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Electromagnetic wave scattering is an active, interdisciplinary area of research with myriad practical applications in fields ranging from atomic physics to medical imaging to geoscience and remote sensing. In particular, the subject of wave scattering by random discrete scatterers and rough surfaces presents great theoretical challenges due to the large degrees of freedom in these systems and the need to include multiple scattering effects accurately. In the past three decades, considerable theoretical progress has been made in elucidating and understanding the scattering processes involved in such problems. Diagrammatic techniques and effective medium theories remain essential for analytical studies; however, rapid advances in computer technology have opened new doors for researchers with the full power of Monte Carlo simulations in the numerical analysis of random media scattering. Numerical simulations allow us to solve the Maxwell equations exactly without the limitations of analytical approximations, whose regimes of validity are often difficult to assess. Thus it is our aim to present in these three volumes a balanced picture of both theoretical and numerical methods that are commonly used for tackling electromagnetic wave scattering problems. While our book places an emphasis on remote sensing applications, the materials covered here should be useful for students and researchers from a variety of backgrounds as in, for example, composite materials, photonic devices, optical thin films, lasers, optical tomography, and X-ray lithography. Introductory chapters and sections are also added so that the materials can be readily understood by graduate students. We hope that our book would help stimulate new ideas and innovative approaches to electromagnetic wave scattering in the years to come.

The increasingly important role of numerical simulations in solving electromagnetic wave scattering problems has motivated us to host a companion web site that contains computer codes on topics relevant to the book. These computer codes are written in the MATLAB programming language and are available for download from our web site at www.emwave.com. They are provided to serve two main purposes. The first is to supply our readers a hands-on laboratory for performing numerical experiments, through which the concepts in the book can be more dynamically relayed. The second is to give new researchers a set of basic tools with which they could quickly build on projects of their own. The fluid nature of the web site would also allow us to regularly update the contents and keep pace with new research developments.
The present volume covers the basic principles and applications of electromagnetic wave scattering and lays the groundwork for the study of more advanced topics in Volumes II and III. We start in Chapter 1 with exact and approximate solutions of wave scattering by a single particle of simple shape. Such problems can be solved exactly by expanding the fields in terms of scalar or vector waves in separable coordinates, depending on the geometry of the scatterer. When the size of the scatterer is small, Rayleigh scattering represents a simple and valid approximation. When scattering is weak, the Born approximation can be applied to the volume integral equation for the internal field. Approximate solutions are also useful when the scatterer lacks perfect symmetry as in the case of a finite cylinder. In Chapter 2, we discuss basic scattering theory. We introduce the Green's function for the wave equation and its various coordinate representations. From the vector Green's theorem, we derive the Huygens' Principle and the extinction theorem, which are especially useful for formulating surface integral equations in scattering problems. The reciprocity principle leads to useful symmetry relations in the scattering amplitudes and the Green's function, while energy conservation leads to the optical theorem. The T-matrix formulation with the extended boundary condition technique is a popular method that can be used to calculate scattering from an arbitrarily shaped object. We give explicit results for dielectric spheres and spheroids.

In Chapter 3, we begin the study of electromagnetic scattering by a random collection of scatterers. The concepts of fluctuating fields and ensemble averaging are of central importance in random media scattering. These and related ideas are explored in this chapter. The specific intensity is often used to describe energy transport through a random medium. The fully polarimetric description of the specific intensity is provided by the Stokes vector. As an application to passive remote sensing, we derive the emissivity of the four Stokes parameters using the fluctuation dissipation theorem. Basic radiative transfer (RT) theory elements including the extinction coefficient and scattering phase matrix are also introduced. In contrast to conventional RT theory, the phase functions are defined in terms of bistatic scattering cross sections. This allows for the development of the dense medium radiative transfer theory (DMRT) to be discussed in Volumes II and III. Many natural media, e.g., snow, vegetation, and ocean surfaces, can be effectively modeled in terms of simple random media. Chapter 4 is devoted to the statistical characterizations of such random discrete media and rough surfaces. Useful characterizations include the pair distribution function for volume scatterers and the power spectrum for rough surfaces.
In Chapter 5, we consider scattering and emission by plane-parallel layered media, which provide simple but very useful models for geophysical remote sensing. We solve this problem in two different ways: the coherent or wave approach, which is exact, versus the incoherent or radiative transfer approach. This gives us some insights into the approximations involved in RT theory. In Chapter 6, we discuss the single scattering approximation, where each particle is assumed to scatter independently. However, we take into account of the phase coherence in the addition of scattered fields. We demonstrate the existence of an interesting correlation effect in random media scattering known as the memory effect. As will be shown in Volumes II and III, this effect persists even when multiple scattering is included. Applications of single scattering to synthetic aperture radar (SAR) and random media scattering are also discussed.

In Chapters 7 and 8, we take a closer look at the radiative transfer equation and its solutions. The iterative method is useful when scattering is weak and provides physical correspondence with different orders of multiple scattering. When scattering is strong, the discrete ordinate eigenanalysis approach can be used to obtain numerically exact solutions. For scattering media with inhomogeneous profiles, the method of invariant imbedding can be applied. Diffusion approximation is useful when, upon multiple scattering, the intensities have been diffused almost uniformly in all directions. We illustrate these solution techniques with extensive examples from active and passive microwave remote sensing.

In Chapter 9, we discuss wave scattering by random rough surfaces. Despite much theoretical and numerical efforts, the two "classical" analytical approximations of small perturbation method and Kirchhoff approach are still the simplest and most widely used analytical methods for solving rough surface problems. Here they are illustrated using one-dimensional rough surfaces with Dirichlet and Neumann boundary conditions. Two-dimensional rough surface scattering problems are discussed extensively in Volumes II and III.

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# Chapter 1

INTRODUCTION TO ELECTROMAGNETIC SCATTERING BY A SINGLE PARTICLE

1. **Basic Scattering Parameters**
   - 1.1 Scattering Amplitudes and Cross Sections
   - 1.2 Scattering Amplitude Matrix

2. **Rayleigh Scattering**
   - 2.1 Rayleigh Scattering by a Small Particle
   - 2.2 Rayleigh Scattering by a Sphere
   - 2.3 Rayleigh Scattering by an Ellipsoid
   - 2.4 Scattering Dyads

3. **Integral Representations of Scattering and Born Approximation**
   - 3.1 Integral Expression for Scattering Amplitude
   - 3.2 Born Approximation

4. **Plane Waves, Cylindrical Waves, and Spherical Waves**
   - 4.1 Cartesian Coordinates: Plane Waves
   - 4.2 Cylindrical Waves
   - 4.3 Spherical Waves

5. **Acoustic Scattering**

6. **Scattering by Spheres, Cylinders, and Disks**
   - 6.1 Mie Scattering
   - 6.2 Scattering by a Finite Length Cylinder Using the Infinite Cylinder Approximation
   - 6.3 Scattering by a Disk Based on the Infinite Disk Approximation

References and Additional Readings
A major topic in this book is the study of propagation and scattering of waves by randomly distributed particles. We first consider scattering by a single particle. This chapter and the next discuss and derive the scattering characteristics of a single particle. Both exact and solutions are studied. Scattering by a single particle is an important subject in electromagnetics and optics. There exist several excellent textbooks on this subject [van de Hulst, 1957; Kerker, 1969; Bohren and Huffman, 1983]. We will treat those topics that are pertinent to later chapters of multiple scattering by random discrete scatterers.

1 Basic Scattering Parameters

1.1 Scattering Amplitudes and Cross Sections

Consider an electromagnetic plane wave impinging upon a particle which has permittivity $\epsilon_p(\vec{r})$ that is different from the background permittivity $\epsilon$ (Fig. 1.1.1). The finite support of $\epsilon_p(\vec{r}) - \epsilon$ is denoted as $V$.

The incident wave is in direction $\vec{k}_i$ and has electric field in direction $\hat{e}_i$ that is perpendicular to $\vec{k}_i$. The electric field of the incident wave is

$$\vec{E}_i = \hat{e}_i E_0 e^{i\vec{k}_i \cdot \vec{r}} \quad (1.1.1)$$

where

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \quad (1.1.2)$$

is the position vector, and

$$k = \omega \sqrt{\mu \epsilon} = \frac{2\pi}{\lambda} \quad (1.1.3)$$

is the wavenumber. In (1.1.3), $\omega$ is the angular frequency, $\mu$ is the permeability, and $\lambda$ is the wavelength. In (1.1.1), $E_0$ is the amplitude of the electric field. The time harmonic dependence $\exp(-i\omega t)$ has been suppressed.

In the far field, the scattered field is that of a spherical wave with dependence $e^{ikr}/r$, where $r$ is the distance from the particle. In general, the particle scatters waves in all directions. Let $\vec{E}_s$ be the far field scattered field in direction of $\vec{k}_s$. Since Maxwell’s equations are linear, we write

$$\vec{E}_s = \hat{e}_s f(\hat{k}_s, \hat{k}_i) E_0 \frac{e^{ikr}}{r} \quad (1.1.4)$$

where $\hat{e}_s$ is perpendicular to $\vec{k}_s$. The proportionality $f(\hat{k}_s, \hat{k}_i)$ is called the scattering amplitude from direction $\hat{k}_i$ into direction $\hat{k}_s$. 

Figure 1.1.1 Scattering of a plane electromagnetic wave $\mathbf{E}_i(\mathbf{r})$ by a particle occupying volume $V$ and having permittivity $\varepsilon_p(\mathbf{r})$. The scattered field is $\mathbf{E}_s(\mathbf{r})$.

The magnetic field associated with the incident wave is

$$\mathbf{H}_i = \frac{1}{\eta} \mathbf{k}_i \times \mathbf{E}_i$$  \hspace{1cm} (1.1.5)

where $\eta = \sqrt{\mu/\varepsilon}$ is the wave impedance. The Poynting vector denoting power flow per unit area is

$$\overline{S}_i = \frac{1}{2} \text{Re} \left( \mathbf{E}_i \times \mathbf{H}_i^* \right) = \frac{|\mathbf{E}_o|^2}{2\eta} \mathbf{k}_i$$  \hspace{1cm} (1.1.6)

Similarly, for the scattered wave, the magnetic field is

$$\mathbf{H}_s = \frac{1}{\eta} \mathbf{k}_s \times \mathbf{E}_s$$  \hspace{1cm} (1.1.7)

and Poynting's vector is

$$\overline{S}_s = \frac{1}{2} \text{Re} \left( \mathbf{E}_s \times \mathbf{H}_s^* \right) = \frac{|\mathbf{E}_s|^2}{2\eta} \mathbf{k}_s$$  \hspace{1cm} (1.1.8)

Using (1.1.4) in (1.1.8) gives

$$\overline{S}_s = \frac{|f(\hat{k}_s, \hat{k}_i)|^2 |\mathbf{E}_o|^2}{r^2} \frac{\hat{k}_s}{2\eta}$$  \hspace{1cm} (1.1.9)

Consider a differential solid angle $d\Omega_s$ in the scattered direction $\hat{k}_s$ (Fig. 1.1.2). In the spherical coordinate system

$$d\Omega_s = \sin \theta_s d\theta_s d\phi_s$$  \hspace{1cm} (1.1.10)

At a distance $r$, the surface area subtended by the differential solid angle $d\Omega_s$ is

$$dA = r^2 d\Omega_s = r^2 \sin \theta_s d\theta_s d\phi_s$$  \hspace{1cm} (1.1.11)
Then the differential scattered power \( dP_s \) through \( dA \) is
\[
dP_s = \left| \vec{S}_s \right| dA = \left| \vec{S}_s \right| r^2 d\Omega_s
\] (1.1.12)

Putting (1.1.9) in (1.1.12) gives
\[
dP_s = \left| f(\hat{k}_s, \hat{k}_i) \right|^2 \frac{|E_o|^2}{2\eta} d\Omega_s
\] (1.1.13)

Using the Poynting vector of the incident wave, from (1.1.6), we have
\[
\frac{dP_s}{\left| \vec{S}_i \right|} = \left| f(\hat{k}_s, \hat{k}_i) \right|^2 d\Omega_s
\] (1.1.14)

The dimension of equation (1.1.14) is area. It is convenient to define a differential scattering cross section \( \sigma_d(\hat{k}_s, \hat{k}_i) \) by
\[
\frac{dP_s}{\left| \vec{S}_i \right|} = \sigma_d(\hat{k}_s, \hat{k}_i) d\Omega_s
\] (1.1.15)

Comparing (1.1.14) and (1.1.15) gives
\[
\sigma_d(\hat{k}_s, \hat{k}_i) = \left| f(\hat{k}_s, \hat{k}_i) \right|^2
\] (1.1.16)

Integrating (1.1.14) over scattered angle gives
\[
\frac{P_s}{\left| \vec{S}_i \right|} = \int d\Omega_s \left| f(\hat{k}_s, \hat{k}_i) \right|^2
\] (1.1.17)
Thus the scattered power is

\[ P_s = \sigma_s |\mathbf{S}_i| \]  

(1.1.18)

where \( \sigma_s \) is the scattering cross section which is

\[ \sigma_s = \int d\Omega_s |f(\hat{k}_s, \hat{k}_i)|^2 = \int d\Omega_s \sigma_d(\hat{k}_s, \hat{k}_i) \]  

(1.1.19)

**Scattering Cross Section and Geometric Cross Section**

The geometric cross section \( \sigma_g \) of a particle is its area projected onto a plane that is perpendicular to the direction of incident wave \( \hat{k}_i \). Thus the power “intercepted” by the particle, \( P_r \), from a geometric optics standpoint, is the product of the geometric cross section and the magnitude of the incident Poynting vector:

\[ P_r = |\mathbf{S}_i| \sigma_g \]  

(1.1.20)

We can compare \( \sigma_g \) to \( \sigma_s \) and \( P_r \) to \( P_s \). Let \( D \) be the size of the object (the maximum distance between two points inside the object). When the size of the object \( D \) is much less than the wavelength, the results of Rayleigh scattering theory indicate that

\[ \frac{\sigma_s}{\sigma_g} = \mathcal{O}\left(\frac{D^4}{\lambda^4}\right) \ll 1 \]  

(1.1.21)

where \( \mathcal{O} \) denotes the order of magnitude. Thus

\[ \frac{P_s}{P_r} = \mathcal{O}\left(\frac{D^4}{\lambda^4}\right) \ll 1 \]  

(1.1.22)

in Rayleigh scattering. This shows that when the particle is small compared with the wavelength, the power scattered by the particle is much less than the product of geometric cross section and incident Poynting vector. In the short wavelength limit, \( D \gg \lambda \). Then

\[ \frac{\sigma_s}{\sigma_g} = \mathcal{O}(1) \]  

(1.1.23)

which is known as the geometric optics limit. It is important to remember that the scattering cross section \( \sigma_s \) also depends on the contrast between \( \epsilon_p \) and \( \epsilon \). In the case of weak scatterers when \( \epsilon_p \simeq \epsilon \), we have

\[ \sigma_s \sim \left| \frac{\epsilon_p}{\epsilon} - 1 \right|^2 \]  

(1.1.24)

where “\( \sim \)” denotes proportional to. Equation (1.1.24) is the result of the Born approximation.
Absorption Cross Section

The particle can also absorb energy from the incoming electromagnetic wave. Let

\[ \epsilon_p(\vec{r}) = \epsilon'_p(\vec{r}) + i\epsilon''_p(\vec{r}) \]  

(1.1.25)

From Ohm’s law, the power absorbed is

\[ P_a = \frac{1}{2}\omega \int_V d\vec{r} \epsilon''_p(\vec{r}) |E_{int}(\vec{r})|^2 \]  

(1.1.26)

where \( E_{int}(\vec{r}) \) denotes internal field which is the electric field inside the particle, \( d\vec{r} = dx\,dy\,dz \), and the integration in (1.1.26) is over the three-dimensional volume of the particle. The absorption cross section \( \sigma_a \) is defined by

\[ \sigma_a = \frac{P_a}{|S_i|} \]  

(1.1.27)

The total cross section of the particle is

\[ \sigma_t = \sigma_a + \sigma_s \]  

(1.1.28)

and the albedo of the particle is

\[ \tilde{\omega} = \frac{\sigma_s}{\sigma_t} \]  

(1.1.29)

Thus \( 0 \leq \tilde{\omega} \leq 1 \). The albedo is a measure of the fraction of scattering cross section in the total cross section.

1.2 Scattering Amplitude Matrix

We next generalize the concept of scattering amplitudes to include polarization effects. For the incident wave, the electric field \( \vec{E}_i \) is perpendicular to the direction of propagation \( \hat{k}_i \). There are two linearly independent vectors that are perpendicular to \( \hat{k}_i \). Let us call them \( \hat{a}_i \) and \( \hat{b}_i \). Then

\[ \vec{E}_i = (\hat{a}_i E_{ai} + \hat{b}_i E_{bi}) e^{ik_i \vec{r}} \]  

(1.1.30)

where \( \vec{k}_i = k\hat{k}_i \). The directions of \( \hat{k}_i, \hat{a}_i, \) and \( \hat{b}_i \) are such that they are orthonormal unit vectors following the right-hand rule.

Similarly for the scattered wave, let \( \hat{k}_s, \hat{a}_s, \) and \( \hat{b}_s \) form an orthonormal system. Then

\[ \vec{E}_s = (\hat{a}_s E_{as} + \hat{b}_s E_{bs}) \frac{e^{ikr}}{r} \]  

(1.1.31)
Figure 1.1.3 Geometry for defining the orthonormal unit system based on scattering plane. The scattering plane contains $\hat{k}_i$ and $\hat{k}_s$. The angle between $\hat{k}_i$ and $\hat{k}_s$ is $\Theta$.

The scattered field components, $E_{as}$ and $E_{bs}$, are linearly related to $E_{ai}$ and $E_{bi}$. The relation can be conveniently represented by a $2 \times 2$ scattering amplitude matrix

$$
\begin{bmatrix}
E_{as} \\
E_{bs}
\end{bmatrix} =
\begin{bmatrix}
f_{aa}(\hat{k}_s, \hat{k}_i) & f_{ab}(\hat{k}_s, \hat{k}_i) \\
f_{ba}(\hat{k}_s, \hat{k}_i) & f_{bb}(\hat{k}_s, \hat{k}_i)
\end{bmatrix}
\begin{bmatrix}
E_{ai} \\
E_{bi}
\end{bmatrix}
$$

(1.1.32)

**Orthonormal Unit Systems for Polarization Description**

There are two common choices of the orthonormal unit systems $(\hat{a}_i, \hat{b}_i, \hat{k}_i)$ and $(\hat{a}_s, \hat{b}_s, \hat{k}_s)$ that describe scattering by a particle.

**A. System Based on Scattering Plane**

Let the angle between $\hat{k}_i$ and $\hat{k}_s$ be $\Theta$ (Fig. 1.1.3). The plane containing the incident direction $\hat{k}_i$ and the scattered direction $\hat{k}_s$ is known as the scattering plane. Let

$$
\hat{a}_i = \hat{i}_i = \hat{i}_s = \hat{a}_s
$$

(1.1.33)

be the unit vectors that are perpendicular to this plane and let

$$
\hat{i}_i = \hat{i}_s = \frac{\hat{k}_s \times \hat{k}_i}{|\hat{k}_s \times \hat{k}_i|}
$$

(1.1.34)

Then by orthonormality,

$$
\hat{b}_i = \hat{2}_i = \hat{k}_i \times \hat{i}_i
$$

(1.1.35)
In this 1-2 system, the scattering amplitude matrix obeys the relation

\[
\begin{bmatrix}
E_{1s} \\
E_{2s}
\end{bmatrix} = \begin{bmatrix}
f_{11} & f_{12} \\
f_{21} & f_{22}
\end{bmatrix} \begin{bmatrix}
E_{1i} \\
E_{2i}
\end{bmatrix}
\] (1.1.37)

The advantage of this system is that the scattering amplitudes can take simple forms for particles with symmetry. The disadvantage for this system is that the directions of \( \hat{1}_i \) and \( \hat{2}_i \) depend on the scattered direction. For example, let \( \vec{E}_i = \hat{x}e^{ikz} \). The incident wave is propagating in the direction \( \hat{k}_i = \hat{z} \) and with \( \hat{x} \) polarization. If \( \hat{k}_s = \hat{y} \), then \( \hat{1}_i = \hat{x} \) and \( \hat{2}_i = \hat{y} \) and the incident wave is \( \hat{1}_i \) polarized. However, if \( \hat{k}_s = \hat{x} \), then \( \hat{1}_i = -\hat{y} \) and \( \hat{2}_i = \hat{x} \) and the incident wave is \( \hat{2}_i \) polarized. The inconvenience is that \( \vec{E}_i \) is \( \hat{x} \) polarized and propagation in \( \hat{z} \) direction, and yet it is \( \hat{1}_i \) or \( \hat{2}_i \) polarized depending on whether the scattered direction \( \hat{k}_s \) is \( \hat{y} \) or \( \hat{x} \).

Some useful relations for this orthonormal system are

\[
\hat{k}_s \cdot \hat{k}_i = \cos \Theta \] (1.1.38)

\[
\hat{2}_s \cdot \hat{2}_i = \hat{k}_s \cdot \hat{k}_i = \cos \Theta \] (1.1.39)

\[
\hat{k}_s \cdot \hat{2}_i = \hat{k}_s \cdot \hat{k}_i \times \hat{1}_i = \hat{k}_s \times \hat{k}_i \cdot \hat{1}_i = \left| \hat{k}_s \times \hat{k}_i \right| = \sin \Theta \] (1.1.40)

**B. Vertical and Horizontal Polarization**

In many problems there is a preferred direction, for example the vertical direction that is labeled \( \hat{z} \). In geophysical probing and earth remote sensing problems, that will be the vertical axis which is perpendicular to the surface of the earth. Then we can have vertical polarization \( \hat{v}_i \) and horizontal polarization \( \hat{h}_i \) that form an orthonormal system with \( \hat{k}_i \). We choose

\[
\hat{b}_i = \hat{h}_i = \frac{\hat{z} \times \hat{k}_i}{|\hat{z} \times \hat{k}_i|}
\] (1.1.41)

that is perpendicular to both \( \hat{z} \) and \( \hat{k}_i \). Then

\[
\hat{a}_i = \hat{v}_i = \hat{h}_i \times \hat{k}_i
\] (1.1.42)

Other names for vertical polarization are TM polarization, parallel polarization, and \( p \) polarization. Other names for horizontal polarization are TE polarization, perpendicular polarization, and \( s \) polarization. If \( \hat{k}_i \) is charac-
Rayleigh Scattering

2 Rayleigh Scattering

2.1 Rayleigh Scattering by a Small Particle

In Rayleigh scattering, the particle size $D$ is much less than wavelength $\lambda$. In such a case an oscillatory dipole with moment $\vec{p}$ is induced inside the particle. The field radiated by the dipole is the scattered field.

The far field radiated by a dipole $\vec{p}$ in the direction $\hat{k}_s$ is

$$\vec{E}_s = -\frac{k^2e^{ikr}}{4\pi\epsilon r} \hat{k}_s \times (\hat{k}_s \times \vec{p})$$

(1.2.1)
Let the electric field inside the particle be denoted by \( \vec{E}_{\text{int}} \), where the subscript \( \text{int} \) denotes internal. The polarization per unit volume inside the particle is

\[
\vec{P}_{\text{int}} = (\epsilon_p - \epsilon) \vec{E}_{\text{int}}
\]  

(1.2.2)

In Rayleigh scattering, the internal field is a constant vector inside the particle. The dipole moment of the particle is

\[
\vec{p} = v_o \vec{P}_{\text{int}}
\]  

(1.2.3)

where \( v_o \) is the volume of the particle. Using (1.2.2) and (1.2.3) in (1.2.1) gives

\[
\vec{E}_s = \frac{k^2 \epsilon e^{ikr}}{4\pi \epsilon r} (\epsilon_p - \epsilon) v_o \hat{k}_s \times \left( \hat{k}_s \times \vec{E}_{\text{int}} \right)
\]

(1.2.4)

Using (1.2.4), the scattering amplitude matrix can be determined by relating the internal field \( \vec{E}_{\text{int}} \) to the incident field \( \vec{E}_i \).

In Rayleigh scattering, the power absorbed by the particle is, from (1.1.26),

\[
P_a = \frac{1}{2} \omega \varepsilon' v_o |\vec{E}_{\text{int}}|^2
\]

(1.2.5)

### 2.2 Rayleigh Scattering by a Sphere

Consider a sphere of radius \( a \ll \lambda \) centered at the origin. Then the internal field inside the particle is

\[
\vec{E}_{\text{int}} = \frac{3\epsilon}{\epsilon_p + 2\epsilon} \vec{E}_i
\]

(1.2.6)

The internal field \( \vec{E}_{\text{int}} \) is parallel to the incident field \( \vec{E}_i \). Then using (1.2.6) in (1.2.4) and \( v_o = 4\pi a^3/3 \),

\[
\vec{E}_s = \frac{e^{ikr}}{r} f_o \left[ \hat{a}_s (\hat{a}_s \cdot \vec{E}_i) + \hat{b}_s (\hat{b}_s \cdot \vec{E}_i) \right]
\]

(1.2.7)

where

\[
f_o = k^2 a^3 \frac{\epsilon_p - \epsilon}{\epsilon_p + 2\epsilon}
\]

(1.2.8)

From (1.2.7)

\[
E_{as} = f_o (\hat{a}_s \cdot \vec{E}_i)
\]

(1.2.9a)

\[
E_{bs} = f_o (\hat{b}_s \cdot \vec{E}_i)
\]

(1.2.9b)
We can calculate the scattering amplitude matrix using the scattering plane orthonormal system.

First let $E_{1i} = 1$ and $E_{2i} = 0$, thus $\overrightarrow{E_i} = \hat{i}_i$. Then $E_{1s} = f_{11}$ and $E_{2s} = f_{21}$. From (1.2.9a)–(1.2.9b), we obtain $E_{1s} = f_o(\hat{1}_s \cdot \hat{i}_i) = f_o$ and $E_{2s} = f_o(\hat{2}_s \cdot \hat{i}_i) = 0$. Hence $f_{11} = f_o$ and $f_{21} = 0$. Next, let $E_{1i} = 0$ and $E_{2i} = 1$ so that $\overrightarrow{E_i} = \hat{2}_i$. Then $E_{1s} = f_{12}$ and $E_{2s} = f_{22}$. From (1.2.9a)–(1.2.9b), $E_{1s} = f_o(\hat{1}_s \cdot \hat{2}_i) = 0$ and $E_{2s} = f_o(\hat{2}_s \cdot \hat{2}_i) = f_o \cos \Theta$. Therefore $f_{12} = 0$ and $f_{22} = f_o \cos \Theta$. Thus the scattering amplitude matrix assumes following simple form in the scattering plane system of coordinates:

$$
\begin{bmatrix}
  f_{11} & f_{12} \\
  f_{21} & f_{22}
\end{bmatrix} =
\begin{bmatrix}
  f_o & 0 \\
  0 & f_o \cos \Theta
\end{bmatrix}
$$

(1.2.10)

Next, we use the orthonormal unit vectors of vertical and horizontal polarization,

$$
\begin{bmatrix}
  E_{vs} \\
  E_{hs}
\end{bmatrix} =
\begin{bmatrix}
  f_{vv} & f_{vh} \\
  f_{hv} & f_{hh}
\end{bmatrix}
\begin{bmatrix}
  E_{vi} \\
  E_{hi}
\end{bmatrix}
$$

(1.2.11)

To get $f_{vv}$ and $f_{hv}$, we let $E_{vi} = 1$ and $E_{hi} = 0$ so that $\overrightarrow{E_i} = \hat{v}_i$. Then from (1.2.9a)–(1.2.9b), $E_{vs} = f_o(\hat{v}_i \cdot \hat{i}_i)$ and $E_{hs} = f_o(\hat{h}_s \cdot \hat{i}_i)$. Thus,

$$
 f_{vv} = f_o(\hat{v}_i \cdot \hat{i}_i) = f_o \left[ \cos \theta_s \cos \theta_i \cos (\phi_s - \phi_i) + \sin \theta_s \sin \theta_i \right] \quad (1.2.12a)
$$

$$
 f_{hv} = f_o(\hat{h}_s \cdot \hat{i}_i) = -f_o \cos \theta_i \sin (\phi_s - \phi_i) \quad (1.2.12b)
$$

To get $f_{vh}$ and $f_{hh}$, we let $E_{vi} = 0$ and $E_{hi} = 1$ so that $\overrightarrow{E_i} = \hat{h}_i$. Then from (1.2.9a)–(1.2.9b), $E_{vs} = f_o(\hat{v}_s \cdot \hat{i}_i)$ and $E_{hs} = f_o(\hat{h}_s \cdot \hat{i}_i)$. Thus,

$$
 f_{vh} = f_o \cos \theta_s \sin (\phi_s - \phi_i) \quad (1.2.13a)
$$

$$
 f_{hh} = f_o \cos (\phi_s - \phi_i) \quad (1.2.13b)
$$

To calculate the scattering cross section, suppose that the incident wave is horizontally polarized. The scattered power is determined by integration over all scattered angles of the sum of the scattered vertical and horizontal polarizations:

$$
\sigma_s = \int_0^\pi d\theta_s \sin \theta_s \int_0^{2\pi} d\phi_s \left( |f_{vh}|^2 + |f_{hh}|^2 \right)
$$

$$
= |f_o|^2 \int_0^\pi d\theta_s \sin \theta_s \int_0^{2\pi} d\phi_s \left[ \cos^2 \theta_s \sin^2 (\phi_s - \phi_i) + \cos^2 (\phi_s - \phi_i) \right]
$$

$$
= \frac{8\pi}{3} |f_o|^2 = \frac{8\pi}{3} k^4 a^6 \left| \frac{\epsilon_p - \epsilon}{\epsilon_p + 2\epsilon} \right|^2
$$

(1.2.14)
If we compare the scattering cross section to the geometric cross section \( \sigma_g = \pi a^2 \) of a sphere, we obtain
\[
\frac{\sigma_s}{\sigma_g} = \frac{8}{3} \left( \frac{ka}{\lambda} \right)^4 \left| \frac{\epsilon_p - \epsilon}{\epsilon_p + 2\epsilon} \right|^2
\] \hspace{1cm} (1.2.15)

Since \( ka \ll 1 \), the scattering cross section is much less than the geometric cross section. It also follows from (1.2.15) that \( \sigma_s/\sigma_g = \mathcal{O}(D^4/\lambda^4) \), in agreement with (1.1.21).

**Absorption Cross Section**

The power absorbed by a small particle is
\[
P_a = \frac{1}{2} \omega \epsilon_p' v_o |\textbf{E}_{int}|^2
\] \hspace{1cm} (1.2.16)

The absorption cross section is
\[
\sigma_a = \frac{P_a}{\frac{1}{2\pi} |\textbf{E}_{i}|^2}
\] \hspace{1cm} (1.2.17)

For the case of a sphere, putting (1.2.6) and (1.2.16) in (1.2.17) gives
\[
\sigma_a = k \frac{\epsilon_p''}{\epsilon} \frac{4\pi a^3}{3} \left| \frac{3\epsilon}{\epsilon_p + 2\epsilon} \right|^2
\] \hspace{1cm} (1.2.18)

If we take the ratio of scattering cross section to absorption cross section, we find
\[
\frac{\sigma_s}{\sigma_a} = \frac{2(ka)^3}{\epsilon_p''} \left| \frac{\epsilon_p - \epsilon}{3\epsilon} \right|^2
\] \hspace{1cm} (1.2.19)

Thus the relative magnitude of \( \sigma_s \) to \( \sigma_a \) depends on the relative values of \((ka)\) and \( \epsilon_p''/\epsilon \).

**2.3 Rayleigh Scattering by an Ellipsoid**

Let \( a, b, \) and \( c \) be the half axes length of the ellipsoid respectively in \( \hat{x}_b, \hat{y}_b, \) and \( \hat{z}_b \) directions where \( a, b, \) and \( c \) are all much less than \( \lambda \). Then the internal electric field \( \textbf{E}_{int} \) induced by the incident electric field \( \textbf{E}_i \) is
\[
\textbf{E}_{int} = \hat{x}_b \left( \frac{\hat{x}_b \cdot \textbf{E}_i}{1 + v_d A_a} \right) + \hat{y}_b \left( \frac{\hat{y}_b \cdot \textbf{E}_i}{1 + v_d A_b} \right) + \hat{z}_b \left( \frac{\hat{z}_b \cdot \textbf{E}_i}{1 + v_d A_c} \right)
\] \hspace{1cm} (1.2.20)

where
\[
v_d = \frac{abc}{2} \left( \frac{\epsilon_p}{\epsilon} - 1 \right)
\] \hspace{1cm} (1.2.21)
Rayleigh Scattering by an Ellipsoid

The integral of the sum of $A_a$, $A_b$, and $A_c$ can be performed analytically to give

$$A_a + A_b + A_c = \frac{2}{abc} \quad (1.2.25)$$

For spheroids, $a = b$ and the integrals in $A_a$, $A_b$, and $A_c$ can be integrated analytically. The axis $\hat{z}_b$ is the axis of symmetry, and

$$A_a = A_b = \frac{1}{2} \left( \frac{2}{a^2c} - A_c \right) \quad (1.2.26)$$

For prolate spheroids, $c > a = b$ and

$$A_c = -\frac{1}{c^3e^3} \left( 2e + \ln \frac{1-e}{1+e} \right) \quad (1.2.27)$$

where

$$e = \sqrt{1 - \left( \frac{a}{c} \right)^2} \quad (1.2.28)$$

is the eccentricity.

For oblate spheroids, $c < a$ and

$$A_c = \frac{2}{c^3f^2} \left( 1 - \frac{1}{f} \tan^{-1} f \right) \quad (1.2.29)$$

where

$$f = \sqrt{\left( \frac{a}{c} \right)^2 - 1} \quad (1.2.30)$$

For a thin disk, we can treat it as a special case of the oblate spheroid by letting $a \gg c$ in (1.2.29)–(1.2.30). Then,

$$f \simeq \frac{a}{c} \gg 1 \quad (1.2.31)$$

$$A_c = \frac{2}{c^3f^2} = \frac{2}{ca^2} \quad (1.2.32)$$

$$A_a = A_b = 0 \quad (1.2.33)$$
Substituting (1.2.32)-(1.2.33) in (1.2.20) gives

\[ \vec{E}_{\text{int}} = \hat{x}_b (\hat{x}_b \cdot \vec{E}_i) + \hat{y}_b (\hat{y}_b \cdot \vec{E}_i) + \hat{z}_b (\hat{z}_b \cdot \vec{E}_i) \frac{\epsilon}{\epsilon_p} \]  

(1.2.34)

The interpretation of (1.2.34) is that the tangential component of the incident electric field penetrates into the thin disk while the normal component of the incident electric field is changed by a factor of \( \frac{\epsilon}{\epsilon_p} \) when it penetrates into the thin disk.

### 2.4 Scattering Dyads

In view of equation (1.2.20), it is convenient to define the scattering dyad

\[ \vec{F} = \frac{k^2 v_0 (\epsilon_p - \epsilon)}{4\pi \epsilon} \left( \frac{\hat{x}_b \hat{x}_b}{1 + v_d A_a} + \frac{\hat{y}_b \hat{y}_b}{1 + v_d A_b} + \frac{\hat{z}_b \hat{z}_b}{1 + v_d A_c} \right) \]  

(1.2.35)

From (1.2.4) and (1.2.20), we have

\[ \vec{E}_s = \frac{k^2 e^{ikr}}{4\pi \epsilon r} v_0 (\epsilon_p - \epsilon) \left( \hat{a}_s \hat{a}_s + \hat{b}_s \hat{b}_s \right) \cdot \vec{E}_{\text{int}} \]

\[ = \frac{k^2 e^{ikr}}{4\pi \epsilon r} v_0 (\epsilon_p - \epsilon) \left( \frac{\hat{x}_b \hat{x}_b}{1 + v_d A_a} + \frac{\hat{y}_b \hat{y}_b}{1 + v_d A_b} + \frac{\hat{z}_b \hat{z}_b}{1 + v_d A_c} \right) \cdot \vec{E}_i \]  

(1.2.36)

Hence

\[ \vec{E}_s = \frac{e^{ikr}}{r} \left( \hat{a}_s \hat{a}_s + \hat{b}_s \hat{b}_s \right) \cdot \vec{F} \cdot \vec{E}_i \]  

(1.2.37)

Since

\[ \begin{bmatrix} E_{vs} \\ E_{hs} \end{bmatrix} = \begin{bmatrix} f_{vv} & f_{vh} \\ f_{hv} & f_{hh} \end{bmatrix} \begin{bmatrix} E_{vi} \\ E_{hi} \end{bmatrix} \]  

(1.2.38)

it follows from (1.2.37) and (1.2.38) that

\[ f_{vv} = \hat{v}_s \cdot \vec{F} \cdot \hat{v}_i \]  

(1.2.39a)

\[ f_{vh} = \hat{v}_s \cdot \vec{F} \cdot \hat{h}_i \]  

(1.2.39b)

\[ f_{hv} = \hat{h}_s \cdot \vec{F} \cdot \hat{v}_i \]  

(1.2.39c)

\[ f_{hh} = \hat{h}_s \cdot \vec{F} \cdot \hat{h}_i \]  

(1.2.39d)

The bistatic scattering cross sections are defined by

\[ \sigma_{vv}(\hat{k}_s, \hat{k}_i) = 4\pi |f_{vv}|^2 \]  

(1.2.40a)
§2.4 Scattering Dyads

\[ \sigma_{vh}(\hat{k}_s, \hat{k}_i) = 4\pi |f_{vh}|^2. \quad (1.2.40b) \]
\[ \sigma_{hv}(\hat{k}_s, \hat{k}_i) = 4\pi |f_{hv}|^2 \quad (1.2.40c) \]
\[ \sigma_{hh}(\hat{k}_s, \hat{k}_i) = 4\pi |f_{hh}|^2 \quad (1.2.40d) \]

In the backscattering direction (monostatic radar)

\[ \hat{k}_s = -\hat{k}_i \quad (1.2.41) \]

so that

\[ \hat{h}_s = -\hat{h}_i \quad (1.2.42) \]

and

\[ \hat{v}_s = \hat{v}_i \quad (1.2.43) \]

Note that there is no negative sign in (1.2.43) while there are negative signs in (1.2.41) and (1.2.42).

**Example 1:**

For spheres, \( v_d = \frac{a^2}{2}(\epsilon_r - 1) \) where \( \epsilon_r = \epsilon_p / \epsilon \) is the relative permittivity.

\[ A_a = A_b = A_c = \frac{1}{3} \left( \frac{2}{a^2} \right) \text{ so that} \]
\[
\frac{1}{1 + v_d A_a} = \frac{3}{\epsilon_r + 2} \quad (1.2.44)
\]

This gives

\[ \vec{F} = f_o (\hat{x}_b \hat{x}_b + \hat{y}_b \hat{y}_b + \hat{z}_b \hat{z}_b) = f_o \overline{I} \quad (1.2.45) \]

In backscattering direction

\[ f_{vv} = \hat{v}_s \cdot f_o \overline{I} \cdot \hat{v}_i = f_o \quad (1.2.46a) \]
\[ f_{vh} = \hat{v}_s \cdot f_o \overline{I} \cdot \hat{h}_i = 0 \quad (1.2.46b) \]
\[ f_{hv} = \hat{h}_s \cdot f_o \overline{I} \cdot \hat{v}_i = 0 \quad (1.2.46c) \]
\[ f_{hh} = \hat{h}_s \cdot f_o \overline{I} \cdot \hat{h}_i = -f_o \quad (1.2.46d) \]

Thus

\[
\begin{pmatrix}
E_{vs} \\
E_{hs}
\end{pmatrix}
= \begin{pmatrix}
f_o & 0 \\
0 & -f_o
\end{pmatrix}
\begin{pmatrix}
E_{vi} \\
E_{hi}
\end{pmatrix}
\quad (1.2.47)
\]

\[ \sigma_{vv}(\hat{k}_i, \hat{k}_i) = 4\pi |f_o|^2 = \sigma_{hh}(\hat{k}_i, \hat{k}_i) \quad (1.2.48) \]

\[ \sigma_{vh} = \sigma_{hv} = 0 \quad (1.2.49) \]