Analysing & Interpreting the Yield Curve

Moorad Choudhry

Foreword by Philippe Priaulet
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Analysing and Interpreting the YIELD CURVE
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# Contents

<table>
<thead>
<tr>
<th>Part I</th>
<th>Introduction to Bond Yield and the Yield Curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter 1</td>
<td>Bond Yield Measurement</td>
</tr>
<tr>
<td>Chapter 2</td>
<td>The Yield Curve</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>A Further Look at Spot and Forward Rates</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Part II</th>
<th>Yield Curve Modeling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter 4</td>
<td>Interest Rate Modeling Part I: An Introduction to Basic Concepts</td>
</tr>
<tr>
<td>Chapter 5</td>
<td>Interest Rate Modeling Part II: The Dynamics of Asset Prices</td>
</tr>
<tr>
<td>Chapter 6</td>
<td>Interest Rate Models: Part I</td>
</tr>
<tr>
<td>Chapter 7</td>
<td>Interest Rate Models: Part II</td>
</tr>
<tr>
<td>Chapter 8</td>
<td>The Index-linked Bond Yield Curve</td>
</tr>
<tr>
<td>Chapter 9</td>
<td>Analysing the Long Bond Yield</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Part III</th>
<th>Fitting the Yield Curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter 10</td>
<td>Estimating and Fitting the Yield Curve: Part I</td>
</tr>
<tr>
<td>Chapter 11</td>
<td>Estimating and Fitting the Yield Curve: Part II</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Part IV</th>
<th>The Yield Curve and Relative-value Trading</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chapter 12</td>
<td>Yield Curves and Relative Value</td>
</tr>
<tr>
<td>Chapter 13</td>
<td>Approaches to Yield Spread Trading Using Government Bonds</td>
</tr>
</tbody>
</table>

| Index | 355 |
For Paula.

You're the best thing that ever happened ...
Foreword

I take great pleasure in introducing this new book *Analysing and Interpreting the Yield Curve* by Moorad Choudhry, who has certainly been among the most prolific authors in the fixed income area; most notably he has written *The Repo Handbook*, *The Bond and Money Markets* and *Capital Market Instruments* during recent years. I thoroughly appreciate the approach of Moorad to make easy otherwise difficult concepts. His method is definitely very pedagogical. Once again, his new book is perfect in this aspect by combining a clear exposition and very useful practical illustrations.

The comprehension of the yield curve or rather of the yield curves, spot and forward, Treasury and swap yield curves, default free and risky curves, zero-coupon and par yield curves, and so on, is the core of this book. Many questions are raised that will interest both MBA and MSc students as well as market practitioners, for instance: “how to fit the yield curve”, “what reveals its shape”, “how it has to be modelled over time?”, “how to interpret forward curves?”, and finally, “how to build strategies to make money”.

The chapter devoted to the yield curve fitting provides innovative ideas to implement efficient relative value models which are determinant to detect bond mispricings. Implementing a model which takes into account the major valuation determinants, is sufficiently aggressive in terms of prices, but above all, gives the accurate hedging portfolio for the seller of the product is the key point for a research team working on derivatives; chapters about interest rate modelling will be very useful to them.

Chapters about forward curves are primordial for strategists whose job is to detect anomalies in forward and volatility curves to propose products at interesting levels. For example, two months ago in May 2003, the EUR forward curve was implying that the slope between 30- and 2-years became negative after 2008, reaching even −60 bp in 2018. Historical precedent suggested that this was very unlikely as since 1999, the flattest that the swap curve has been was in August 2000 when it reached +48 bp. In that context entering a 5-year forward swap to pay during 10 years the 2-year CMS flat and receiving the 30-year CMS flat was an excellent opportunity...

Moorad gives a lot of interesting solutions to just these kinds of questions. I highly recommend the book that will be of immense interest both for students on fixed-income securities to understand the yield curve bases, and for practitioners to provide them with very subtle answers.

Philippe Priaulet
Fixed-Income Strategist, HSBC-CCF, Paris
Associate Professor, Department of Mathematics, University of Evry, France
August 2,
As Sir Arthur Conan Doyle would have put it, so elementary a form of literature as the textbook on financial economics hardly deserves the dignity of a preface. It is possible though, to bring some instant clarity to the purpose of such a book if we open with a few words here.

In my book *The Bond and Money Markets*, I try to explain, from first principles, just how important the global debt market is, and describe the various participants that interact with each other in this market. Given the importance of the global bond market, one can never learn too much about it. However, this is not a book about the bond market, rather it is about a very specific, and important part of the bond markets. In developed markets, as well as a fair number of developing ones, there is usually a large number of bonds trading at one time, at different yields and with varying terms to maturity. Investors and traders closely examine the relationship between the yields on bonds that are in the same class. Plotting yields of bonds that differ only in their term to maturity, produces what is known as the *yield curve*. The yield curve represents the bond market. It is sometimes referred to as the term structure of interest rates, but as we shall see later in this book, this expression refers to only one specific type of yield curve. There are lots of different yield curves. We shall examine them all in detail later.

Much of the analysis and pricing activity that takes place in the bond market revolves around the yield curve. The primary yield curve in any domestic capital market is the government bond yield curve, so for example in the US market it is the US Treasury yield curve. In this book we focus mainly, but not exclusively, on the government bond yield curve. Because the author spent over 5 years as a UK government bond trader (or gilt-edged market-maker), most of the examples are from the gilt market. However, the principles remain the same across all markets. It is the importance of the yield curve to just about every aspect of finance that has been the motivation behind writing this book. Our objective is to:

- describe what the yield curve is;
- explain what it tells us;
- explain why it assumes certain shapes;
- show how we can use it;
- introduce how it is modeled;
- show how it is “fitted” from market rates.
In order to set the scene, we begin with some basic description of bonds and bond mathematics. We assume a basic knowledge and familiarity with bonds and market institutions, and concentrate on the yield curve. It is an arcane, specialist topic but well worth getting familiar with. We explain term structure theory, describe the most popular mathematical approaches used to model the yield curve, and show how to fit the yield curve using econometric techniques. This knowledge is of great use to just about anyone involved in the bond markets—traders, bond salespersons, fund managers, research analysts, issuers of bonds—in fact issuers, investors and all the middlemen in between. Investors in the equity markets can also benefit from an understanding of the yield curve, as it enables them to gain a better insight into market sentiment.

We must necessarily be quite focused and specialist in our discussion of the yield curve. Hopefully the more technical material is presented in good order so that it remains accessible. There are numerous textbooks available for the complete beginner, which are recommended in the reading lists at the end of each chapter, along with further reading.

**Layout of the Book**

This book is organized into thirteen chapters. They are structured in four parts, covering:

- the concept of bond yield and spot and forward rates;
- interest rate models;
- fitting the yield curve using spline techniques;
- yield curve relative-value trading.

At each stage we reference the most important literature and also provide necessary background in appendices. A more technical background can be found in the relevant readings noted at the end of each chapter.

**Website**

Research on the debt capital markets may be downloaded from the dedicated fixed income website at

www.YieldCurve.com

This site also lists details of other books and articles written by Moorad Choudhry and YieldCurve.com associates.
Acknowledgements

My thanks to Nick Wallwork at Wiley Asia, a gentleman who also had faith in my ability to deliver a half-decent manuscript! Thanks also to Pauline and Malar at Wiley, true professionals both.

Special thanks to Philippe Priaulet for the excellent foreword, an honour for me who has been reading and learning from his books for some time now.

A big thanks to Kenneth Garbade, at the Federal Reserve Bank of New York, for his foreword to my book Derivative Instruments. As I had his book Fixed Income Analytics on the desk with me when I was trading gilts some years ago at ABN Amro Hoare Govett and Hambros Bank, this was a nice touch!

Moorad Choudhry
Surrey, England
30 September 2003
Moorad Choudhry is Head of Treasury at KBC Financial Products in London. He was educated at Claremont Fan Court school, the University of Westminster and the University of Reading, before joining the London Stock Exchange in 1989. From there he joined Hoare Govett Securities Limited (later ABN Amro Hoare Govett Limited), where he was a gilt-edged market-maker and Treasury trader. Subsequently he was a sterling bond proprietary trader at Hambros Bank Limited. He later joined the structured finance services team at JP Morgan Chase Bank in London.

Moorad is a Fellow of the Securities Institute of the UK and a Visiting Professor at the Department of Economics, Finance and International Business at London Metropolitan University.

Moorad was born in Bangladesh and lives in Surrey, England.
“Education never ends, Watson. It’s a series of lessons, with the greatest for the last.”

— The Adventure of the Red Circle

His Last Bow
Sir Arthur Conan Doyle
(1859–1930)
Introduction to Bond Yield and the Yield Curve

In Part I we introduce the concept of bond yields using traditional analysis. We assume that most readers will already have a good grounding in the concepts of net present value and internal rate of return. We then describe the yield curve itself. The bulk of the discussion is in Chapter 2 which looks at the different types of yield curve and, more importantly, introduces the main theories of the yield curve. We also look at interpreting the curve. The language is non-specialist and should be accessible to anyone with an involvement in the bond markets. This is followed by a discussion on spot and forward rates, and the derivation of such rates from market yields.

Yield curve analysis and the modeling of the term structure of interest rates is one of the most heavily-researched areas of financial economics. The treatment here and in the rest of the book is kept as concise as possible, which sacrifices some detail, but bibliographies at the end of each chapter will direct interested readers on to what the author feels are the most accessible and readable references in this area.
In the Preface to this book, we noted the importance of the yield curve to an understanding of the bond markets. But before we discuss the yield curve, we must be familiar with the concept of bond yields and bond yield measurement. So in this chapter, we will introduce this subject for beginners.

From an elementary understanding of financial arithmetic we will know how to calculate the price of a bond using an appropriate discount rate known as the bond’s yield. This is the same as calculating a net present value of the bond’s cash flows at the selected discount rate. We can reverse this procedure to find the yield of a bond where the price is known, which is equivalent to calculating the bond’s internal rate of return (IRR). There is no equation for this calculation and a solution is obtained using numerical iteration. The IRR calculation is taken to be a bond’s yield to maturity or redemption yield and is one of various yield measures used in the markets to estimate the return generated from holding a bond. In this chapter, we will consider these various measures as they apply to plain vanilla bonds.

In most markets, bonds are generally traded on the basis of their prices but because of the complicated patterns of cash flows that different bonds can have, they are generally compared in terms of their yields. This means that a market-maker will usually quote a two-way price at which he will buy or sell a particular bond, but it is the yield at which the bond is trading that is important to the market-maker’s customer. This is because a bond’s price does not actually tell us anything useful about what we are getting. Remember that in any market there will be a number of bonds with different issuers, coupons and terms to maturity. Even in a homogeneous market such as the UK government bond (gilt) market, different gilts trade according to their own specific characteristics. To compare bonds in the market,
Therefore, we need the yield on any bond and it is yields that we compare, not prices. A fund manager who is quoted a price at which he can buy a bond is instantly aware of what yield that price represents, and whether this yield represents fair value.

So it is the yield represented by the price that is the important figure for bond traders. We can illustrate this by showing the gilt prices from a newspaper, reproduced below. Figure 1.1 is an extract from the Financial Times as at 9 June 2003 and shows both prices and yields of UK gilts. It allows us to compare returns from different bonds. If it listed only prices for stocks it would be useful only to, say, private investors who have purchased stock at a certain price and now wish to see where it is trading—it does not allow us to compare returns from the different bonds listed. Note from Figure 1.1 that the Financial Times extract also shows us the yield spread to the zero-coupon curve, and the yields of three gilt strips, which are zero-coupon gilts. Comparing the spread of gilt yields to the zero-coupon curves enables us to see where theoretical fair value of a coupon bond lies. We will discuss this concept in greater detail later in the book.

**UK GILTS - cash market**

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Source: Unit Management Office (UMO) Gilt Office. Non-residents on application. + X dividend cashing out.

Closing mid prices are shown in pounds per £1 nominal of stock. Weekly percentage changes are calculated on a Friday to Friday basis. Gilt 'numbers', the benchmarks and most makeable stocks, are shown in bold type.

**Figure 1.1** United Kingdom gilts section from the Financial Times newspaper, closing prices from 6 June 2003. Reprinted from The Financial Times, 9 June 2003. © Financial Times. Reproduced with permission.
Chapter 1 Bond Yield Measurement

The yield on any investment is the interest rate that will make the present value of the cash flows from the investment equal to the initial cost (price) of the investment. Mathematically, the yield on any investment, represented by \( r \), is the interest rate that satisfies equation (1.1):

\[
P = \sum_{n=1}^{N} \frac{C_n}{(1 + r)^n}
\]

(1.1)

where

\( C_n \) is the cash flow in year \( n \);
\( P \) is the price of the investment;
\( n \) is the number of years.

The yield calculated from this relationship is the internal rate of return.

But as we have noted, there are other types of yield measure used in the market for different purposes. The most important of these are bond redemption yields, spot rates and forward rates. We will now discuss each type of yield measure and show how they are computed, followed by a discussion of the relative usefulness of each measure.

**Current Yield**

The simplest measure of the yield on a bond is the current yield, also known as the flat yield, interest yield or running yield. The running yield is given by (1.2):

\[
rc = \frac{C}{P} \times 100
\]

(1.2)

where;

\( C \) is the bond coupon;
\( rc \) is the current yield;
\( P \) is the clean price of the bond.

In (1.2) \( C \) is not expressed as a decimal. Current yield ignores any capital gain or loss that might arise from holding and trading a bond and does not consider the time value of money. It essentially calculates the bond coupon income as a proportion of the price paid for the bond, and to be accurate would have to assume that the bond was more like an annuity rather than a fixed-term instrument. It is not really an “interest rate”, though.

The current yield is useful as a “rough-and-ready” interest rate calculation. It is often used to estimate the cost of or profit from a short-term
holding of a bond. For example, if other short-term interest rates such as the one-week or three-month rates are higher than the current yield, holding the bond is said to involve a running cost. This is also known as negative carry or negative funding. The term is used by bond traders and market-makers and leveraged investors. The carry on a bond is a useful measure for all market practitioners as it illustrates the cost of holding or funding a bond. The funding rate is the bondholder’s short-term cost of funds. A private investor could also apply this to a short-term holding of bonds.

**Example 1.1 Running yield**

A bond with a coupon of 6% is trading at a clean price of 97.89. What is the current yield of the bond?

\[
rc = \frac{6.00}{98.76} \times 100 = 6.129\%
\]

What is the current yield of a bond with 7% coupon and a clean price of 103.49?

\[
rc = \frac{7}{103.49} \times 100 = 6.76\%
\]

Note from the above that the current yield of a bond will lie above the coupon rate if the price of the bond is below par, and vice versa if the price is above par.

Mr. Badur buys a bond with coupon of 10% at a price of 110.79, with funds borrowed at 8.75% via a special-rate credit card offer and holds the bond for three months. Has he made money during the course of the investment (ignoring transaction costs)?

The running yield on the bond is 9.026%, while Badur has paid interest on his borrowed funds at 8.75%. Therefore he has earned approximately 0.276% net carry or funding return on his investment, ignoring any capital gain or loss he may have suffered when he sold the bond.

**Simple Yield to Maturity**

The simple yield to maturity makes up for some of the shortcomings of the current yield measure by taking into account capital gains or losses. The assumption made is that the capital gain or loss occurs evenly over the remaining life of the bond. The resulting formula is:
where;

\[ P \] is the clean price;

\[ rs \] is the simple yield to maturity;

\[ n \] is the number of years to maturity.

For the bond discussed in Example 1.1 and assuming \( n = 5 \) years:

\[
rs = \frac{6.00}{97.89} + \frac{100 - 97.89}{5 \times 97.89} = 0.06129 + 0.00431 = 6.560% 
\]

The simple yield measure is useful for rough-and-ready calculations. However its main drawback is that it does not take into account compound interest or the time value of money. Any capital gain or loss resulting is amortized equally over the remaining years to maturity. In reality, as bond coupons are paid they can be reinvested, and hence interest can be earned. This increases the overall return from holding the bond. As such, the simple yield measure is not overly useful and it is not commonly encountered in say, the gilt market. However it is often the main measure used in the Japanese government bond market.

**Yield to Maturity**

**Calculating bond yield to maturity**

The *yield to maturity* (YTM) or *gross redemption* yield is the most frequently used measure of return from holding a bond.\(^1\) Yield to maturity takes into account the pattern of coupon payments, the bond's term to maturity and the capital gain (or loss) arising over the remaining life of the bond. These elements are all related and are important components determining a bond's price. If we set the IRR for a set of cash flows to be the rate that applies from a start-date to an end-date we can assume the IRR to be the YTM for those cash flows. The YTM therefore is equivalent to the *internal rate of return* on the bond, the rate that equates the value of the discounted cash flows on the bond to its current price. The calculation assumes that the bond is held until maturity and therefore it is the cash flows

\(^1\) In this book the terms *yield to maturity* and *gross redemption yield* are used synonymously. The latter term is common in sterling markets.
to maturity that are discounted in the calculation. It also employs the concept of the time value of money.

As we would expect, the formula for YTM is essentially that for calculating the price of a bond. For a bond paying annual coupons, the YTM is calculated by solving equation (1.4), and we assume that the first coupon will be paid exactly one interest period now (which, for an annual coupon bond is exactly one year from now).

\[
P_d = \frac{C}{(1 + rm)^1} + \frac{C}{(1 + rm)^2} + \frac{C}{(1 + rm)^3} + \cdots + \frac{C}{(1 + rm)^n} + \frac{M}{(1 + rm)}
\] (1.4)

where;

\(P_d\) is the bond dirty price;
\(M\) is the par or redemption payment (100);
\(n\) the number of interest periods;
\(C\) is the coupon rate;
\(rm\) is the annual yield to maturity (the YTM).

Note that the number of interest periods in an annual coupon bond is equal to the number of years to maturity, and so for these bonds \(n\) is equal to the number of years to maturity.

We can simplify (1.4) using \(\Sigma\):

\[
P = \sum_{n=1}^{N} \frac{C}{(1 + rm)^n} + \frac{M}{(1 + rm)^n}.
\] (1.5)

Note that the expression at (1.5) has two variable parameters, the price \(P_d\) and yield \(rm\). It cannot be rearranged to solve for yield \(rm\) explicitly and must be solved using numerical iteration. The process involves estimating a value for \(rm\) and calculating the price associated with the estimated yield. If the calculated price is higher than the price of the bond at the time, the yield estimate is lower than the actual yield, and so it must be adjusted until it converges to the level that corresponds with the bond price.\(^2\)

For YTM for a semi-annual coupon bond, we have to adjust the formula to allow for the semi-annual payments. Equation (1.5) is modified as shown by (1.6) again assuming there are precisely six months to the next coupon payment:

\[^2\text{Bloomberg® also uses the term } yield-to-workout \text{ where “workout” refers to the maturity date for the bond.}\]
Chapter 1 Bond Yield Measurement

\[
d = \frac{\sum_{n=1}^{N} \frac{C/2}{(1 + \frac{1}{2} m)^n} + \frac{M}{(1 + \frac{1}{2} m)^n}}{\frac{C}{(1 + \frac{1}{2} m)^{n+1}}}
\]  

(1.6)

where \( n \) is the number of interest periods in the life of the bond and therefore equal to the number of years to maturity multiplied by 2.

For yield calculations carried out by hand (“long-hand”), we can simplify (1.5) and (1.6) to reduce the amount of arithmetic. For a semi-annual coupon bond with an actual/365 day-base count, (1.6) can be written out long-hand and rearranged to give us (1.7):

\[
d = \left( \frac{1}{(1 + \frac{1}{2} rm)^{N_{nc}/182.5}} \right) \times \left( \frac{C}{rm} \left( 1 + \frac{1}{2} rm \right) - \frac{1}{(1 + \frac{1}{2} rm)^{n-1}} \right) + \frac{M}{(1 + \frac{1}{2} rm)^{n-1}}
\]  

(1.7)

where;

- \( P_{dc} \) is the dirty price of the bond;
- \( \gamma \) is the yield to maturity;
- \( N_{nc} \) is the number of days between the current date and the next coupon date;
- \( n \) is the number of coupon payments before redemption. If \( T \) is the number of complete years before redemption, then \( n = 2T \) if there is an even number of coupon payments before redemption, and \( n = 2T + 1 \) if there is an odd number of coupon payments before redemption.

All the YTM equations above use \( rm \) to discount a bond’s cash flows back to the next coupon payment and then discount the value at that date back to the date of the calculation. In other words \( rm \) is the internal rate of return (IRR) that equates the value of the discounted cash flows on the bond to the current dirty price of the bond (at the current date). The internal rate of return is the discount rate, which, if applied to all of the cash flows, solves for a number that is equal to the dirty price of the bond (its present value). By assuming that this rate will be unchanged for the reinvestment of all the coupon cash flows, and that the instrument will be held to maturity, the IRR can then be seen as the yield to maturity. In effect both measures are identical—the assumption of uniform reinvestment rate allows us to calculate the IRR as equivalent to the redemption yield. It is common for the IRR measure to be used by corporate financiers for project appraisal, while
the redemption yield measure is used in bond markets. The solution to the equation for \( r_m \) cannot be found analytically and has to be solved through numerical iteration, that is, by estimating the yield from two trial values for \( r_m \) then solving by using the formula for linear interpolation. It is more common nowadays to use a spreadsheet programme or a programmable calculator such as the Hewlett-Packard calculator.

For the equation at (1.7) we have altered the exponent used to raise the power of the discount rate in the first part of the formula to \( N/182.5 \). This is a special case and is only applicable to bonds with an actual/365 day-count base. The YTM in this case is sometimes referred to as the consortium yield, which is a redemption yield that assumes exactly 182.5 days between each semi-annual coupon date. As most developed-country bond markets now use actual/actual day bases, it is not common to encounter the consortium yield equation.

**Example 1.2 Yield to maturity for semi-annual coupon bond**

- A semi-annual paying bond has a dirty price of £98.50, an annual coupon of 6% and there is exactly one year before maturity. The bond therefore has three remaining cash flows, comprising of two coupon payments of £3 each and a redemption payment of £100. Equation (1.7) can be used with the following inputs:

\[
98.50 = \frac{3.00}{(1 + \frac{1}{2} r_m)^2} + \frac{103.00}{(1 + \frac{1}{2} r_m)^2}.
\]

Note that we use half of the YTM value \( r_m \) because this is a semi-annual paying bond. The expression above is a quadratic equation, which is solved using the standard solution for quadratic equations, which is noted below.

\[
ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

In our expression, if we let \( x = (1 + m/2) \) we can rearrange the expression as follows:

\[
98.50x^2 - 3.00x - 103.00 = 0
\]
We then solve for a standard quadratic equation, and as such there will be two solutions, only one of which gives a positive redemption yield. The positive solution is \( rm/2 = 0.037929 \), so that \( rm = 7.5859\% \).

As an example of the iterative solution method, suppose that we start with a trial value for \( rm \) of \( r_1 = 7\% \) and insert this into the right-hand side of equation (1.7). This gives a value for the right-hand side of \( \text{RHS}_1 = 99.05 \) which is higher than the left-hand side (\( \text{LHS} \approx 98.50 \)). The trial value for \( rm \) was therefore too low. Suppose that we then try \( r_2 \approx 8\% \) and use this as the right-hand side of the equation. This gives \( \text{RHS}_2 \approx 98.114 \) which is lower than the LHS. Because \( \text{RHS}_1 \) and \( \text{RHS}_2 \) lie on either side of the LHS value, we know that the correct value for \( rm \) lies between 7\% and 8\%. Using the formula for linear interpolation:

\[
rm = r_1 + (r_2 - r_1) \frac{\text{RHS}_1 - \text{LHS}}{\text{RHS}_1 - \text{RHS}_2}
\]

our linear approximation for the redemption yield is \( rm \approx 7.587\% \), which is near the exact solution.

**Example 1.3**

We wish to calculate the gross redemption yield for the bond in Example 1.1. If we assume that the analysis is performed with precisely five years to maturity, with a settlement date of 3 August 1999 and that it is a semi-annual coupon bond, the bond will comprise cash flows of ten coupon payments of £3 every six months and a redemption payment of £100 five years from now. In order to calculate the redemption yield \( rm \) long-hand, we need to try different trial levels for the discount rate \( rm \) until we obtain the cash flows' present value total of 97.89. We know that the YTM must be greater than the coupon rate of 6\% because the bond is trading at a price below par. In the table below, we use different trial values for \( rm \) until we reach the semi-annual discount rate of 3.25\%, which is equal to a YTM of 6.50\%. In practice, we would have obtained two rates that gave present value totals above and below the price of 97.89 and then used the formula for numerical iteration to solve for \( rm \).
### When calculating yields long-hand we can use the following formulas to calculate cash flow present values, where \( n \) is the number of interest periods during the life of the bond.

**Present value of coupon payments:**
- For annual coupon bonds
  \[
  C \left( \frac{1 - (1/(1 + rm)^n)}{rm} \right)
  \]
- For semi-annual coupon bonds
  \[
  C \left( \frac{1 - (1/(1 + rm/2)^n)}{rm/2} \right)
  \]

**Present value of redemption payment:**
- For annual coupon bonds
  \[
  M \left( \frac{1}{(1 + rm)^n} \right)
  \]
- For semi-annual coupon bonds
  \[
  M \left( \frac{1}{(1 + (rm/2)^n)} \right)
  \]

Note that the redemption yield as discussed in this section is the **gross redemption yield**, the yield that results from payment of coupons without deduction of any withholding tax. The **net redemption yield** is obtained by multiplying the coupon rate \( C \) by \((1 - \text{marginal tax rate})\). The net yield is what will be received if the bond is traded in a market where bonds pay coupon net, which means net of a withholding tax. The net redemption yield is always lower than the gross redemption yield.

### Using the redemption yield calculation

We have already alluded to the key assumption behind the YTM calculation, namely that the rate \( rm \) remains stable for the entire period of the life of the bond. By assuming the same yield, we can say that all coupons are reinvested at the same yield \( rm \). For the bond in Example 1.3 this means that if all the cash flows are discounted at 6.5% they will have a total present value...
value or NPV of 97.89. At the same time, if all the cash flows received during the life of the bond are reinvested at 6.5% until the maturity of the bond, the final redemption yield will be 6.5%. This is unrealistic since we can predict with virtual certainty that interest rates for instruments of similar maturity to the bond at each coupon date will not remain at 6.5% for five years. In practice, however, investors require a rate of return that is equivalent to the price that they are paying for a bond and the redemption yield is, to put it simply, as good a measurement as any. A more accurate measurement might be to calculate present values of future cash flows using the discount rate that is equal to the market’s view on where interest rates will be at that point, known as the forward interest rate. However forward rates are implied interest rates, and a YTM measurement calculated using forward rates can be as speculative as one calculated using the conventional formula. This is because the actual market interest rate at any time is invariably different from the rate implied earlier in the forward markets. Therefore a YTM calculation made using forward rates would not be realized in practice either. We shall see later in this chapter how the zero-coupon interest rate is the true interest rate for any term to maturity, however the YTM is, despite the limitations presented by its assumptions, still the main measure of return used in the markets.

**Example 1.4 Comparing the different yield measures**

The examples in this section illustrate a five-year bond with a coupon of 6% trading at a price of 97.89. Using the three common measures of return we have:

- Running yield = 6.129%
- Simple yield = 6.560%
- Redemption yield = 6.50%

The different yield measures are illustrated graphically in Figure 1.2 below.

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3 However, such an approach is used to price interest rate swaps.
Calculating redemption yield between coupon payments

The yield formula (1.4) can be used whenever the settlement date for the bond falls on a coupon date, so that there is precisely one interest period to the next coupon date. If the settlement date falls in between coupon dates, the same price/yield relationship holds and the YTM is the interest rate that equates the NPV of the bond’s cash flows with its dirty price. However the formula is adjusted to allow for the uneven interest period, and this is given by (1.8) for an annual coupon bond:

\[
P_d = \frac{C}{(1 + rm)^w} + \frac{C}{(1 + rm)^{1+w}} + \frac{C}{(1 + rm)^{2+w}} + \cdots \\
+ \frac{C}{(1 + rm)^{n-1+w}} + \frac{M}{(1 + rm)^{n-1+w}}
\]

(1.8)

where

- \(w\) is the number of days between the settlement date and the next coupon date
- \(n\) is number of days in the interest period
- \(N\) is the number of coupon payments remaining in the life of bond

and the other parameters are as before. As before the formula can be shortened as given by (1.9):

\[
P_d = \frac{\sum_{n=1}^{N} C}{(1 + rm)^{n-1+w}} + \frac{M}{(1 + rm)^{n-1+w}}
\]

(1.9)