Credit Derivatives Pricing Models

Models, Pricing and Implementation

Philipp J. Schönbucher
Credit Derivatives Pricing Models
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Philipp J. Schönbucher
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Preface

This book grew out of a series of training courses for practitioners that I taught on credit derivatives pricing, credit risk modelling and credit portfolio modelling. The aim of the book is to show how currently traded credit derivatives can be priced and risk-managed. As there is no such thing as the credit risk, and given the large variety of traded credit derivatives, different models will be appropriate for different tasks. I aimed to present the most important modelling approaches and to clearly point out the strengths and potential weaknesses of the different modelling approaches.

Even more than in other areas, a clear intuitive understanding of the mechanics of a pricing model is essential in credit markets. As credit markets are usually incomplete, it is dangerous to rely on a model that is not fully understood. One important function of the pricing model is to provide an aid to the intuition by reducing the problem of building an opinion on a complex quantity (like the price of a credit derivative) to the forming of an opinion on more fundamental quantities which are much better understood. Furthermore, a formal pricing model makes this process consistent so that arbitragers across the prices of different instruments are not possible. For this reason much time is spent on the explanation of the implications of different modelling assumptions and the meaning of the models’ parameters.

A second function of credit derivatives pricing models is to provide a framework for the assessment of the risk involved in one particular credit derivative transaction or a whole portfolio of such instruments. Here the model’s output is not just one number (the price) but rather a whole return distribution. If the model is to be used for hedging, the output is a consistent description of the price dynamics of the credit derivative and the underlying hedge instruments.

Like the market for credit derivatives, the field of credit risk modelling is expanding rapidly and still in flux. Thus, only in rare cases is it possible to give a definitive recipe for a given problem, and again an intuitive understanding of the essential features of a given modelling approach is necessary in order to assess other models that may be encountered later on.

In the past few years, credit derivatives have fundamentally transformed the way banks and other financial institutions view and manage credit risks.

The key feature of credit derivatives is that they allow the transfer of credit risks in an efficient, simple and standardised way and open up a market for these risks in which everybody can participate. Credit derivatives allow active risk transfer for credit risk managers, the management (some say arbitrage) of regulatory capital requirements for banks, new funding opportunities for banks through the securitisation of loan portfolios, portfolio optimisation for
bond and loan portfolio managers, highly leveraged investment vehicles for speculators and
depend on the structured of previously unavailable risk–return profiles.

Thus it is no surprise that the market of credit derivatives has seen exponential growth rates
with no signs of slowdown. The 14 firms alone that participated in the recent survey by Risk
magazine (Patel, 2002) had a notional volume of $1398 billion credit derivatives contracts

Traditionally, a bank could only manage its credit risks at origination. Once the risk was
originated, it remained on the books until the loan was paid off or the obligor defaulted. There
was no efficient and standardised way to transfer this risk to another party, to buy or sell
protection, or to optimise the risk–return profile of the portfolio. Consequently, the pricing
of credit risks was in its infancy, spreads on loans only had to be determined at origination
and were often determined by non-credit considerations such as the hope of cross-selling
additional business in the corporate finance sector. There was no need to become more efficient
because the absence of a transparent market meant that the mode of operation was more like an
oligopoly than an efficient competition. Whether a loan was mispriced or not was impossible
to determine with certainty, it all depended on the individual subjective assessment of the
obligor’s default risk. The main “cost” of extending a loan was the cost of the regulatory risk
capital as prescribed by the rules of the Basel I capital accord, and this is the point where credit
derivatives came in.

Credit derivatives allow banks to transfer the credit risk (and thus a large part of the regulatory
capital) off their books while keeping the loan business; the borrower would never need to
know that the bank had hedged his exposure. They can also substitute “good” risks for “bad”
risks (and some cash) while keeping the required capital constant, and can actively exploit
many of the other loopholes that exist in the regulatory framework. Using creatively structured
credit derivatives, some banks were able to reduce their regulatory capital requirements so far
that these requirements almost lost their meaning. Ultimately this led to the collapse of the
Basel I capital accord and to the development of the new Basel II rules designed to take these
new possibilities into account.

It can be argued that the regulatory capital rules of Basel I fuelled the growth of the market
for credit derivatives in its initial stages and helped these instruments to attain the critical mass
of participants and liquidity. But credit derivatives had and still have other applications, even
with changed regulatory capital rules they are here to stay.

In recent years, credit exposure on a large number of obligors has become a traded asset.
And even if a particular credit exposure is not among these traded credits, it nevertheless is a
potentially traded asset. It only requires filling another name in the ISDA standard confirmation
to open up a new credit market. So, at least potentially, almost every important credit exposure
has its market.

This gives us the most important reason why credit derivatives will remain a growing sector:
credit is now a traded asset. While credit portfolios used to resemble an insurance portfolio,
they are now closer to a portfolio of traded assets like equities. The management of equity
portfolios involves marking-to-market, active trading, risk–return analysis and tradeoffs, active
portfolio optimisation. All these activities have their credit analogue in the trading of credit
derivatives.

The most common instrument by far is the credit default swap (CDS). A CDS is a swap
contract in which one counterparty (the protection buyer) pays a regular fee (the CDS fee) and
the other counterparty (the protection seller) must pay a default payment if a credit event should
occur with respect to the reference credit. The default payment is designed to approximate the loss that a holder of a bond issued by the reference credit would suffer at the default event. (The exact specification of these terms will be described later on, they have been standardised by the ISDA.) In the course of time, other instruments have emerged, most notably portfolio credit derivatives which pose entirely new modelling problems, and credit derivatives with option features embedded in them.

The book begins with a presentation of the most important credit derivatives and their payoff structures in Chapter 2. There are other, more specialised books on the details of the documentation of credit derivatives so we focus on those features that are relevant for the pricing and risk management of credit derivatives. Thus we focus on the cash flows of the credit derivatives and describe some rough, but model-independent, hedging strategies. These strategies can be made more precise using the spread curve and bond-based pricing techniques presented in the following chapter. This chapter provides models that connect the markets for credit default swaps and defaultable bonds of the same issuer. Furthermore, the problem of pricing single-name credit default swaps is reduced to the pricing of a set of basic building block securities: defaultable zero-coupon bonds and payoffs at default.

For the more advanced models some mathematical background is necessary, which is given in Chapter 4. In particular, useful facts on point processes and stopping times are summarised. Chapter 5 then picks up where Chapter 3 left off and provides an introduction to the intensity-based approach to the pricing of credit risk and credit derivatives. Intensity-based models are the most popular class of credit derivatives pricing models, they are easily calibrated to market prices and provide realistic dynamics of defaultable bond prices.

Chapter 6 treats the modelling of recovery risk, an important input variable of all credit derivatives pricing models. In the literature on intensity-based credit risk models, several different approaches to the modelling of positive recovery have been proposed. In this chapter, these approaches are compared and their implications for the prices of defaultable coupon bonds and credit default swaps are analysed. Recovery rate risk and default risk are frequently not separable using market data, therefore historical experience is often used to fix one of the two parameters and the last part of this chapter treats the results of some of the most recent studies on defaulted bond recoveries.

Based upon the analysis of the previous chapters, we can tackle the question of how to concretely specify and implement intensity-based models in Chapter 7. Here a variety of different approaches are presented: two concrete specifications with analytical tractability (a multifactor Gaussian model and a multifactor square-root-based model of the Cox et al., 1985b type); a model for the full term structure of credit spreads based upon the Heath et al. (1992) approach; and a variety of numerical implementation methods, amongst them a tree-based implementation algorithm, and a discussion of p.d.e.-based methods and Monte Carlo methods that can be used to implement intensity-based default risk models.

The credit rating models of Chapter 8 can be viewed as an extension of the intensity-based models treated so far. Here, default intensities can also depend on the rating transition process. In Chapter 9, the firm’s value-based approach to credit risk modelling is explored. These models have much appeal as they provide a link between the equity and credit markets. The analogies between this approach and pricing models for equity derivatives are shown, and empirical evidence of the accuracy of the firm’s value-based models is presented.

Finally, Chapter 10 treats the modelling of portfolio credit risk and default correlation. While single-name credit risk models often show great analogies to interest rate models (in the
Preface

intensity-based approach) or exotic equity options (in the firm’s value models), portfolio and basket credit derivatives pose entirely new modelling problems. Starting from an analysis of the nature of these problems, we investigate different approaches to these modelling issues. Starting from simple static one-time-step models like Moody’s binomial expansion technique and the Vasicek (1987) model, we then analyse the possibilities to directly extend the firm’s value models and the intensity-based models to the multi-obligor case. As these direct attempts to cover portfolio credit risk all have their problems and disadvantages, we then also introduce copula-based models, the most promising new approach in this area. To this end, basic facts about copula functions are introduced and the models are built up step-by-step from simple static copula default models to semi-dynamic models and finally fully dynamic models.

Many of the credit derivatives pricing models presented in this book are extensions or modifications of pricing models that were developed in other contexts, most frequently interest rate modelling or exotic equity options. Therefore, modelling credit risk can be a quite advanced subject in terms of the background knowledge that is presupposed. While I tried to avoid unnecessary mathematical complications and chose the simpler option whenever this was possible without losing realism, it was not possible to present the material without assuming some previous knowledge in mathematical finance.

On the finance side, I assume previous knowledge of the level of a typical introductory book on mathematical finance in continuous time, like the books by Hull (1989), Wilmott et al. (1993) or Nielsen (1999). On the mathematical side it is assumed that the reader is familiar with the basics of the theory of stochastic processes. In particular, I assume familiarity with probability spaces, measurability, filtrations, the stochastic integral with respect to Brownian motion and diffusion processes, quadratic variation for continuous processes and Itô’s lemma for diffusion processes. Basic knowledge of concepts like adapted and predictable stochastic processes, martingales and the Doob–Meyer decomposition will also be helpful. There are now a number of good and not too difficult books on the subject, e.g. the books by Neftci (1996) or Lamberton and Lapeyre (1996). Mathematical concepts that cannot be found in these books will be explained here with as little technicality as possible. I will give references to the mathematical literature for those readers who would like to go into more depth on these issues.
Acknowledgements

This book would not have been possible without input and support from many sides. Special thanks are due to Professor Dr. Dieter Sondermann and all my colleagues at the Department of Statistics, Bonn University who created the environment in which a project such as this book could flourish; to my editor Samantha Whittaker at Wiley for her patience and support; to Martin Helm, Greg Gupton, Ebbe Rogge, Erik Schlögl, Lutz Schlögl and Paul Wilmott for carefully reading the whole or parts of the manuscript and providing valuable comments and guidelines for improvement. I am further indebted to all fellow researchers on credit risk modelling issues who helped push forward the frontiers of the area, and to many practitioners and participants at my training courses who helped me stay focused on the realism and applicability of the models I developed and who provided fascinating research questions.

Needless to say, while I am extremely grateful for all the support I received during the time it took to write this book, I still claim sole responsibility for any errors that occur.
Abbreviations

ANC       affine combination of non-central chi-squared random variables
bn        billion (American definition: 1 bn = 1000m)
bp        basis points (1 bp = 1/100 percentage points)
CBO       collateralised bond obligation
CDO       collateralised debt obligation
CDS       credit default swap
CIR       Cox, Ingersoll, Ross (1985) interest-rate model
CLN       credit-linked note
CLO       collateralised loan obligation
CSO       credit spread option
DDS       default digital swap
DiD       distance to default (KMV model)
E2C       ’equity-to-credit’ default-risk model
EDF       expected default frequency (KMV model)
EUR       euro
Euribor    European interbank offered rate
FR        fractional recovery model
FRA       forward rate agreement
FiD       first-to-default swap
HJM       Heath, Jarrow, Morton interest rate model
ISDA      International Swap Dealers Association
JLT       Jarrow, Lando, Turnbull rating transition model
JPY       Japanese yen
KMV model default risk model marketed by Moody’s KMV
LGD       loss given default
Libor     London interbank offered rate
LMM       Libor market model
m         million
MD        multiple defaults model
NPV       net present value
nDiD      nth-to-default swap
OTC       over-the-counter (not traded on an organised exchange)
PDE       partial differential equation
<table>
<thead>
<tr>
<th>Abbreviations</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>repo</td>
<td>repurchase (referring to repurchase transactions)</td>
</tr>
<tr>
<td>RMV</td>
<td>recovery of market value model</td>
</tr>
<tr>
<td>RP</td>
<td>recovery of par value model</td>
</tr>
<tr>
<td>RT</td>
<td>recovery of treasury model</td>
</tr>
<tr>
<td>S&amp;P</td>
<td>Standard and Poor’s</td>
</tr>
<tr>
<td>SPV</td>
<td>special purpose vehicle</td>
</tr>
<tr>
<td>StD</td>
<td>second-to-default swap</td>
</tr>
<tr>
<td>t</td>
<td>thousand</td>
</tr>
<tr>
<td>TRS</td>
<td>total return swap</td>
</tr>
<tr>
<td>USD</td>
<td>US dollar</td>
</tr>
<tr>
<td>ZCB</td>
<td>zero-coupon bond</td>
</tr>
<tr>
<td>ZR</td>
<td>zero recovery model</td>
</tr>
</tbody>
</table>
Notation

Counterparties

A  protection buyer
B  protection seller
C  reference obligor

Loans and bonds

A      value of default-free annuity
c      coupon
τ      coupon (defaultable bond)
C      coupon-bearing bond, default-free
C      coupon-bearing bond, defaultable
K      notional

Rates and spreads

L      Libor
r_{repo} repo rate
s      interest-rate swap rate (fixed-for-floating)
s^A    asset swap rate
s^{par} par spread of defaultable bonds
s^{TRS} TRS rate
τ      CDS rate
τ^{DDS} default digital swap rate
τ^{FtD} FtD rate

Default-free term structure of bond prices

b      default-free continuously compounded bank account
B      zero-coupon bond price
β      discount factor
f      forward rate, continuous compounding
<table>
<thead>
<tr>
<th>xx</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>forward rate, effective compounding</td>
</tr>
<tr>
<td>$r$</td>
<td>instantaneous short rate, continuously compounded</td>
</tr>
<tr>
<td>$R$</td>
<td>short rate, effective compounding</td>
</tr>
</tbody>
</table>

**Defaultable term structure of bond prices**

| $\bar{B}$ | zero-coupon bond price, zero recovery             |
| $\beta$   | discount factor                                    |
| $\bar{f}$ | forward rate, continuous compounding              |
| $\bar{F}$ | forward rate, effective compounding               |
| $\bar{r}$ | instantaneous short rate, continuously compounded |
| $\bar{R}$ | short rate, effective compounding                 |

**Term structure dynamics**

| $\mu$ | drift, forward rates                              |
| $\bar{\mu}$ | drift, defaultable forward rates                  |
| $\sigma$ | volatility, forward rates                         |
| $\bar{\sigma}$ | volatility, defaultable forward rates             |

**Default model**

| $I$  | survival indicator function                        |
| $\lambda$ | default intensity                                 |
| $P$  | survival probability                               |
| $\bar{P}$ | pseudo-survival probability                       |
| $P^{\text{def}}$ | default probability                              |
| $\tau$ | time of default                                    |

**Recovery models**

| $c$  | recovery rate, recovery of treasury                |
| $\pi$ | recovery rate, recovery of par                     |
| $q$  | recovery rate, fractional recovery                 |
| $Q$  | payoff quota, multiple defaults                    |

**Rating transition models**

| $\{1, \ldots, K\}$ | credit rating classes                             |
| $I$                 | identity matrix                                   |
| $\Lambda$           | generator matrix                                  |
| $Q(t, T)$           | transition probability matrix                     |
| $R(t)$              | rating at time $t$                                |
Firm’s value models

\( \overline{K} \) default barrier
\( S \) share price of the firm
\( V \) firm’s value

Other notation

\( \Delta t > 0 \) time step in discretisation
\( (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, Q) \) filtered probability space
1

Introduction

1.1 THE WORLD OF CREDIT RISK

Credit risk\(^1\) is the risk that an obligor does not honour his payment obligations.

Besides being an obvious topic for the introduction of a book on credit risk models, the proper definition of “credit risk” or “default risk” is also an important point in the documentation and definition of credit derivatives. Default risk is intrinsically linked to the payment obligation which the obligor ought to honour. An obligor who does not have any payment obligations, does not have any default risk either. (And no-one would be interested in his default risk anyway.) Therefore this definition only covers the default risk of a payment obligation, but not the default risk of the obligor himself. In principle, an obligor could not pay one of his obligations, but honour another.

This behaviour by the obligor is prevented by legal rules: the bankruptcy codes and contract law. Thanks to these, we can speak of the default risk of an obligor without specifying a particular payment obligation, because the obligor has to honour all his payment obligations as long as he is able to. If he is not able to do so, a workout procedure is entered, the obligor loses control of all of his assets and an independent agent tries to find ways to pay off the creditors using the obligor’s assets. The bankruptcy code ensures that all creditors of the obligor are treated fairly and in accordance with a predetermined procedure. In particular, it is ensured that a default on one obligation entails a default on all other obligations, so the obligor cannot choose which claims he honours. Furthermore, because the workout procedure usually involves significant losses by the obligor, his incentives are such that he will try to ensure his solvency: a default only occurs if the obligor really cannot pay his obligations. But then it is no surprise that a default almost invariably entails a loss to the creditor.

It is usually very hard to predict these losses in default. They contain several unpredictable components: the question whether the obligor’s business will be shut down and liquidated or sold to another investor, what assets there are to liquidate, the liquidation value or the sales price of the business, the vagaries of the legal proceedings, to name just a few.

Obligors who are not bound by bankruptcy codes, e.g. sovereign borrowers and borrowers in countries without a properly functioning legal system, frequently make use of the possibility to default only on selected of their obligations, sometimes without being in real financial distress. In these cases, the connection of default risk to the particular underlying payment obligation must not be ignored. Creditors often include cross-acceleration (or cross-default) clauses in the loan or bond contracts of such borrowers to ensure that a default on one obligation triggers an immediate repayment (or a default) of the debtor’s other obligations. The severity of the losses in these cases is even more unpredictable than in the cases which are governed by a legal procedure. They depend almost entirely on the outcome of workout negotiations between the lenders and the obligor.

\(^1\) Throughout the book, the terms “default risk” and “credit risk” will be used as synonyms.
2 Credit Derivatives Pricing Models

This discussion already shows some important properties of default risk which make their quantitative modelling difficult:

- Default events are rare.
- They may occur unexpectedly.
- Default events involve significant losses.
- The size of these losses is unknown before default.

If default risk cannot be defined without a payment obligation, the converse is also true: there is no payment obligation that does not entail some credit risk. Default risk is ubiquitous: firms might default on principal or interest payments on loans or bonds, on trade receivables, on tax obligations, on the wages of their employees, or on the invoices of their business partners, to name just a few. Basically, everything that represents a payment obligation carries default risk for exactly that reason.

In Section 2.3 we give a short overview of the most important defaultable payment obligations. They were selected for their relevance to credit risk modelling and credit derivatives. Thus we concentrate on large, financial obligations like loans and bonds which are usually of a long-term nature, while ignoring most of the others. These securities are used as underlying assets for credit derivatives, to define credit events, payoff times and payoffs of credit derivatives, and as hedge instruments to hedge the risks of uncovered credit derivatives positions.

1.2 THE COMPONENTS OF CREDIT RISK

We have already identified the most important components of credit risk.

**Arrival risk** is a term for the uncertainty whether a default will occur or not. To enable comparisons, it is specified with respect to a given time horizon, usually one year. The measure of arrival risk is the probability of default. The probability of default describes the distribution of the indicator variable default before the time horizon.

**Timing risk** refers to the uncertainty about the precise time of default. Knowledge about the time of default includes knowledge about the arrival risk for all possible time horizons, thus timing risk is more detailed and specific than arrival risk. The underlying unknown quantity (random variable) of timing risk is the time of default, and its risk is described by the probability distribution function of the time of default. If a default never happens, we set the time of default to infinity.

**Recovery risk** describes the uncertainty about the severity of the losses if a default has happened. In recovery risk, the uncertain quantity is the actual payoff that a creditor receives after a default. It can be expressed in several ways which will be discussed in a later chapter. Market convention is to express the recovery rate of a bond or loan as the fraction of the notional value of the claim that is actually paid to the creditor. Recovery risk is described by the probability distribution of the recovery rate, i.e. the probabilities that the recovery rate is of a given magnitude. This probability distribution is a conditional distribution, conditional upon default.

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2 Later on we will also speak of a default-free payoff but this is a convenient abstraction. Even the US government can default on its obligations (and many of the US states have indeed defaulted in history), we only assume that its default risk is negligible compared to the credit risk of some other obligor.
Market risk describes a different kind of risk, the risk of changes in the market price of a defaultable asset, even if no default occurs. Apart from other market factors that also affect the prices of default-free claims, market risk is also driven by changes in timing and recovery risks, or at least changes in the market’s perception of these risks. This risk might be called risk change risk.

Market risk models are dynamic models, thus they add an additional layer of complexity. To avoid arbitrage opportunities in the model, market risk must be modelled in consistency with timing and recovery risk, and the changes in these risks, and in consistency with other market prices. For standard credit default swaps, changes in credit risk are in fact the dominant driver of market risk, while defaultable coupon bonds are also strongly influenced by changes in the default-free interest rates.

The impact of default risk can be affected by the behaviour of other market variables like movements in default-free interest rates, exchange rates, etc. These may influence the value of the defaultable claim, for instance if counterparty risk is considered in derivatives transactions. But it is also present in classical loans: for a given recovery rate, a default on a fixed-coupon loan in a high-interest-rate environment is less severe than a default on the same loan in a low-interest-rate environment, because the net present value of the lost claim is lower in the former case.

The term market price correlation risk covers this type of risk: the risk that defaults (and default likelihoods) are correlated with price movements of the defaultable asset.

While arrival risk and timing risk are usually specific to one defaultable obligor, recovery risk, market risk and market price correlation risk are specific to a particular payment obligation of a given obligor, or at least to a particular class of payment obligations.

If we consider the risk of joint defaults of several obligors, an additional risk component is introduced. Default correlation risk describes the risk that several obligors default together. Again here we have joint arrival risk which is described by the joint default probabilities over a given time horizon, and joint timing risk which is described by the joint probability distribution function of the times of default.

From a theoretical point of view, it is desirable to include as many of the different faces of default risk as possible. This comes at the cost of additional complexity in the model, implementation problems and slower runtime. Therefore, the first question a modeller has to ask himself is: Which risks do I need to include in the model, and which risks can I ignore without major losses in realism? For example, dynamic models of market risk are necessary to risk-manage and mark-to-model credit derivatives and tradeable defaultable bonds on a frequent basis. For a static book of loans this may be less important than having an accurate model of the default correlations. A second constraint is given by the available data. If there is no data to base a sophisticated model upon, one might as well choose a simpler version of the model which requires fewer inputs.

But this comes with a warning: every simplification involves implicit assumptions about the risks that are modelled, and these assumptions may have consequences that are not always obvious. One aim of this book is to give the quantitative modeller a good enough understanding of the properties of the different models, the risks that are modelled and the risks that are ignored, and the effects that changes in the inputs have on the model’s outputs. This should enable him to make modelling decisions more confidently, and it should tell him which model is suitable to answer which question and in which circumstances to be wary of the model’s outputs.
1.3 MARKET STRUCTURE

The market for credit derivatives was created in the early 1990s in London and New York. It is the market segment of derivative securities which is growing fastest at the moment. Table 1.1 shows estimates of the development of the market. The real size of the market will probably be even larger as both surveys only reached a subset of the market participants. Despite the downturn of most other markets, the credit derivatives market still exhibits strong growth rates. Nevertheless, viewed as a fraction of the total credit exposure in the world the market share of credit derivatives is minute – at least on this front the market is far from its limits.

According to the recent study by Risk magazine (Table 1.2; Patel, 2002), the largest share of the market is taken up by credit default swaps (CDSs). While the large majority of these will be “vanilla” CDSs based upon ISDA standards, there can still be significant variations and complications in the structures like cancellation or extension options or coupon stepups. First-to-default swaps (FtDs) also seemed to be grouped into this category. The second largest group are portfolio-related credit derivatives like collateralised loan obligations (CLOs), portfolio tranche protection, and synthetic collateralised debt obligations (CDOs). Finally, there are more exotic credit derivatives like credit spread options and hybrid instruments (i.e. instruments that combine features of other derivatives with a CDS) but – as mentioned before – many of the “exotic” features may already be present in a CDS.

Among newly traded CDSs, the most common specification has five years’ maturity, a notional amount between 5m and 20m USD (or EUR), and is written on a large investment-grade reference credit at the lower end of the investment grade rating scale (A to BBB). Market share between European and US obligors is about equal, with 40% from Europe and 44% from the USA. Asian obligors had 12% market share in the Risk survey and emerging market obligors (excluding Asia) made up 5% of the market share. (As no Japanese bank took part in the Risk survey, the Asian market share may be misrepresented.) In many risks, the market for CDSs is already liquid enough to serve as a benchmark for marking-to-market or risk management.

<table>
<thead>
<tr>
<th>Table 1.1</th>
<th>Size of the market for credit derivatives according to surveys by the British Bankers’ Association and Risk (Patel, 2002)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year</td>
<td>1997</td>
</tr>
<tr>
<td>Outstanding notional (USD bn)</td>
<td>170</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 1.2</th>
<th>Market share by instrument type (rounded numbers)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instrument</td>
<td>Share (%)</td>
</tr>
<tr>
<td>Credit default swaps (including FtDs)</td>
<td>67</td>
</tr>
<tr>
<td>Synthetic balance sheet CLOs</td>
<td>12</td>
</tr>
<tr>
<td>Tranched portfolio default swaps</td>
<td>9</td>
</tr>
<tr>
<td>Credit-linked notes, asset repackaging, asset swaps</td>
<td>7</td>
</tr>
<tr>
<td>Credit spread options</td>
<td>2</td>
</tr>
<tr>
<td>Managed synthetic CDOs</td>
<td>2</td>
</tr>
<tr>
<td>Total return swaps</td>
<td>1</td>
</tr>
<tr>
<td>Hybrid credit derivatives</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Source: Risk (Patel, 2002).
Examples are sovereign credit risk, most banks and financials, the large telecoms, and large industrial firms.

Participants in the market for credit derivatives are mostly banks (47%) who are motivated by regulatory capital arbitrage, funding arbitrage, or trading motives. Insurances and reinsurances (23%) and investment funds (5%) also have a large market share and often are the ultimate suppliers of credit protection to the market. They use credit derivatives as investments, or for the credit risk management of bond portfolios. Insurances feel comfortable with tranched portfolio transactions because many of the structures are designed to resemble reinsurance transactions. Hedge funds (8%) are entering the credit derivatives business in increasing numbers, motivated by the opportunities to gain on relative value trades between different markets (bonds and CDSs) and by the high leverage that many credit derivatives transactions allow. Not many industrials (4%) use credit derivatives up to this point, if they do they do it to hedge counterparty or country risk in isolated cases.

Trading takes place over-the-counter, but there are some virtual meeting places for buyers and sellers. The large investment banks routinely quote prices on their Bloomberg and internet pages, there are large specialised brokers (e.g. GFInet), and also two electronic exchanges for credit default swaps (creditex and credittrade). Nevertheless, these are only means to bring the counterparties together, in the end the contract is signed between the two counterparties and there is no central counterparty as there would be on an organised exchange.