = 0$ automatically gives the DC or continuous AC response of that circuit, respectively. Furthermore most circuit elements or components can in some way be represented by a circuit model involving passive components and current or voltage sources. Thus the behaviour of such elements or components can be analysed when subject to any type of circuit stimulus or signal. In this way the behaviour of semiconductor circuits, electrical machines, transformers, gas discharge devices, etc., can be analysed using relatively straightforward algebraic techniques.

The basic idea of the method is to convert both the electrical circuit or network and the circuit stimulus into their complex frequency equivalent forms. From this the algebraic relationships for voltage or current are derived in terms of $\mathbf{s}$ using standard network techniques and theorems. The resulting expressions are then simplified into a form where the inverse transform can be found, usually with the help of a set of Laplace transform tables. The method has often been compared to the use of logarithms for the multiplication or division of two large numbers. A difficult calculation is essentially made easier by converting the two numbers into logarithms, performing the easier operations of addition or subtraction and then taking the antilogarithm of the result to get the answer to the problem. This then gives the desired response of the circuit in the time domain. Figure 1.4 is a diagram which describes the basis of the method.

The conversion of resistors, capacitors and inductors into their complex frequency impedances can be achieved using the integro-differential relationships for these components as given in Equations (1.1), (1.2) and (1.3), and by the application of a voltage signal of the form given by Equation (1.44). For example, in the case of a capacitor

$$i(t) = C\frac{dv(t)}{dt} \tag{1.62}$$

which becomes

$$\mathscr{Re}(V(\mathbf{s})e^{\mathbf{s}t}) = \frac{1}{\mathbf{s}C}\mathscr{Re}(I(\mathbf{s})e^{\mathbf{s}t}) \tag{1.63}$$

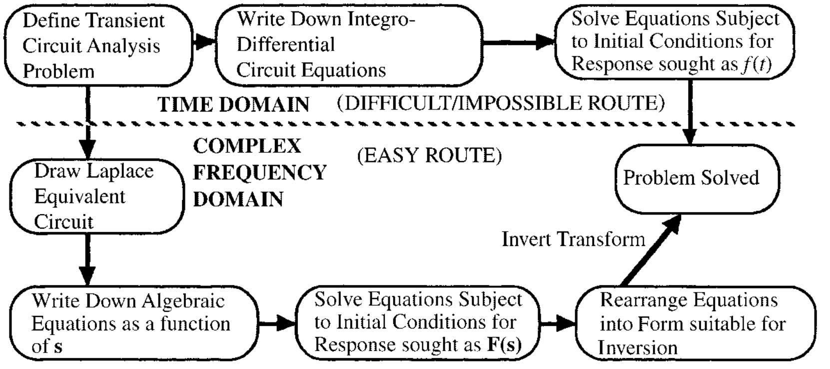

Figure 1.4 The Laplace transform concept

assuming that the capacitor is initially uncharged. If the factor e^{st} is suppressed and the $\mathscr{Re}$ is ignored, the complex frequency impedance of a capacitor becomes

$$\mathbf{Z(s)} = \frac{1}{\mathbf{s}C} \tag{1.64}$$

By reference to Equations (1.1) and (1.3) it is easily seen that the complex frequency impedances of an inductor and capacitor are $\mathbf{s}L$ and R, respectively.

In order to take the Laplace method further it is necessary to examine the transformation, into the complex frequency domain, of a number of signals (forcing functions) that are likely to be applied to an electrical circuit. These signals are sketched in Figure 1.5.

1.4.2 Laplace Transforms of Some Basic Signals

The unit step

The unit step function is defined as

$$\begin{aligned} u(t) &= 0 \qquad t < 0 \\ &= 1 \qquad t \geq 0 \end{aligned} \tag{1.65}$$

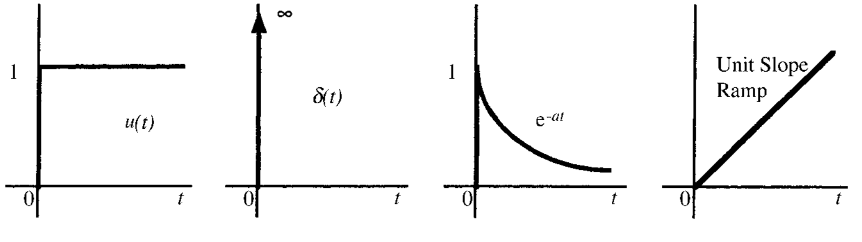

Figure 1.5 Some basic signals

Thus from Equation (1.60) which defines the Laplace transform of a function of time $f(t)$, the transform of the unit step is found to be

$$\begin{aligned}\mathscr{L}\{u(t)\} = \mathbf{F}(\mathbf{s}) &= \int_{0_-}^{\infty} u(t)e^{-\mathbf{s}t}\,dt \\ &= \left[-\frac{1}{\mathbf{s}}e^{-\mathbf{s}t}\right]_{0_-}^{\infty} \\ &= \frac{1}{\mathbf{s}}\end{aligned} \tag{1.66}$$

The delta function

The delta function or unit impulse function may be defined as

$$\left.\begin{aligned}\delta(t) &= 0 \qquad t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)\,dt &= 1 \\ \text{or,} \quad \int_{0_-}^{0_+} \delta(t)\,dt &= 1\end{aligned}\right\} \tag{1.67}$$

Thus using the definition for the Laplace transformation gives

$$\mathscr{L}\{\delta(t)\} = \mathbf{F}(\mathbf{s}) = \int_{0_-}^{\infty} \delta(t)e^{-\mathbf{s}t}\,dt = 1 \tag{1.68}$$

The exponential function

An exponential function that is suddenly applied a time $t = 0$ is written as $e^{-at}u(t)$. Thus

$$\left.\begin{aligned}\mathscr{L}\{e^{-at}u(t)\} &= \int_{0_-}^{\infty} e^{-(\mathbf{s}+a)t}\,dt \\ &= \left[-\frac{1}{\mathbf{s}+a}e^{-(\mathbf{s}+a)t}\right]_{0_-}^{\infty} \\ \mathbf{F}(\mathbf{s}) &= \frac{1}{\mathbf{s}+a}\end{aligned}\right\} \tag{1.69}$$

The ramp function

As with the sudden application of the exponential function, a ramp function applied at a time $t = 0$ is written as $tu(t)$. Thus

$$\mathscr{L}\{tu(t)\} = \mathbf{F}(\mathbf{s}) = \int_{0_-}^{\infty} te^{-\mathbf{s}t}\,dt = \frac{1}{\mathbf{s}^2} \tag{1.70}$$

1.4.3 Some Properties of the Laplace Transformation

The development of the Laplace transforms of some more difficult signals can be made easier if some of the properties of the Laplace transform are examined. From this it is then possible to construct a table of the transforms of the signals which are most commonly encountered in electrical circuits.

Linearity

The symbol $\mathscr{L}$ which represents the Laplace transform operation is a linear operator, and consequently

$$\mathscr{L}\left\{\sum_i f_i(t)\right\} = \sum_i \mathscr{L}\{f_i(t)\} = \sum_i \mathbf{F}_i(\mathbf{s}) \tag{1.71}$$

Constant product or scaling

This is simply written as

$$\mathscr{L}\{Kf(t)\} = K\mathscr{L}\{f(t)\} = K\mathbf{F}(\mathbf{s}) \tag{1.72}$$

where K is an arbitrary constant.

Differentiation

Given that the Laplace transform of a function of time $f(t)$ exists, what is the Laplace transform of its derivative $f'(t)$? Again, from the definition of the Laplace transformation and using integration by parts

$$\begin{aligned}\mathscr{L}\{f'(t)\} &= \int_{0_-}^{\infty} f'(t)e^{-\mathbf{s}t}\,dt \\ &= [f(t)e^{-\mathbf{s}t}]_{0_-}^{\infty} + \mathbf{s}\int_{0_-}^{\infty} f(t)e^{-\mathbf{s}t}\,dt\end{aligned} \tag{1.73}$$

It is easily seen that the term on the left is just $f(0_-)$ and the term on the right is $\mathbf{s}$ times the Laplace transform of $f(t)$, i.e.

$$\mathscr{L}\{f'(t)\} = \mathbf{sF}(\mathbf{s}) - f(0_-) \tag{1.74}$$

It can also be shown in the case of higher derivatives that

$$\mathscr{L}\{f^n(t)\} = \mathbf{F}^n(\mathbf{s}) = \mathbf{s}^n\mathbf{F}(\mathbf{s}) - \mathbf{s}^{n-1}f(0_-) - \mathbf{s}^{n-2}f'(0_-) - \cdots - f^{n-1}(0_-) \tag{1.75}$$

Differentiation with respect to t in the time domain corresponds to multiplication by $\mathbf{s}$ in the complex frequency domain with initial conditions being taken into account by the terms of the type $f^n(0_-)$. This, in part, helps to explain why integro-differential equations in the time domain, associated with a particular circuit analysis problem, become much simpler to handle in the complex frequency domain.

Integration

In this case it is necessary to find the effect, in the complex frequency domain, of integrating a given time-dependent function. Once again, using integration by parts gives

$$\begin{aligned}\mathscr{L}\left\{\int_{0_-}^{t} f(\tau)\,d\tau\right\} &= \int_{0_-}^{\infty}\left[\int_{0_-}^{t} f(\tau)\,d\tau\right]e^{-st}\,dt \\ &= \left[-\frac{e^{-\mathbf{s}t}}{\mathbf{s}}\int_{0_-}^{t} f(\tau)\,d\tau\right]_{0_-}^{\infty} + \frac{1}{\mathbf{s}}\int_{0_-}^{\infty} f(t)e^{-\mathbf{s}t}\,dt \\ &= \frac{\mathbf{F}(\mathbf{s})}{\mathbf{s}} \end{aligned} \tag{1.76}$$

Integration in the time domain thus corresponds to division by **s** in the complex frequency domain.

Change of scale

This is a useful property which can be used to extend the standard table of Laplace transforms

$$\left.\begin{aligned}\mathscr{L}\{f(at)\} &= \int_{0_-}^{\infty} f(at)e^{-\mathbf{s}t}\,dt \\ &= \int_{0_-}^{\infty} f(u)e^{-\mathbf{s}\left(\frac{u}{a}\right)}\,d\left(\frac{u}{a}\right) \\ &= \frac{1}{a}\int_{0_-}^{\infty} f(u)e^{-\mathbf{s}\left(\frac{u}{a}\right)}\,du \\ \mathbf{F}(\mathbf{s}) &= \frac{1}{a}f\left(\frac{\mathbf{s}}{a}\right)\end{aligned}\right\} \tag{1.77}$$

Translation in time

Often it is desirable to apply a second signal to a circuit at some time delay after the application of an initial signal. This can be achieved by translation of the second signal in time such that the function is shifted to a new position on the time axis without any distortion or change in the signal under translation. To explain how this may be done, consider the shifting of a parabola described by $f(t) = t^2$ as shown in Figure 1.6(a) to a new starting point at a time $t = t_1$ as shown in Figure 1.6(c). Writing a new function $f(t) = (t - t_1)^2$ gives the curve shown in Figure 1.6(b) which clearly does not have the correct form. However if $f(t)$ is multiplied by a unit step function which is also shifted to a new starting point at the same time $t = t_1$, the correct definition of the defined function results. To shift the unit step, the definition of the unit step function given by Equation (1.65) is modified to become

$$\begin{aligned}u(t - t_1) &= 0 \quad t < t_{1_-} \\ &= 1 \quad t \geq t_1\end{aligned} \tag{1.78}$$