Time Series Analysis
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To the memory of

Gwilym M. Jenkins

Gregory C. Reinsel
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Preface to the Fourth Edition

It may be of interest to briefly recount how this book came to be written. Gwilym Jenkins and I first became friends in the late 1950s. We were intrigued by an idea that a chemical reactor could be designed that optimized itself automatically and could follow a moving maximum. We both believed that many advances in statistical theory came about as a result of interaction with researchers who were working on real scientific problems. Helping to design and build such a reactor would present an opportunity to further demonstrate this concept.

When Gwilym Jenkins came to visit Madison for a year, we discussed the idea with the famous chemical engineer Olaf Hougen, then in his eighties. He was enthusiastic and suggested that we form a small team in a joint project to build such a system. The National Science Foundation later supported this project. It took three years, but suffice it to say, that after many experiments, several setbacks, and some successes the reactor was built and it worked.

As expected this investigation taught us a lot. In particular we acquired proficiency in the manipulation of difference equations that were needed to characterize the dynamics of the system. It also gave us a better understanding of nonstationary time series required for realistic modeling of system noise. This was a happy time. We were doing what we most enjoyed doing: interacting with experimenters in the evolution of ideas and the solution of real problems, with real apparatus and real data.

Later there was fallout in other contexts, for example, advances in time series analysis, in forecasting for business and economics, and also developments in statistical process control (SPC) using some notions learned from the engineers.

Originally Gwilym came for a year. After that I spent each summer with him in England at his home in Lancaster. For the rest of the year, we corresponded using small reel-to-reel tape recorders. We wrote a number of technical reports and published some papers but eventually realized we needed a book. The first two editions of this book were written during a period in which Gwilym was, with extraordinary courage, fighting a debilitating illness to which he succumbed sometime after the book had been completed.

Later Gregory Reinsel, who had profound knowledge of the subject, helped to complete the third edition. Also in this fourth edition, produced after his untimely
death, the new material is almost entirely his. In addition to a complete revision and
updating, this fourth edition resulted in two new chapters: Chapter 10 on nonlinear
and long memory models and Chapter 12 on multivariate time series.

This book should be regarded as a tribute to Gwilym and Gregory.
I was especially blessed to work with two such gifted colleagues.

GEORGE E. P. BOX

Madison, Wisconsin
March 2008

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G. E. P. B.
Preface to the Third Edition

This book is concerned with the building of stochastic (statistical) models for time series and their use in important areas of application. This includes the topics of forecasting, model specification, estimation, and checking, transfer function modeling of dynamic relationships, modeling the effects of intervention events, and process control. Coincident with the first publication of Time Series Analysis: Forecasting and Control, there was a great upsurge in research in these topics. Thus, while the fundamental principles of the kind of time series analysis presented in that edition have remained the same, there has been a great influx of new ideas, modifications, and improvements provided by many authors.

The earlier editions of this book were written during a period in which Gwilym Jenkins was, with extraordinary courage, fighting a slowly debilitating illness. In the present revision, dedicated to his memory, we have preserved the general structure of the original book while revising, modifying, and omitting text where appropriate. In particular, Chapter 7 on estimation of ARMA models has been considerably modified. In addition, we have introduced entirely new sections on some important topics that have evolved since the first edition. These include presentations on various more recently developed methods for model specification, such as canonical correlation analysis and the use of model selection criteria, results on testing for unit root nonstationarity in ARIMA processes, the state space representation of ARMA models and its use for likelihood estimation and forecasting, score tests for model checking, structural components, and deterministic components in time series models and their estimation based on regression-time series model methods. A new chapter (12) has been developed on the important topic of intervention and outlier analysis, reflecting the substantial interest and research in this topic since the earlier editions.

Over the last few years, the new emphasis on industrial quality improvement has strongly focused attention on the role of control both in process monitoring as well as in process adjustment. The control section of this book has, therefore, been completely rewritten to serve as an introduction to these important topics and to provide a better understanding of their relationship.

The objective of this book is to provide practical techniques that will be available to most of the wide audience who could benefit from their use. While we have tried
to remove the inadequacies of earlier editions, we have not attempted to produce here a rigorous mathematical treatment of the subject.

We wish to acknowledge our indebtedness to Meg (Margaret) Jenkins and to our wives, Claire and Sandy, for their continuing support and assistance throughout the long period of preparation of this revision.

Research on which the original book was based was supported by the Air Force Office of Scientific Research and by the British Science Research Council. Research incorporated in the third edition was partially supported by the Alfred P. Sloan Foundation and by the National Aeronautics and Space Administration. We are grateful to Professor E. S. Pearson and the Biometrika Trustees for permission to reprint condensed and adapted forms of Tables 1, 8, and 12 of Biometrika Tables for Statisticians, Vol. 1, edited by E. S. Pearson and H. O. Hartley, to Dr. Casimer Stralkowski for permission to reproduce and adapt three figures from his doctoral thesis, and to George Tiao, David Mayne, Emanuel Parzen, David Pierce, Granville Wilson, Donald Watts, John Hampton, Elaine Hodkinson, Patricia Blant, Dean Wichern, David Bacon, Paul Newbold, Hiro Kanemasu, Larry Haugh, John MacGregor, Bovas Abraham, Gina Chen, Johannes Ledolter, Greta Ljung, Carole Leigh, Mary Esser, and Meg Jenkins for their help, in many different ways, in preparing the earlier editions.

George Box and Gregory Reinsel
CHAPTER ONE

Introduction

A time series is a sequence of observations taken sequentially in time. Many sets of data appear as time series: a monthly sequence of the quantity of goods shipped from a factory, a weekly series of the number of road accidents, hourly observations made on the yield of a chemical process, and so on. Examples of time series abound in such fields as economics, business, engineering, the natural sciences (especially geophysics and meteorology), and the social sciences. Examples of data of the kind that we will be concerned with are displayed as time series plots in Figure 4.1. An intrinsic feature of a time series is that, typically, adjacent observations are dependent. The nature of this dependence among observations of a time series is of considerable practical interest. Time series analysis is concerned with techniques for the analysis of this dependence. This requires the development of stochastic and dynamic models for time series data and the use of such models in important areas of application.

In the subsequent chapters of this book we present methods for building, identifying, fitting, and checking models for time series and dynamic systems. The methods discussed are appropriate for discrete (sampled-data) systems, where observation of the system occurs at equally spaced intervals of time.

We illustrate the use of these time series and dynamic models in five important areas of application:

1. The forecasting of future values of a time series from current and past values
2. The determination of the transfer function of a system subject to inertia—the determination of a dynamic input–output model that can show the effect on the output of a system of any given series of inputs
3. The use of indicator input variables in transfer function models to represent and assess the effects of unusual intervention events on the behavior of a time series

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4. The examination of interrelationships among several related time series variables of interest and determination of appropriate multivariate dynamic models to represent these joint relationships among variables over time

5. The design of simple control schemes by means of which potential deviations of the system output from a desired target may, so far as possible, be compensated by adjustment of the input series values

1.1 FIVE IMPORTANT PRACTICAL PROBLEMS

1.1.1 Forecasting Time Series

The use at time $t$ of available observations from a time series to forecast its value at some future time $t + l$ can provide a basis for (1) economic and business planning, (2) production planning, (3) inventory and production control, and (4) control and optimization of industrial processes. As originally described by Holt et al. [157], Brown [79], and the Imperial Chemical Industries monograph on short-term forecasting [263], forecasts are usually needed over a period known as the lead time, which varies with each problem. For example, the lead time in the inventory control problem was defined by Harrison [143] as a period that begins when an order to replenish stock is placed with the factory and lasts until the order is delivered into stock.

We suppose that observations are available at discrete, equispaced intervals of time. For example, in a sales forecasting problem, the sales $z_t$ in the current month $t$ and the sales $z_{t-1}$, $z_{t-2}$, $z_{t-3}$, ... in previous months might be used to forecast sales for lead times $l = 1, 2, 3, \ldots, 12$ months ahead. Denote by $\hat{z}_t(l)$ the forecast made at origin $t$ of the sales $z_{t+l}$ at some future time $t + l$, that is, at lead time $l$. The function $\hat{z}_t(l)$, which provides the forecasts at origin $t$ for all future lead times, based on the available information from the current and previous values $z_t$, $z_{t-1}$, $z_{t-2}$, $z_{t-3}$, ... through time $t$, will be called the forecast function at origin $t$. Our objective is to obtain a forecast function such that the mean square of the deviations $z_{t+l} - \hat{z}_t(l)$ between the actual and forecasted values is as small as possible for each lead time $l$.

In addition to calculating the best forecasts, it is also necessary to specify their accuracy, so that, for example, the risks associated with decisions based upon the forecasts may be calculated. The accuracy of the forecasts may be expressed by calculating probability limits on either side of each forecast. These limits may be calculated for any convenient set of probabilities, for example, 50 and 95%. They are such that the realized value of the time series, when it eventually occurs, will be included within these limits with the stated probability. To illustrate, Figure 1.1 shows the last 20 values of a time series culminating at time $t$. Also shown are forecasts made from origin $t$ for lead times $l = 1, 2, \ldots, 13$, together with the 50% probability limits.

Methods for obtaining forecasts and estimating probability limits are discussed in detail in Chapter 5. These forecasting methods are developed based on the assumption that the time series $z_t$ follows a stochastic model of known form.
Consequently, in Chapters 3 and 4 a useful class of such time series models that might be appropriate to represent the behavior of a series $z_t$, called autoregressive integrated moving average (ARIMA) models, are introduced and many of their properties are studied. Subsequently, in Chapters 6, 7, and 8 the practical matter of how these models may be fitted to actual time series data is explored, and the methods are described through the three-stage procedure of tentative model identification or specification, estimation of model parameters, and model checking and diagnostics.

1.1.2 Estimation of Transfer Functions

A topic of considerable industrial interest is the study of process dynamics [22, 162]. Such a study is made (1) to achieve better control of existing plants and (2) to improve the design of new plants. In particular, several methods have been proposed for estimating the transfer function of plant units from process records consisting of an input time series $X_t$ and an output time series $Y_t$. Sections of such records are shown in Figure 1.2, where the input $X_t$ is the rate of air supply and the output $Y_t$ is the concentration of carbon dioxide produced in a furnace. The observations were made at 9-second intervals. A hypothetical impulse response function $v_j$, $j = 0, 1, 2, \ldots$, which determines the transfer function for the system through a dynamic linear relationship between input $X_t$ and output $Y_t$ of the form $Y_t = \sum_{j=0}^{\infty} v_j X_{t-j}$, is also shown in the figure as a bar chart. Transfer function models that relate an input process $X_t$ to an output process $Y_t$ are introduced in Chapter 11 and many of their properties are examined.

Methods for estimating transfer function models based on deterministic perturbations of the input, such as step, pulse, and sinusoidal changes, have not always
been successful. This is because, for perturbations of a magnitude that are relevant and tolerable, the response of the system may be masked by uncontrollable disturbances referred to collectively as *noise*. Statistical methods for estimating transfer function models that make allowance for noise in the system are described in Chapter 12. The estimation of dynamic response is of considerable interest in economics, engineering, biology, and many other fields.

Another important application of transfer function models is in forecasting. If, for example, the dynamic relationship between two time series $Y_t$ and $X_t$ can be determined, past values of both series may be used in forecasting $Y_t$. In some situations this approach can lead to a considerable reduction in the errors of the forecasts.

### 1.1.3 Analysis of Effects of Unusual Intervention Events to a System

In some situations it may be known that certain exceptional external events, *intervention events*, could have affected the time series $z_t$ under study. Examples of such intervention events include the incorporation of new environmental regulations, economic policy changes, strikes, and special promotion campaigns. Under such circumstances we may use transfer function models, as discussed in Section 1.1.2, to account for the effects of the intervention event on the series $z_t$, but where the "input" series will be in the form of a simple indicator variable taking only the values 1 and 0 to indicate (qualitatively) the presence or absence of the event.

In these cases, the intervention analysis is undertaken to obtain a quantitative measure of the impact of the intervention event on the time series of interest. For example, Box and Tiao [73] used intervention models to study and quantify the impact of air pollution controls on smog-producing oxidant levels in the Los Angeles area and of economic controls on the consumer price index in the United States. Alternatively, the intervention analysis may be undertaken to adjust for any unusual values in the series $z_t$ that might have resulted as a consequence of the intervention event. This will ensure that the results of the time series analysis of the series, such as the structure of the fitted model, estimates of model parameters, and forecasts of future values, are not seriously distorted by the influence of