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Preface

Lasers are paradigmatic examples of nonlinear systems and have played a decisive role in the development of nonlinear dynamics into a cross disciplinary subject over the past 40 years. Already a free running laser represents a nontrivial nonlinear system, but even more interesting phenomena arise when lasers are subjected to feedback or coupled to build large networks. Some of these phenomena already found their way to industrial applications, for example, the creation of ultrashort pulses with the mode locking technique by using integrated multisection devices or the stabilization of laser outputs with optical injection. The technological advances in semiconductor processing technologies also allow to produce a variety of lasers with nanostructured active regions that give rise to interesting physics and allow designing new innovative devices.

Nowadays, nonlinear laser dynamics is a still growing field of active research, and this book focuses and reviews recent advances in this area. In an interdisciplinary approach, it will concentrate on mathematical, physical, as well as experimental aspects. By discussing problems such as the modeling of integrated devices, the creation of networks, exploitation of chaotic lasers for secure communication, and the use of nanostructured lasers for logic gates and memory elements, it will enter innovative grounds and hopefully inspire future research on that topic.

On the occasion of the sixtieth birthday of Prof. Eckehard Schöll, this book is also intended to recognize the work during his scientific career, as he always enforced the connection between rigorous mathematical analysis and physical modeling. For this reason, the contributors are former and future collaborators of Prof. Eckehard Schöll.

The book is separated into three parts. Within the first part, “Nanostructured devices”, the dynamic properties and modeling aspects of Quantum Dot Lasers, Vertical Cavity Surface Emitting Lasers, and Quantum Cascade Lasers are reviewed, while the second part “Coupled Laser Devices” focusses on the complex dynamics and bifurcations induced by self coupling, delay coupling, or mode coupling of lasers. The third part, “Synchronization and Cryptography”, discusses the chaotic dynamics of excitable systems and their application for secure communication or for the generation of synchronized cluster states in networks.
I am grateful to the group of Prof. Schöll for their enduring support during the compilation of this volume and to the staff from Wiley VCH for their excellent help.

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Part I
Nanostructured Devices
1
Modeling Quantum-Dot-Based Devices

Kathy Lüdge

1.1 Introduction

During the past decades, the performance of semiconductor lasers has been dramatically improved from a laboratory curiosity to a broadly used light source. Owing to their small size and low costs, they can be found in many commercial applications ranging from their use in DVD players to optical communication networks. The rapid progress in epitaxial growth techniques allows to design complex semiconductor laser devices with nanostructured active regions and, therefore, interesting dynamical properties. Future high-speed data communication applications demand devices that are insensitive to temperature variations and optical feedback effects, and provide features such as high modulation bandwidth and low chirp, as well as error-free operation. Currently, self-organized semiconductor quantum dot (QD) lasers are promising candidates for telecommunication applications [1]. For an introduction to QD-based devices, their growth process, and their optical properties, see, for example, [2, 3].

This review focuses on the modeling of these QD laser devices and on the discussion of their dynamic properties. It uses a microscopically based rate equation model that assumes a classical light field but includes microscopically calculated scattering rates for the collision terms in the carrier rate equations, as introduced in [4–8]. Following the hierarchy of different semiconductor modeling approaches (for an overview, see [9]), this model aims to be sophisticated enough to permit a quantitative modeling of the QD laser dynamics but still allows an analytic treatment of the dynamics. Different levels of complexity will be explored to enable comprehensive insights into the underlying processes.

In order to reduce the numeric effort and still allow for analytic insights, a variety of effects have been neglected. This way, a different approach has to be chosen if, for example, the photon statistics of the emitted light [10] or changes in the emission wavelength due to Coulomb enhancement effects [11, 12] are to be of interest. For the analysis of ultrafast phenomena, as, for example, the gain recovery in QD-based optical amplifiers [13], coherent effects resulting from the dynamics of the microscopic polarization become important, and the model has to be extended.
to semiconductor Bloch equations. This has been intensively studied in [14, 15] in good agreement with experimental results [16], but it will not be discussed in this review. Note that later on in this book, the experimental results obtained with QD lasers under optical injection are presented in Chapter 3 (by Sciamanna [17]), and the results regarding the sensitivity of QD lasers to optical feedback [18] are discussed in Chapter 6 by Erneux et al. [19].

After a detailed introduction to the microscopical modeling aspects in Section 1.2, the turn-on and switching dynamics of a QD laser with two confined levels is discussed in Sections 1.3 and 1.4, and temperature effects are analyzed in Section 3.1. In Section 1.5, the results of an asymptotic analysis of the rate equation systems are presented, which allows to give analytic expression to relaxation oscillation (RO) frequency and damping of the turn-on dynamics, and thus allows to predict the modulation properties of the laser. Resulting from the analytic predictions, the effect of using a doped carrier reservoir on the laser dynamics is investigated in Section 1.6. At the end, in Section 1.7, the results are discussed and compared to quantum well (QW) laser devices.

1.2 Microscopic Coulomb Scattering Rates

A schematic view of the QD laser structure is shown in Figure 1.1a. The active area of the p–n heterojunction is a dot-in-a-well (DWELL) structure that consists of several InGaAs QW layers that have a height of about 4 nm, and contain embedded QDs that are confined in all three dimensions having a size of approximately 4 nm x 18 nm x 18 nm. During laser operation, an electric current is injected into

![Figure 1.1](image-url)
the QW layers. They form the carrier reservoir where carrier–carrier scattering events take place because of Coulomb interaction and lead to a filling (or depletion) of the confined QD levels. As a result, carrier inversion is reached first between the lowest confined QD levels in the conduction band and its counterpart in the valence band. Since the size and the composition of the zero-dimensional QD structures determine the energetic position of the QD levels, it is possible to design lasers with different emission wavelengths. The lasers discussed here have a ground state (GS) emission wavelength of 1.3 \( \mu \text{m} \), as needed for optical data communication.

For high carrier densities in the reservoir, that is, during electrical pumping, the Coulomb interaction (carrier–carrier Auger scattering) will dominate the scattering rate into (and out of) the QDs, whereas the scattering events resulting from carrier–phonon interaction are negligible [20]. Inside the QD, two confined energy levels are modeled. Thus, direct capture processes for electrons \((b = e)\) and holes \((b = h)\) into or out of the GS labeled as \(S_{\text{cap}}^b\), into or out of the excited state (ES) labeled as \(S_{\text{cap}}^b\), and relaxation processes between GSs and ESs named \(S_{\text{rel}}^b\) are considered as depicted in Figure 1.1b, where gray arrows indicate the in-scattering events.

Section 1.2.1 systematically describes and quantifies the different Auger processes before they are incorporated into the dynamic rate equation model in Section 1.3. Note that although phonon scattering between the carrier reservoir (QW) and the QDs is neglected, the fast phonon-assisted carrier relaxation processes within the QW states are taken into account by assuming a quasi-Fermi distribution with quasi-Fermi levels \(F_{\text{QW}}^e\) and \(F_{\text{QW}}^h\) for electrons in the conduction band and holes in the valence band of the QW, respectively.

1.2.1 Carrier–Carrier Scattering

If the Coulomb interaction is treated in the second-order Born approximation in the Markov limit up to second order in the screened Coulomb potential [21, 22], a Boltzmann equation for the collision terms, which describe the change in the occupation probability in the QD states, can be derived, and subsequently easily incorporated into laser rate equation models (for details, see also [15]). The striking difference from the standard rate equation models is that there are no constant relaxation times. Instead, the detailed modeling of the scattering events inside the reservoir leads to scattering times that are nonlinearly dependent on the carrier densities in the reservoir.

Figure 1.2 gives a systematic overview of all processes leading to in-scattering into the QD electron levels. The gray arrows denote electron transitions of the scattering partners. Panels I and III show pure \(e–e\) processes, while panels II and IV display mixed \(e–h\) processes. The corresponding processes for in-scattering into the QD hole levels are obtained by exchanging all electron and hole states. The out-scattering processes are obtained by inverting all arrows of the electron transitions. The exchange processes of pure \(e–e\) capture processes contributing to the scattering rates are not shown, since there is no qualitative difference from that of the direct processes. In case of mixed \(e–h\) processes (II, IV), the exchange processes lead to
transitions across the band gap, which are neglected since they are unlikely to occur. Note that the process shown in panel III of Figure 1.2b is the exchange process of the one in panel I. In the following, the scattering events shown in Figure 1.2 are decomposed into contributions originating from direct carrier capture from the QW into the QD levels $R^\text{cap}_m$ (Figure 1.2a) and relaxation processes between the QD states with one and two intra-QD transitions $R^{\text{rel}'}_b$ and $R^{\text{rel}''}_b$, respectively (Figure 1.2b). Processes involving three QD states are neglected. Thus, the collision term in the Boltzmann equation for the carrier occupation probability in the QD states $\rho_{mb}$, where $m$ labels the quantum number of the 2D angular momentum of the confined QD states ($m = E$ for the first ES; $m = G$ for the GS) reads:

$$\frac{\partial \rho_{mb}}{\partial t}\big|_{\text{col}} = R^\text{cap}_{b,m} + R^{\text{rel}'}_b + R^{\text{rel}''}_b$$ (1.1)

The contribution to Eq. (1.1) from direct capture processes (Figure 1.2a) can be expressed as

$$R^\text{cap}_{b,m} = S^\text{in,cap}_{b,m} (1 - \rho^m_b) - S^\text{out,cap}_{b,m} \rho^m_b$$ (1.2)

where the direct capture Coulomb scattering rates for in- and out-scattering $(S^\text{in,cap}_{b,m})$ and $(S^\text{out,cap}_{b,m})$ are defined as

$$S^\text{in,cap}_{b,m} = \sum_{k_1k_2k_3,k_3'} \mathcal{W}^{b}_{k_1k_2k_3,k_3'} f_{k_1} f_{k_2} (1 - f_{k_3'}) \left( 1 - f_{k_1} \right) \left( 1 - f_{k_3} \right) \left( f_{k_2} \right) \left( f_{k_3'} \right)$$ (1.3)

$$S^\text{out,cap}_{b,m} = \sum_{k_1k_2k_3,k_3'} \mathcal{W}^{b}_{m,k_2k_3,k_3'} f_{k_1} (1 - f_{k_2}) \left( 1 - f_{k_3} \right) \left( 1 - f_{k_1} \right) \left( f_{k_2} \right) \left( f_{k_3'} \right) \left( f_{k_1} \right)$$ (1.4)

States in the QW are labeled by the in-plane carrier momentum $k^b_i$ ($b = e$ and $b = h$ indicate conduction and valence band states, respectively). For both bands in the QW, $f_{k^b_i}$ indicates the electron occupation probability. The transition probability $\mathcal{W}^{b}_{k_1k_2k_3,k_3'}$ for a process where two carriers scatter from initial states $k_1$ and $k_3$ to the final states $m$ and $k_2$, respectively, $(k_1 \rightarrow m, k_3 \rightarrow k_2)$ contains the
screened Coulomb matrix elements for direct and exchange interactions, and the energy-conserving \( \delta \)-function [6, 15]. Owing to the microscopic reversibility of the Coulomb matrix elements, the transition probability is equal for reversed direction

\[
W_{b \rightarrow b'} = W_{b' \rightarrow b}.
\]

The relaxation processes shown in Figure 1.2b describe a redistribution of carriers within the intra-QD levels. The contribution from processes I and II to Eq. (1.1) is given by

\[
R_{b \rightarrow b'}^\text{rel} = S_{\text{in}, \text{rel}}^b (1 - \rho_{b}^G) - S_{\text{out}, \text{rel}}^b (1 - \rho_{b}^E) \rho_{b}^G. \tag{1.5}
\]

The relaxation in-scattering rate is given by

\[
S_{\text{in}, \text{rel}}^b = \sum_{k_2 k_3} W_{b \rightarrow b'}^{k_2 k_3' k_2' k_3} (1 - f_{b'}^G) f_{b'}^E (E \rightarrow G, k_3 \rightarrow k_2). \tag{1.6}
\]

The dynamical equations for the processes III and IV (\( R_{b \rightarrow b'}^\text{rel''} \)) in Figure 1.2b can be obtained in a similar manner as in Eq. (1.5).

For the calculation of the Coulomb scattering rates, a quasiequilibrium within the QW states (fast phonon scattering inside one band) but nonequilibrium between the QW electrons and the QD electrons, the QW holes, and the QD holes is assumed. As a result, the electron occupation probability \( f_{b} \) in the conduction \((b = e)\) and valence band \((b = h)\) of the QW can be expressed by a quasi-Fermi distribution given by

\[
f_{b} = \left[ \exp \left( \frac{E_b - F_b^{\text{QW}}}{kT} \right) + 1 \right]^{-1} \quad (b = e, h). \tag{1.7}
\]

The quasi-Fermi levels \( F_b^{\text{QW}} \) are determined by the total carrier density in the respective band via the relation given in Eq. (1.8), as shown in [7, 23],

\[
F_b^{\text{QW}} (w_b) = E_b^{\text{QW}} \pm kT \ln \left[ \exp \left( \frac{w_b}{D_{b} kT} \right) - 1 \right] \tag{1.8}
\]

where the + and − signs correspond to electrons and holes, respectively. Furthermore, \( D_b = m_b / (\pi \hbar^2) \) is the 2D density of states, with the effective masses \( m_b \) of electrons \((b = e)\) and holes \((b = h)\), respectively. \( E_b^{\text{QW}} \) are the QW band edges of conduction and the valence band, respectively. Note that the analytic expression Eq. (1.8) is only valid for a 2D electron gas, where the integrals

\[
w_e = \int_{E_e^{\text{QW}}}^{E_e^{\text{QW}}} dE_k D_e f_{k}^e \quad \text{and} \quad w_h = \int_{-E_h^{\text{QW}}}^{E_h^{\text{QW}}} dE_k D_h (1 - f_{k}^h) \tag{1.9}
\]

can be solved. As a result, the quasi-Fermi distributions \( f_{k}^e \) and \( f_{k}^h \) are determined by the QW carrier densities \( w_e \) and \( w_h \), and thus, the scattering rates given in Eqs. (1.3) and (1.6) are calculated as functions of \( w_e \) and \( w_h \). Besides that, the scattering rates parametrically depend on the effective masses of the carriers in the QW bands and on the band structure given by the energetic distances \( \Delta E_b \) and \( \Delta_b \), as indicated in Figure 1.1b. The resulting rates are shown in Figure 1.3 as a function of \( w_e \) along the line \( w_h / w_e = 1.5 \). For the relaxation rates, the sum of all relaxation processes...
Figure 1.3 Coulomb scattering rates of the QDs-in-a-well system versus QW electron density \(w_e\) (\(w_h/w_e = 1.5\)). (a) Intra-QD relaxation rates for electrons (gray) and holes (black); (b) and (c) direct capture rates into the GS (dashed line) and ES (dotted line) for holes and electrons, respectively. Top and bottom panels show in- and out-scattering rates, respectively. Parameters as in Table 1.1.

is plotted but note that the rates involving a transition within the QD accompanied by a QW transition (rel') are much larger than the rates involving two QW–QD transitions (rel''). The relaxation rates are characterized first by a sharp increase and later by a decrease in higher carrier densities because of the effect of Pauli blocking. These relaxation scattering events are on a ps time scale, whereas the direct capture rates plotted in Figure 1.3b,c for holes and electrons are an order of magnitude smaller for small carrier densities. Owing to their small effective mass, the rate for electron capture is much smaller, although the dependence on \(w_e\) is similar to that of the hole rate. For small electron densities inside the QW, the capture rates increase quadratically with \(w_e\), which is expected from mass action kinetics.

1.2.2 Detailed Balance

In thermodynamic equilibrium, there is a detailed balance between the in- and out-scattering rates of the QD level. This allows one to relate the rate coefficients of in- and out-scattering even for nonequilibrium carrier densities [24].

For a single scattering process between two carriers of type \(b\) and \(b'\), the in-scattering rate for capture into the GS (\(m = G\)) or ES (\(m = E\)) is defined in Eq. (1.3), and can be rewritten as

\[
W_{k_1'k_3'}^b k_1 k_2 m f_{k_1k_3}^b(1 - f_{k_2})
\]

\[
= W_{k_1'k_3'}^b k_1 k_2 m (1 - f_{k_1})(1 - f_{k_3}) f_{k_1'}^b f_{k_2'}^b \frac{f_{k_1}}{1 - f_{k_1}} \frac{f_{k_3}}{1 - f_{k_3}} \frac{1 - f_{k_2'}}{f_{k_2'}}
\]

\[
= W_{k_1'k_3'}^b k_1 k_2 m (1 - f_{k_1})(1 - f_{k_3}) f_{k_1'}^b \exp \left[ \frac{F_{c}^{QW} - E_{k_1} - E_{k_3} + E_{k_2'}}{kT} \right]
\]