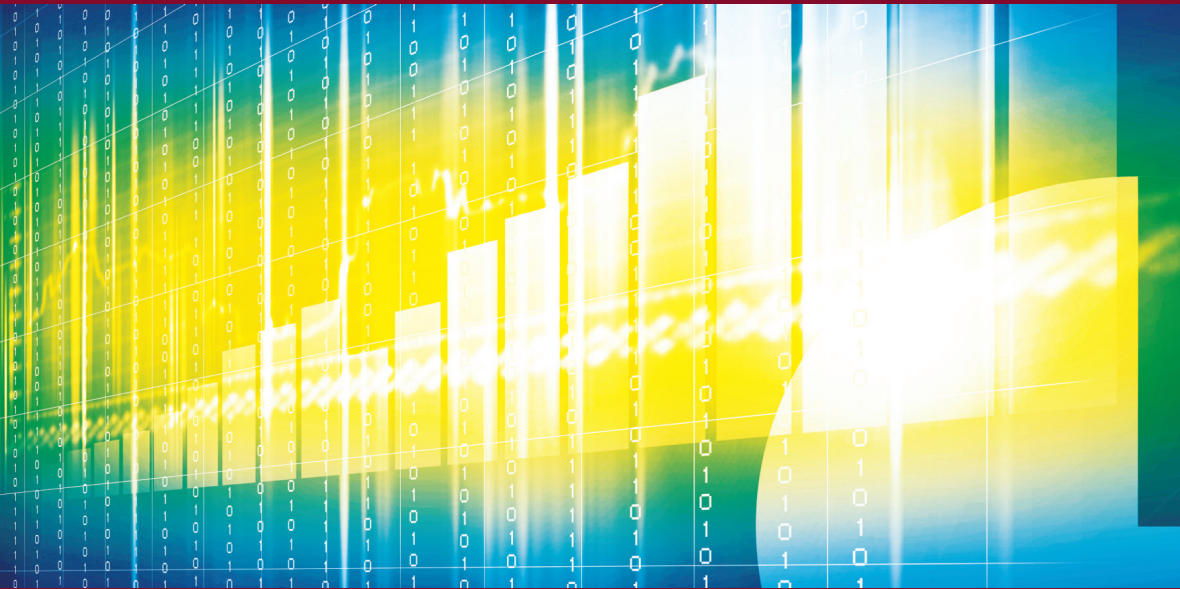


DIGITAL SIGNAL AND IMAGE PROCESSING SERIES

Mathematical Foundations of Image Processing and Analysis 2

Jean-Charles Pinoli



ISTE

WILEY

Mathematical Foundations of Image
Processing and Analysis 2

To
Blandine, Flora and Pierre-Charles

Series Editor
Jean-Pierre Goure

Mathematical Foundations of Image Processing and Analysis 2

Jean-Charles Pinoli

ISTE

WILEY

First published 2014 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

Apart from any fair dealing for the purposes of research or private study, or criticism or review, as permitted under the Copyright, Designs and Patents Act 1988, this publication may only be reproduced, stored or transmitted, in any form or by any means, with the prior permission in writing of the publishers, or in the case of reprographic reproduction in accordance with the terms and licenses issued by the CLA. Enquiries concerning reproduction outside these terms should be sent to the publishers at the undermentioned address:

ISTE Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

www.wiley.com

© ISTE Ltd 2014

The rights of Jean-Charles Pinoli to be identified as the author of this work have been asserted by him in accordance with the Copyright, Designs and Patents Act 1988.

Library of Congress Control Number: 2014939771

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN 978-1-84821-748-5



Printed and bound in Great Britain by CPI Group (UK) Ltd., Croydon, Surrey CR0 4YY

Contents

PREFACE	xvii
INTRODUCTION	xxv
PART 5. TWELVE MAIN GEOMETRICAL FRAMEWORKS FOR BINARY IMAGES	1
CHAPTER 21. THE SET-THEORETIC FRAMEWORK	3
21.1. Paradigms	3
21.2. Mathematical concepts and structures	3
21.2.1. Mathematical disciplines	3
21.3. Main notions and approaches for IPA	4
21.3.1. Pixels and objects	4
21.3.2. Pixel and object separation	4
21.3.3. Local finiteness	4
21.3.4. Set transformations	5
21.4. Main applications for IPA	6
21.4.1. Object partition and object components	6
21.4.2. Set-theoretic separation of objects and object removal	6
21.4.3. Counting of separate objects	7
21.4.4. Spatial supports border effects	7
21.5. Additional comments	7
CHAPTER 22. THE TOPOLOGICAL FRAMEWORK	9
22.1. Paradigms	9
22.2. Mathematical concepts and structures	9
22.2.1. Mathematical disciplines	9
22.2.2. Special classes of subsets of \mathbb{R}^n	9
22.2.3. Fell topology for closed subsets	10

22.2.4. Hausdorff topology for compact subsets	10
22.2.5. Continuity and semi-continuity of set transformations	12
22.2.6. Continuity of basic set-theoretic and topological operations	12
22.3. Main notions and approaches for IPA	12
22.3.1. Topologies in the spatial domain \mathbb{R}^n	12
22.3.2. The Lebesgue–(Čech) dimension	13
22.3.3. Interior and exterior boundaries	13
22.3.4. Path-connectedness	14
22.3.5. Homeomorphic objects	14
22.4. Main applications to IPA	14
22.4.1. Topological separation of objects and object removal	14
22.4.2. Counting of separate objects	15
22.4.3. Contours of objects	16
22.4.4. Metric diameter	16
22.4.5. Skeletons of proper objects	16
22.4.6. Dirichlet–Voronoi diagrams	18
22.4.7. Distance maps	19
22.4.8. Distance between objects	19
22.4.9. Spatial support’s border effects	20
22.5. Additional comments	20
CHAPTER 23. THE EUCLIDEAN GEOMETRIC FRAMEWORK	23
23.1. Paradigms	23
23.2. Mathematical concepts and structures	23
23.2.1. Mathematical disciplines	23
23.2.2. Euclidean dimension	24
23.2.3. Matrices	24
23.2.4. Determinants	26
23.2.5. Eigenvalues, eigenvectors and trace of a matrix	27
23.2.6. Matrix norms	28
23.3. Main notions and approaches for IPA	29
23.3.1. Affine transformations	29
23.3.2. Special groups of affine transformations	30
23.3.3. Linear and affine sub-spaces and Grassmannians	31
23.3.4. Linear and affine spans	31
23.4. Main applications to IPA	32
23.4.1. Basic spatial transformations	32
23.4.2. Hyperplanes	33
23.4.3. Polytopes	33
23.4.4. Minkowski addition and subtraction	34
23.4.5. Continuity and semi-continuities of Euclidean transformations	34
23.5. Additional comments	35

CHAPTER 24. THE CONVEX GEOMETRIC FRAMEWORK	37
24.1. Paradigms	37
24.2. Mathematical concepts and structures	37
24.2.1. Mathematical disciplines	37
24.3. Main notions and approaches for IPA	37
24.3.1. Convex objects	37
24.3.2. Hausdorff topology for compact convex objects	40
24.3.3. Compact poly-convex objects	41
24.3.4. Star-shaped objects	41
24.3.5. Simplices	42
24.4. Main applications to IPA	42
24.4.1. Convex deficiency set and concavities	42
24.4.2. Functions related to convex and star-shaped objects	43
24.4.3. Delaunay triangulation	44
24.5. Additional comments	44
CHAPTER 25. THE MORPHOLOGICAL GEOMETRIC FRAMEWORK	47
25.1. Paradigms	47
25.2. Mathematical concepts and structures	47
25.2.1. Mathematical disciplines	47
25.3. Mathematical notions and approaches for IPA	47
25.3.1. Morphological dilation and erosion	48
25.3.2. Morphological closing and opening	48
25.3.3. Set properties of morphological dilation, erosion, closing and opening	49
25.3.4. Morphological regular objects	49
25.3.5. Continuity of the morphological operations	50
25.4. Main notions and approaches for IPA	51
25.4.1. Morphological transformations	51
25.4.2. Parallel objects	51
25.4.3. Federer sets	52
25.5. Main applications to IPA	52
25.5.1. Object contours and morphological boundaries	52
25.5.2. Object filtering and morphological smoothing	53
25.5.3. Morphological skeleton	54
25.5.4. Ultimate erosion	55
25.5.5. Morphing	55
25.6. Additional comments	55
CHAPTER 26. THE GEOMETRIC AND TOPOLOGICAL FRAMEWORK	57
26.1. Paradigms	57
26.2. Mathematical concepts and structures	57

26.2.1. Mathematical disciplines	57
26.2.2. Manifolds or locally Euclidean spaces	58
26.2.3. Manifolds with border	58
26.2.4. Submanifolds	59
26.2.5. Compact and closed manifolds	59
26.2.6. Lipschitz manifolds and Lipschitz sets	60
26.3. Mathematical approaches for IPA	60
26.3.1. Unit ball and unit cube, torii and annulii	60
26.3.2. Points, curves and surfaces	61
26.3.3. Hypersurfaces	62
26.3.4. Homeomorphic and homotopic objects	62
26.4. Main applications to IPA	63
26.4.1. Contour	63
26.4.2. Topological content	64
26.4.3. The Lebesgue(-Čech) dimension of homeomorphic or homotopic objects	65
26.4.4. The Descartes–Euler–Poincaré’s number and the Betti numbers	65
26.4.5. Some particular basic manifolds	66
26.5. Additional comments	67

CHAPTER 27. THE MEASURE-THEORETIC GEOMETRIC FRAMEWORK 71

27.1. Paradigms	71
27.2. Mathematical concepts and structures	71
27.2.1. Mathematical disciplines	71
27.2.2. The Gauss measure	72
27.2.3. The Peano–Jordan measures	72
27.2.4. Measures and contents	73
27.2.5. Outer measures and Borel sets	74
27.2.6. Finite and σ -finite measures	75
27.2.7. Null sets, negligible sets and complete measures	75
27.2.8. Atoms and atomic measures	75
27.2.9. The n -dimensional Lebesgue measure	75
27.2.10. The m -dimensional Hausdorff measure	78
27.2.11. Jordan sets	79
27.3. Main approaches for IPA	79
27.3.1. Rectifiable objects	79
27.3.2. Parallel dilated objects	81
27.3.3. The Minkowski contents	82
27.3.4. The Fréchet–Nikodym–Aronszajn distance	83
27.3.5. Caccioppoli sets	84
27.4. Applications to IPA	84
27.4.1. Perimeter measures	84

27.4.2. Invariant measures	85
27.4.3. The m -dimensional Favard measure	86
27.4.4. Comparison of objects	87
27.5. Additional comments	87
CHAPTER 28. THE INTEGRAL GEOMETRIC FRAMEWORK	89
28.1. Paradigms	89
28.2. Mathematical concepts and structures	89
28.2.1. Mathematical disciplines	89
28.2.2. Geometric functionals	90
28.2.3. Intrinsic volumes and Minkowski functionals on compact convex objects	90
28.2.4. Content functionals on finite unions of compact convex objects . .	92
28.2.5. Hadwiger's characterization theorem	93
28.2.6. Particular m -dimensional content functionals	93
28.2.7. Continuity of geometric functionals	94
28.3. Main approaches for IPA	95
28.3.1. The Favard measure and Cauchy–Crofton's formulas	95
28.3.2. Cauchy–Crofton's formulas for compact, poly-convex objects	96
28.3.3. Cauchy–Crofton's formulas for a k -dimensional countably rectifiable manifold	97
28.3.4. Intersections with lower dimensional affine subspaces	98
28.3.5. The covariogram of a measurable object	99
28.4. Applications to IPA	100
28.4.1. p -dimensional affine sections	100
28.4.2. m -dimensional content functionals for $n = 1, 2$ and 3	100
28.4.3. Steiner's formulas for $n = 1, 2$ and 3	101
28.4.4. Cauchy–Crofton's formulas in dimension 2 and 3	102
28.4.5. Feret diameters and areas	104
28.4.6. Other diameters	105
28.4.7. Cauchy's projection formulas	105
28.4.8. Cabo–Baddeley's lineal transformation	107
28.4.9. Crofton–Hadwiger's chord power formula	107
28.4.10. Miles–Lantuéjoul's correction method	108
28.4.11. Hadwiger's recursive formula for the Descartes–Euler–Poincaré (DEP) number	109
28.5. Additional comments	109
CHAPTER 29. THE DIFFERENTIAL GEOMETRIC FRAMEWORK	111
29.1. Paradigms	111
29.2. Mathematical concepts and structures	111
29.2.1. Mathematical disciplines	111

29.2.2. Differential manifolds	112
29.2.3. Tangent spaces	112
29.2.4. Tangent cones and normal cones	113
29.2.5. Orientable manifolds	113
29.2.6. Diffeomorphisms	114
29.2.7. Vector fields	114
29.2.8. Riemanian manifolds	114
29.2.9. Principal curvatures of manifolds of co-dimension 1	115
29.2.10. The Gauss map and the Weingarten map	116
29.2.11. Lipschitz–Killing curvatures	117
29.2.12. Weyl’s tube formula	118
29.2.13. Differentiable manifold, positive reach and convexity	118
29.3. Main approaches for IPA	119
29.3.1. Planar curves	119
29.3.2. Space curves	120
29.3.3. Surfaces	121
29.3.4. Geodesic curves	123
29.4. Main applications for IPA	123
29.4.1. Classification of pixels on a curve or a surface	123
29.4.2. Dupin indicatrix	124
29.4.3. Numerical approximations of curvatures	125
29.4.4. The winding number of a closed curve	125
29.4.5. The genus of surfaces	126
29.4.6. Deformable contours	126
29.5. Additional comments	127
CHAPTER 30. THE VARIATIONAL GEOMETRIC FRAMEWORK	129
30.1. Paradigms	129
30.2. Mathematical concepts and structures	129
30.2.1. Mathematical disciplines	129
30.3. Main approches for IPA	130
30.3.1. Curves in dimension 2	130
30.3.2. Surfaces in dimension 3	130
30.4. Main applications for IPA	131
30.4.1. Object disocclusion	131
30.5. Additional comments	133
CHAPTER 31. THE STOCHASTIC GEOMETRIC FRAMEWORK	135
31.1. Paradigms	135
31.2. Mathematical concepts and structures	135
31.2.1. Mathematical disciplines	135
31.2.2. Random closed objects and Choquet capacity	136
31.2.3. Spatial fraction	138

31.2.4. The m -point covariance function	139
31.2.5. Contact distribution functions	140
31.2.6. Lineal-path, chord-length, and pore-size distribution functions	141
31.3. Main approaches for IPA	141
31.3.1. Random point models	141
31.3.2. Boolean models	145
31.3.3. Random tessellations	149
31.4. Applications to IPA	150
31.4.1. Simulation	150
31.4.2. Expanding measurement windows	150
31.4.3. Estimation of characteristics	150
31.4.4. Integral-geometric formulas and spatial fractions	151
31.4.5. The covariance function	152
31.4.6. Testing the stochastic model hypothesis	153
31.4.7. Distance-based methods for random point fields	154
31.4.8. Roses of directions	155
31.4.9. Davy–Miles’ formulae	156
31.5. Additional comments	156

CHAPTER 32. THE STEREOLOGICAL FRAMEWORK 159

32.1. Paradigms	159
32.2. Mathematical structures	159
32.2.1. Mathematical disciplines	159
32.2.2. Stereological vocabulary	160
32.2.3. Statistical sampling	160
32.2.4. Stratified sampling	161
32.2.5. Statistics and estimators	162
32.3. Main approaches for IPA	163
32.3.1. Stereological vocabulary	163
32.3.2. Two dual stereological approaches	164
32.3.3. Unbiasedness, sampling and precision of stereological estimators	165
32.3.4. The Bertrand paradox	165
32.4. Applications to IPA	166
32.4.1. Stereological functionals	166
32.4.2. Spatial fractions	166
32.4.3. Fiber random fields	166
32.4.4. Surface random fields	167
32.4.5. Stereological functionals for geometric features	167
32.4.6. Convex bodies	169
32.4.7. Geometric sampling effects	170
32.4.8. Counting methods	171
32.4.9. Thin sections	172
32.5. Additional comments	173

PART 6. FOUR SPECIFIC GEOMETRICAL FRAMEWORKS FOR BINARY IMAGES	177
CHAPTER 33. THE GRANULOMETRIC GEOMETRIC FRAMEWORK	179
33.1. Paradigms	179
33.2. Mathematical concepts and structures	179
33.2.1. Mathematical disciplines	179
33.3. Mathematical notions and approaches for IPA	179
33.3.1. Sizes, granularity, and granulometry	179
33.3.2. Direct granulometry	180
33.3.3. General granulometries	182
33.4. Main notions and approaches for IPA	184
33.4.1. Mathematical morphology	184
33.4.2. Granulometric moments	185
33.4.3. Sieving residues	185
33.5. Applications to IPA	186
33.5.1. Size distribution descriptors	186
33.5.2. Perfect-grain model	186
33.5.3. Frame of measurement debiasing	187
33.5.4. Morphological pattern spectrum vs. Fourier spectrum	187
33.6. Additional comments	187
CHAPTER 34. THE MORPHOMETRIC GEOMETRIC FRAMEWORK	189
34.1. Paradigms	189
34.2. Mathematical concepts and structures	189
34.2.1. Mathematical disciplines	189
34.2.2. Geometric inequalities	190
34.2.3. Isodiametric inequalities	190
34.2.4. Isoperimetric inequalities	191
34.2.5. Minkowski inequalities	193
34.2.6. The Urysohn inequality	193
34.2.7. Extremal objects	194
34.2.8. Plateau's problem	195
34.2.9. Favard's problem	196
34.3. Approaches for image analysis	196
34.3.1. Symmetrization	196
34.3.2. Morphometric functionals	197
34.4. Applications to IPA	198
34.4.1. Global shape descriptors	198
34.4.2. Shape diagrams	204
34.4.3. Comparison of shapes	207
34.5. Additional comments	207

CHAPTER 35. THE FRACTAL GEOMETRIC FRAMEWORK	211
35.1. Paradigms	211
35.2. Mathematical structures	211
35.2.1. Mathematical disciplines	211
35.2.2. The Peano spaces	212
35.2.3. Continuous nowhere-differentiable functions	212
35.2.4. Fractal objects and dimensions	213
35.2.5. Self-similar objects	213
35.3. Main approaches for IPA	214
35.3.1. Hausdorff–Besicovitch’s dimension	214
35.3.2. Relevant definition for a fractal object	217
35.3.3. Tricot’s packing dimensions	217
35.3.4. Richardson–Mandelbrot’s dimension	217
35.3.5. Minkowski–Bouligand’s dimension	218
35.3.6. The Pontrjagin–Schnirelmann’s dimension	220
35.3.7. Fractional dimension inequalities	220
35.3.8. Distance sets	220
35.4. Applications to IPA	221
35.4.1. Examples of theoretical fractal objects	221
35.4.2. Examples of natural fractal objects	222
35.4.3. Multifractal analysis	223
35.4.4. Box counting methods	224
35.5. Additional comments	224
CHAPTER 36. THE TEXTURAL GEOMETRIC FRAMEWORK	229
36.1. Paradigms	229
36.2. Mathematical concepts and structures	229
36.2.1. Mathematical disciplines	229
36.2.2. Lebesgue density	230
36.2.3. Measure-theoretic interior, closure and boundary	231
36.2.4. The Lebesgue–Hausdorff density	231
36.2.5. Lebesgue–Hausdorff density and m -dimensional rectifiable objects	233
36.3. Main approaches for IPA	233
36.3.1. Textural functionals	233
36.3.2. Rugosity	233
36.3.3. Lacunarity	235
36.4. Applications to IPA	237
36.4.1. Object rugosity	237
36.4.2. Object lacunarity	237
36.4.3. Box counting methods	237
36.5. Additional comments	237

PART 7. FOUR ‘HYBRID’ FRAMEWORKS FOR GRAY-TONE AND BINARY IMAGES	241
CHAPTER 37. THE INTERPOLATIVE FRAMEWORK	243
37.1. Paradigms	243
37.2. Mathematical concepts and structures	243
37.2.1. Mathematical disciplines	243
37.2.2. Nodes and interpolant	244
37.2.3. Extrapolation	244
37.3. Main approaches for IPA	244
37.3.1. Nearest-neighbor interpolation	244
37.3.2. Polynomial interpolation	245
37.3.3. Spline interpolation	246
37.3.4. Sampling and reconstruction	247
37.3.5. Nyquist–Shannon’s sampling theorem	247
37.3.6. Oversampling, undersampling, downsampling and upsampling	248
37.3.7. Kriging	248
37.4. Main applications for IPA	250
37.4.1. Image resizing	250
37.4.2. Curve fitting	250
37.4.3. Active contours	250
37.5. Additional comments	250
CHAPTER 38. THE BOUNDED-VARIATION FRAMEWORK	253
38.1. Paradigms	253
38.2. Mathematical structures	253
38.2.1. Mathematical disciplines	253
38.2.2. Gray-tone-valued measures	254
38.2.3. Gray-tone functions of bounded variation	255
38.2.4. Some basic properties of <i>BV</i> functions	255
38.3. Main approaches for IPA	257
38.3.1. Jumps and the Radon–Nikodym–Lebesgue decomposability	257
38.3.2. Special <i>BV</i> gray-tone functions	259
38.3.3. Caccioppoli or finite perimeter sets	259
38.4. Main applications for IPA	261
38.4.1. Image restoration	261
38.4.2. Image segmentation	263
38.4.3. Object perimeter measurement	266
38.4.4. Natural gray-tone images are not of bounded variation	266
38.5. Additional comments	267

CHAPTER 39. THE LEVEL SET FRAMEWORK	269
39.1. Paradigms	269
39.2. Mathematical concepts and structures	269
39.2.1. Mathematical disciplines	269
39.2.2. Level-sets	269
39.2.3. BV gray-tone functions	271
39.2.4. The generalized derivative of a <i>BV</i> gray-tone function	272
39.2.5. Implicit mappings	273
39.3. Main approaches for IPA	273
39.3.1. The gray-level-set method	273
39.3.2. The gray-level-set motion equation	274
39.3.3. Thresholded random fields	275
39.4. Applications to IPA	277
39.4.1. Image segmentation	277
39.4.2. Image characterization	278
39.4.3. Geometric pattern modeling and simulation	278
39.5. Additional comments	278
CHAPTER 40. THE DISTANCE-MAP FRAMEWORK	281
40.1. Paradigms	281
40.2. Mathematical structures	281
40.2.1. Mathematical disciplines	281
40.2.2. Distance-maps	282
40.2.3. Signed-distance-maps	282
40.2.4. Differentiable properties of signed-distance-maps	283
40.2.5. Integral properties of signed-distance-maps	284
40.2.6. Singularities, skeletons and cracks	284
40.2.7. Crack-free objects	286
40.2.8. Eikonal equation	286
40.2.9. (Signed)-distance transformations and their approximations on point grids	286
40.3. Main approaches for IPA	287
40.3.1. Smooth Jordan sets	287
40.3.2. Approximation of an object by its dilated sets	287
40.3.3. Sobolev sets	288
40.3.4. Sets with bounded curvature	288
40.3.5. Federer's sets of positive reach	289
40.3.6. Convex sets	290
40.4. Applications to IPA	291
40.4.1. Dirichlet–Voronoi's diagrams	291
40.4.2. Skeleton of objects and the eikonal equation	291
40.4.3. Characterization and classification of binary objects	291
40.5. Additional comments	292

CONCLUDING DISCUSSION AND PERSPECTIVES	295
APPENDICES	301
TABLES OF NOTATIONS AND SYMBOLS	303
TABLE OF ACRONYMS	341
TABLE OF LATIN PHRASES	347
BIBLIOGRAPHY	349
INDEX OF AUTHORS	435
INDEX OF SUBJECTS	445

Preface

The era of imaging sciences and technologies

The important place of *images* in the modern world is undeniable. They are intimately integrated into our organic life (“visual perception” is particularly well developed in human beings). They are frequently involved in our daily life (magazines, newspapers, telephones, televisions and video games, etc.), personal life (medical imaging, biological imaging and photographs, etc.), professional life (plant control, office automation, remote monitoring, scanners and video conferencing), etc. They are not confined to the various technological sectors, but they are vectors of observations and investigations of matter at very small scales (electron microscopes and scanning probe microscopes, etc.), or of the universe at very large scales (telescopes and space probes, etc.), sometimes leading major scientific discoveries. Mankind is now able to see images of other worlds without going there (e.g. distant planets, stars and galaxies, or the surface terrain of the Earth) and worlds within (e.g. human organs, geological imaging, or atomic and molecular structures at the nanoscale level). From a technological point of view, this importance is enhanced by the performance of the systems of investigation by imaging and the powers of calculation of computers, which expanded considerably in the second half of the 20th Century, and that are still progressing, with both hardware and software advances.

The scope of *Imaging Sciences and Technologies* is broad and multidisciplinary. It involves all the theories, methods, techniques, devices, equipment, applications, software and systems, etc. relating to images in order to obtain information and qualitative and/or quantitative knowledge, in order to investigate, analyze, measure, understand, interpret and finally to decide. The range of applications is broad in contemporary sciences and technologies. The scientific and technical disciplines that are concerned or that use it are numerous: Astronomy, Biology, Electronics, Metallurgy, Geology, Medicine, Neurology, Optics, Physics,

Perceptual Psychology and Robotics, etc. and others too numerous to name, and of course Mathematics, with their strengths and their limitations.

Mathematical Imaging

When dealing with image processing and analysis, the most surprising point at first glance, not only for many engineers or scientists, but also for academics and mathematicians, is the key role of Mathematics. Although the image processing and analysis field was historically largely applied and still partly remains so, it is not limited to an engineering field. Indeed, it has attracted the attention of many scientists during the past three decades, and the fundamentals that it requires are becoming strong and of high-level, in particular from a mathematical viewpoint.

The so-called **Mathematical Imaging** is currently a rapidly growing field in applied Mathematics, with an increasing need for theoretical Mathematics. More and more mathematicians are interested in carrying out their research into image processing and analysis. In fact, image processing and analysis have created tremendous opportunities for Mathematics and mathematicians. The contemporary field of image processing and analysis is very attractive because it has very interesting application issues, is closely related to the fascinating Human Vision and requires advanced mathematical bases.

Historically, input from mathematicians has had a fundamental impact on many scientific, technological and engineering disciplines. When accurate, robust, stable and efficient models and tools were required in more traditional areas of science and technology, Mathematics often played an important role in helping to supply them. No doubt, the same will be true in the case of imaging sciences. Mathematical Imaging has become a critical, enthusiastic and even exciting, but still in-progress, branch in contemporary sciences.

Author claims

Nowadays, there exist several good books or monographs, each dealing with one or some mathematical fundamentals for image processing and analysis purposes, but a textbook completely focused on the mathematical foundations of image processing and analysis does not currently exist.

The proposed textbook is intended:

- to fill a niche by providing a self-contained, (relatively) complete and informative review of the mathematical foundations of image processing and analysis;
- to emphasize with an (as far as possible) accessible style, the role of Mathematics as a rigorous basis for imaging sciences;

- to be a review of mathematics that are necessary for imaging sciences, often existing only in the (generally hidden) background for non-mathematicians;
- to help mathematicians to become more familiar with image processing and analysis;
- to be a mathematical companion for image processing and analysis students, scientists, researchers, scholars, engineers and even practitioners.

Textbook aims

This textbook aims to provide a comprehensive and convenient overview of the key mathematical concepts, notions, tools and frameworks involved in the various fields of gray-tone and binary image processing and analysis. It establishes a bridge between pure and applied mathematical disciplines, and the processing and analysis of gray-tone and binary images. It is accessible to readers who have neither extensive mathematical training, nor peer knowledge in image processing and analysis. The notations will be simplified as much as possible in order to be more explicative and consistent throughout the textbook. The explanations provided will be sufficiently accurate for one such statement. The mathematical aspects will systematically be discussed in the image processing and analysis context, through practical examples or concrete illustrations. Conversely, the discussed applicative issues allow the role held by Mathematics to be highlighted.

The author would greatly appreciate if the present textbook could help mathematicians to become more familiar with image processing and analysis, and likewise, image processing and image analysis scientists and engineers to get a better understanding of mathematical notions and concepts.

The proposed book is not:

- an introductory book, treatise, or textbook on image processing and analysis;
- a long textbook with extensive treatments on Mathematical Imaging;
- a monograph or a textbook on some mathematical aspects for image processing and analysis;
- a mathematical book with too heavy a jargon and detailed technical developments or complete proofs.

The proposed book is:

- a two-volume, self-contained textbook on the mathematical notions, concepts, operations, structures and frameworks that constitute the foundations of image processing and analysis, emphasizing the role of Mathematics as a rigorous basis for imaging sciences.

Organization of the textbook

This textbook is organized into an introduction, a concluding discussion with perspectives, a textbody, appendices with two tables and three indexes and a detailed bibliography.

The textbook is split over two volumes, made up of 7 main parts divided into 40 chapters and sub-divided into 207 sections.

Part 1 entitled “An Overview of Image Processing and Analysis (IPA)” presents the basic terms and notions for gray-tone and binary imaging (Chapters 1 and 3, respectively), a first overview dealing with the main image processing and image analysis fields and subfields for gray-tone images (Chapter 2), and a second overview dealing with the main image processing and image analysis fields and subfields for binary images (Chapter 4). Then, the key notions and concepts for image processing and analysis are exposed, followed by comments on how and why mathematical imaging frameworks are presented in this textbook (Chapters 5 and 6, respectively).

Part 2 entitled “Basic Mathematical Reminders for Gray-Tone and Binary Image Processing and Analysis” is devoted to basic elements in Mathematics, mainly in set theory, algebra, topology and functional analysis, that can possibly be skipped by the reader well-versed in Mathematics.

Part 3 entitled “The Main Mathematical Notions for the Spatial and Tonal Domains” focuses on the first-level mathematical notions for the spatial and tonal domains (Chapters 9 and 10).

Parts 4, 5, 6 and 7 present the functional and geometrical mathematical frameworks for image processing and analysis, and comprise a total of 30 chapters.

Part 4 entitled “Ten Main Functional Frameworks for Gray Tone Images” focuses on the main mathematical (functional) frameworks for gray-tone image processing and analysis, detailed in 10 chapters.

Part 5 and 6, entitled “Twelve Main Geometrical Frameworks for Binary Images” and “Four Specific Geometrical Frameworks for Binary Images”, respectively, focus on the main mathematical (geometric) frameworks for binary image processing and analysis, detailed in 12 chapters and 4 chapters, respectively.

Part 7, entitled “Four ‘Hybrid’ Frameworks for Gray-Tone and Binary Images”, is a further extension and supplementation focusing in 4 chapters on four mixed functional and geometric mathematical frameworks for gray-tone or/and binary images.

The textbook will be organized following two main entries:

– “*The Imaging entry*”: from an image processing and analysis viewpoint, the straightforward way to read this textbook is to start from Part 1 and then Part 3.

– “*The Mathematics entry*”: the reading of Part 2 is not required. The reader can refer to it if necessary. Part 4 is primarily based on the concepts and tools of functional analysis. Parts 5 and 6 rely primarily on the concepts and tools of geometry. The reading of Parts 5 and 6 are (almost) independent. Part 7 is mathematically advanced and needs the readings of Parts 4, 5 and 6.

The mathematical frameworks for image processing or analysis purposes are presented in separate chapters following a “*generic organization form*”, with four sections appearing successively: (1) paradigms, (2) mathematical notions and structures, (3) main approaches for image processing or analysis and (4) main applications to image processing or analysis.

Most chapters end with a section entitled “*additional comments*”, in which readers will find some historical comments, several main references: introductory or overview journal articles, seminal and historical articles, textbooks and monographs, bibliographic notes and additional readings, suggested further topics and recommended readings, and finally (often) some references on applications to image processing and analysis, all with short comments.

Important lists or tables are presented in the appendices as follows:

– a detailed and extended appendix on notation is organized in 23 *tables of notations and symbols*; special effort has been put into alleviating the notations and symbols, making them easier to read and understand, promoting genericity and declination, and avoiding confusion and inconsistencies;

– a *table of acronyms*;

– a *table of Latin phrases*;

– a complete *list of referenced authors*, with a few pieces of information (dates of birth and death, nationality, main discipline(s) of expertise). This list is of more cultural interest and will allow the readers to locate in time and space the cited scientists;

– a detailed and extended *list of subjects and keyterms*; this list will often be a real entry for any reader, who wants to search the meaning and use of a particular subject or keyterm.

A large *bibliography* is also proposed, including as far as possible historical references and seminal papers, current reviews, and cornerstone published works.

Intended audiences

This textbook is written for a broad audience: students, mathematicians, image processing and analysis specialists, and even for other scientists and practitioners.

The author hopes that the individual reader should come up with his or her own comfortable usage of the textbook.

Students

This textbook is primarily intended for 3rd/4th year undergraduate, graduate, post-graduate and doctorate students in image processing and analysis, and in Mathematics who are interested in the mathematical foundations of image processing and analysis. These students will be provided with a comprehensive and convenient summary of the mathematical foundations, that they should use or refer to throughout undergraduate, Master of Science (MSc), Master of Engineering (MEng), or PhD courses.

Mathematicians

This textbook is also intended for applied, but also ‘pure’ mathematicians. There are a still growing number of mathematicians in applied and computational Mathematics, but also in pure Mathematics, who have either little or no previous involvement in image processing and image analysis, but wish to broaden their own horizon of view, scope of knowledge, and fields of application. The author recommends that they follow the proposed logical structure of the current textbook. Those readers will find, on the one hand, an overview of image processing and analysis fields and subfields, and, on the other hand, a review of the main mathematical frameworks involved in imaging sciences.

Image processing and analysis specialists

This textbook will serve as a two-volume textbook for practitioners, researchers lecturers or scholars in image processing and analysis that aims at overviewing the mathematical foundations of image processing and analysis. It is hoped that this textbook will become the useful mathematical companion to anybody reading image processing and analysis books or articles, writing research or technical articles, preparing a lecture or a course, or for teaching.

Other scientists and practitioners

As secondary audiences, this textbook should also be of interest to many scientists of various disciplines too numerous to name who make use of images and are thus faced with image processing and analysis problems and tools. They may have an occasional need of this textbook for a better understanding of a mathematical notion.

The textbook is also intended for research and development, or industrial engineers, or project leaders, scientists, technical or scientific directors, wishing to discover or improve their knowledge of the scientific aspects of image processing and analysis, and the role of Mathematics in image processing and analysis.

Underlying matter

This textbook has been written starting from two scientific articles published in French by the Scientific and Technical Encyclopedia “Techniques de l’Ingénieur” in 2012:

– “Mathématiques pour le traitement et l’analyse d’images à tons de gris”, *Techniques de l’Ingénieur*, [E6610], 25 pages, February 2012 (Jean-Charles Pinoli) [PIN 12b];

– “Mathématiques pour le traitement et l’analyse d’images binaires”, *Techniques de l’Ingénieur*, [E6612], 25 pages, September 2012 (Jean-Charles Pinoli) [PIN 12c];

– Several extensions have been presented and new developments included (e.g. Parts 2, 6 and 7). Four unpublished chapters have been added, together with five important detailed and commented lists or tables: 23 tables of notations and symbols, a table of Latin phrases, a list of acronyms, a list of referenced authors and a list of subjects.

This textbook is also an outgrowth of PhD, Master of Engineering and Master of Science courses, which have been given for many years by the author.

Notes for the textbook reading

“*Italics*” will be used to mark a passage in a foreign language, including in particular Latin phrases, that are briefly defined and explained in the Table of Latin Phrases in Appendices.

Key terms and subject matters will appear in “***slanted bold***” in the body of the textbook. They are collected in the Appendices in the List of Subjects.

Quotation marks or inverted commas (informally referred to as quotes) are punctuation marks surrounding a word or phrase with a specific meaning or use. *Single quotes* ‘...’ will be used to indicate a different meaning, or a direct, rough or even abusive speech. *Double quotes* “...” will emphasize that an instance of a word refers to the word itself rather than its associated concept. The so-called “use-mention distinction” is necessary to make a clear distinction between using a word or phrase and mentioning it.

As a rule, a whole publication (e.g. a book title) would be both slanted and double quoted, while a citation will be both italicized and double quoted.

JEAN-CHARLES PINOLI
June, 2014

Introduction

I.1. Imaging sciences and technologies

The last few decades have largely been the dawning years of the era of **Imaging Sciences and Technologies**, which is a multidisciplinary field concerned with the (by alphabetical order) acquisition, analysis, collection, display, duplication, generation, modeling, modification, processing, reconstruction, recording, rendering, representation, simulation, synthesis and visualization, etc., of images.

From a computer science viewpoint, there are two dual fields: (1) **Computer Vision**, which tries to reconstruct the 3D world from observed 2D images, and (2) **Computer Graphics**, which pursues the opposite direction by designing suitable 2D scene images to simulate our 3D world. Image processing is the crucial middle way connecting the two. Image synthesis in the computer graphics field being the dual of image analysis treated in computer vision.

As the human visual system has been achieved by mother nature, there is nowadays a tremendous need for developing so-called **Artificial Vision** systems. Such systems consist of four more or less independent stages: (1) image acquisition, (2) image processing, (3) image analysis and (4) image interpretation.

“**Image acquisition**” mainly focuses on the physical and technological mechanisms and systems by which imaging devices generate spatial observations, but it also involves mathematical and computational models and methods implemented on computers, integrated into and/or associated to such imaging systems. The term “**image processing**”, is usually understood as all kinds of operations or transformations performed onto images (or sequences of images), in order to increase their quality, restore their original content, emphasize some particular aspects of the information content, optimize their transmission, or perform radiometric and/or spatial analysis. The term “**image analysis**” is usually understood as all kinds of operations or operators performed on images (or sequences of images), in order to

extract qualitative and/or quantitative information content, perform various measurements, and apply statistical analysis. All these methods and techniques have of course a wide range of applications in our daily world: biological imaging, industrial vision, materials imaging, medical imaging, multimedia applications, quality control, satellite imaging, traffic control and so on. “**Image interpretation**” is roughly speaking, the inverse stage of image acquisition. The latter deals with the 2D or 3D imaging of spatial structures that are investigated. The former, however, aims at understanding the observed 3D world from generally 2D images.

I.2. Historical elements on image processing and image analysis

The first digital pictures dated back to the early 1920s [MCF 72]. Then, practical works and more theoretical research mainly focused on picture coding and compression for transmission applications, and then for television image signals (see, e.g. [MER 34, GOL 51]) [SCH 67].

Historically, the “**Image Processing and Analysis (IPA)**” field has emerged early from the 1950s (see, e.g. [KOV 55] or [KIR 57]), and mainly from the 1960s (see, e.g. [GRA 67, SCH 67, ROS 69a, ROS 69b, ROS 73c] and many references therein), in works carried out and published by researchers and engineers belonging to several academic and professional communities, and from different scientific trainings, mainly “Applied Physics” (Electrical Engineering and Signal Processing), “Computer Sciences” (Computer Vision, Pattern Recognition and Artificial Intelligence), and “Mathematics” (mainly, Statistics, Applied Functional Analysis and (generally discrete) Geometry and Topology).

The first textbook entitled “*Picture Processing by Computer*” [ROS 69a] was written in 1969 by Azriel Rosenfeld, a mathematician, who was then regarded as a pioneer, and even “the” pioneer of image processing and image analysis, and as a leading researcher in the world in the field of computer image processing and analysis. Another book appeared soon after, with a similar title “*Computer Techniques in Image Processing*” [AND 72], by Harry C. Andrews, an applied physicist and computer scientist.

Several other *pioneering textbooks* were published later in the 1970s and early 1980s, mainly: “*Digital Image Restoration*” (1977) [AND 77] by Harry C. Andrews, and Bobby R. Hunt, applied physicists and computer scientists, “*Digital Image Processing*” (1977) [GON 87; 1st ed., 1977], by Rafael C. Gonzalez and Paul Wintz, electrical engineering specialists, “*Digital Image Processing*” (1978) [PRA 07; 1st ed., 1978] by William K. Pratt, an applied physicist, “*A Computational Investigation into the Human Representation and Processing of Visual Information*” (1982) [MAR 82] by David Marr, a computer scientist, “*Algorithms for Graphics and Image Processing*” (1982) [PAV 12; 1st ed., 1982] by Theo Pavlidis, a computer scientist,

“*Image Analysis and Mathematical Morphology*” (1982) [SER 82] by Jean Serra, an applied mathematician.

Concerning technical, engineering and scientific journals, deserving of special mention are two journals that early on published papers on picture processing. One of these journals, the “*Proceedings of the IRE*” (the journal of the “*Institute of Radio Engineers*”), was founded in 1913 and was renamed in 1963 as the “*Proceedings of the IEEE*” (the journal of the “*Institute of Electrical and Electronics Engineers (IEEE)*”), when the “*American Institute of Electrical Engineers (AIEE)*” and the “*Institute of Radio Engineers (IRE)*” merged to form the “*Institute of Electrical and Electronic Engineers (IEEE)*”). The other journal, “*Pattern Recognition*” (the journal of the “*Pattern Recognition Society*”), was founded in 1968. In this connection, *The Journal of the ACM* (the journal of the *Association for Computing Machinery (ACM)*, established in 1954) should also be mentioned, which published several papers on image processing and analysis in the 1960s and 1970s. The series of volumes on “*Machine Intelligence*”, initiated in 1967, and of the journal “*Artificial Intelligence*”, founded in 1970, should also be noted.

The first scientific journals dedicated to, completely or partially, image processing and analysis were published during the 1970s (e.g. “*Computer Graphics, Vision and Image Processing*” in 1972 and “*IEEE Transactions on Pattern Analysis and Machine Intelligence*” in 1979). After that period of pioneers, the field of image processing and analysis started its growth from about the middle of the 1980s. In Europe, “*Acta Stereologica*” was founded in 1982 by the “*International Society for Stereology*” and was renamed “*Image Analysis and Stereology*” in 1999. Many papers dealing with image analysis were and still are currently published.

In addition, significant contributions to image processing and even more to image analysis were also made by researchers or practitioners from other disciplines, such as for example the cytometrists, geologists, metallographs and mineralogists, just to name a few (e.g. [COS 86, WEI 81, RIG 89]).

The first international scientific conferences focusing only on image processing and analysis appeared at end of the 1980s (i.e. “*International Conference on Computer Vision (ICCV)*” in 1987) and at the beginning of the 1990s (i.e. “*International Conference on Image Processing (ICIP)*” in 1994).

The first mathematical imaging journal explicitly on both Mathematics and Image Processing and Analysis only appeared in the early 1990s (i.e. “*Journal of Mathematical Imaging and Vision*” in 1992). Very recently, the SIAM society (“*Society for Industrial and Applied Mathematics*”) published its first mathematical journal in Mathematical Imaging (i.e. “*SIAM Journal on Imaging Sciences*”) in 2008.

However, although presented in this short introductory, historical discussion under the joint name “Image Processing and Image Analysis”, it is important to note that on one side “Image Processing”, and on the other side “Image Analysis” have been addressed by researchers and engineers generally from different scientific communities. This is still often the case even if an interpenetration of the two fields is in progress. Earlier, some mathematicians focused on Image Analysis in the 1960s and 1970s. More mathematicians became interested in Image Processing from the 1980s, and even more in the 1990s. One of the main scientific reasons, if not the most important, is that image analysis required knowledge of geometry and topology, that were and still are often too poorly taught in MSc courses, and therefore are less prevalent than those most used in mathematical analysis, especially due to the strong interest in Mathematical Physics in general, during the 1980s, and in particular for image problem modeling using partial differential equations and their numerical resolutions. The following statement then appears as a logical consequence:

There exist nowadays a (relatively) large number of books dealing with image processing, but mainly on a or some particular field(s), and often in the form of edited books rather than monographs. On the contrary, only a small number of books are dealing with image analysis.

I.3. Mathematical Imaging

Early mathematical contributions and/or reviews were authored by researchers of the Electrical Engineering and Signal Processing community (see, e.g. [JAI 81]), and Discrete Geometry community (see, e.g. [ROS 66, GRA 71]).

Several areas of Mathematics have contributed to and in fact increasingly contribute to essential progress of Image Processing and Image Analysis. Mathematics provide the fundamentals for image processing and image analysis frameworks, operations, models, techniques and methods.

However:

– there is no single “mathematical theory of image processing and image analysis”. Quite often, different approaches exist to model the same problem, using notions coming from different disciplines of Mathematics. Those disciplines underlying and/or involved in Image Processing and Analysis range from Algebra to Analysis, from Set Theory and Topology to Geometry, from Functional Analysis to Calculus of Variations, from Probability Theory to Statistics, and so on;

– the ties between Image Sciences and Mathematics are still not strong enough. International conferences are very often organized by a specific scientific community. Very few symposiums are organized to promote interaction between researchers of image sciences and mathematicians.