Advances in Modeling and Design of Adhesively Bonded Systems

Edited by S. Kumar and K.L. Mittal
Advances in Modeling and Design of Adhesively Bonded Systems
Adhesion and Adhesives: Fundamental and Applied Aspects

The topics to be covered include, but not limited to, basic and theoretical aspects of adhesion; modeling of adhesion phenomena; mechanisms of adhesion; surface and interfacial analysis and characterization; unraveling of events at interfaces; characterization of interphases; adhesion of thin films and coatings; adhesion aspects in reinforced composites; formation, characterization and durability of adhesive joints; surface preparation methods; polymer surface modification; biological adhesion; particle adhesion; adhesion of metallized plastics; adhesion of diamond-like films; adhesion promoters; contact angle, wettability, and adhesion; superhydrophobicity and superhydrophilicity. With regards to adhesives, the Series will include, but not limited to, green adhesives; novel and high-performance adhesives; and medical adhesive applications.

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Preface

Adhesively bonded systems find applications in a wide spectrum of industries (e.g., aerospace, electronics, construction, ship-building, biomedical, etc.) for various purposes. Emerging adhesive materials with improved mechanical properties has allowed adhesion strength approaching that of the bonded materials themselves. Owing to advances in adhesive materials and many potential merits adhesive bonding offers, adhesive bonding has replaced other joining methods in many applications. More recently there has been a high tempo of interest in bonding composite materials. The need for innovative joints and a variety of material combinations is inevitable to realize more efficient, cost-effective structural systems.

There are many aspects to proper fabrication and successful implementation of adhesive joints including adequate surface preparation, proper control of variables dictating the performance, durability and reliability. In this vein, the modeling and design of adhesively bonded joints is of cardinal importance in predicting the reliability and life of such joints.

This book containing 9 articles written by world-renowned experts deals with the advances in modeling (theoretical and computational), and the design and experimental aspects of adhesively bonded structural systems. Advances in stress analysis and strength prediction of adhesively bonded structural systems considering a range of material models under a variety of loading conditions are discussed. Finite element modeling using macro-elements is elaborated. Recent developments in modeling and experimental aspects of bonded systems with graded adhesive layer and dual adhesives are described. Simulation of progressive damage in bonded joints is addressed. A novel vibration-based approach to detect disbonds and delamination in composite joints is also discussed.

In essence, this book represents a commentary on some of the advances which have been made in the arena of modeling and design of adhesively bonded systems. All signals indicate that
the interest in this topic will continue unabated and innovative approaches to modeling and design of adhesively bonded systems will be taken in the future which will help in expanding the utilization of bonded systems in a host of applications with increased confidence.

It should be recorded that all manuscripts were rigorously peer-reviewed, properly edited and suitably revised before inclusion in this book. So this book is not a mere collection of papers but articles which have passed muster.

This book should be of interest to both academic researchers engaged in the mechanics of structural adhesive joints as well as to R&D personnel in various industries which rely on structural adhesive bonding for a variety of purposes.

Also we hope this book will serve as a fountainhead for new research ideas in modeling and design of adhesively bonded systems.
First of all, we are beholden to the authors for their contribution, interest, enthusiasm and cooperation without which this book would not have been possible. Second, we are very thankful to the reviewers for their time and effort in providing critical and constructive comments, as the comments from peers are *sine qua non* to maintain the highest standard of a publication. Also it is our pleasure to extend our appreciation to Martin Scrivener (Scrivener Publishing) for his steadfast interest in this book project and unwavering support in more ways than one.

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Stress and Strain Analysis of Symmetric Composite Single Lap Joints Under Combined Tension and In-Plane Shear Loading

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Abstract
An analysis is presented that predicts adhesive shear and peel stresses in adhesively bonded composite single lap joints. The single lap joint is under combined tension and in-plane shear loading, and accounts for moments induced by geometric eccentricity. These eccentricity moments primarily contribute to the peel stress. When shear, tension, and eccentricity moments are simultaneously applied to a joint, a combined multiaxial stress state (two shear stress components and peel) in the adhesive can be calculated. Example calculations presented in this paper show that the predicted stress profiles are well matched with finite element analysis (FEA) predictions. The von Mises yield criterion is applied to predict the elastic limit of the adhesive for a lap joint under combined loading. This approach allows the calculation of an envelope of combined loading conditions under which the joint is expected to behave elastically.

Keywords: Adhesive bonding, combined load, multiaxial stress, peel, elastic limit
1.1 Introduction

A closed-form model is derived that predicts shear and peel stress profiles in adhesively bonded symmetric single lap joints under multiaxial loading: in-plane shear load $N_{xy}$ and in-plane tension load $N_x$. Edge moments induced from the geometric eccentricity have been accounted for when formulating shear and peel governing equations corresponding to in-plane tension load $N_x$. Shear stress components are computed based on shear-lag assumptions and peel stress is obtained from a beam on elastic foundation (BOEF) approach.

Classical analyses, based on shear-lag, have been previously developed to predict only the adhesive shear stress in bonded joints of uniform bondline thickness for a symmetric joint subjected to tension loading only [1, 2]. Improvements to the classical model include predicting peel stress and edge moments in single lap joints [3–6], accounting for plasticity in the adhesive prior to failure [7, 8], and allowing for transverse shear deformation of the symmetric adherends [9]. Delale et al. [10] extended Goland and Reissner’s approach for symmetric joints by formulating the adhesive shear stress equation to account for asymmetric adherends. Similar approaches for the asymmetric joints are presented by Yang and Pang [11], Bigwood and Crocombe [12], and Wu et al. [13].

Adhesively bonded lap geometries loaded by in-plane shear have been discussed by Hart-Smith [2], van Rijn [14], and the Engineering Sciences Data Unit [15]. The authors of these works indicate that shear loading can be analytically accounted for by simply replacing the adherend Young’s modulus in the tensile loaded lap joint solution with the respective adherend shear modulus. This assumption is valid only for simple cases with one-dimensional loading, whereas in-plane shear loaded joints are generally two- or three-dimensional. A closed-form solution for combined multiaxial loading is presented by Mortensen and Thomson [16], although the boundary conditions are treated as input parameters and the solution is not validated by FEA or experiment. To the authors’ best knowledge, there are no closed-form analytical works that are applicable to symmetric joints under combined shear loading and tension loading with self-induced eccentricity moments. Previous work by Lee and Kim [17] predicts adhesive shear and peel stress profiles for a generally asymmetric joint and includes the effects of eccentricity moments. Kim and Kedward [18] have
computed failure envelopes for combined tension and shear but did not account for adherend bending and peel stress. Mathias et al. [19] and Adams and Peppiatt [20] have also developed stress analyses predicting the multi-axial stress state from bi-directional loading and Poisson’s ratio effects. Like the work of Kim and Kedward [18], however, these did not account for the bending moments due to load path eccentricity.

This work is the combination of recent tension/bending calculations [17] with the prediction of stresses due to in-plane shear [21]. The presented analysis accounts for uncoupled bending rigidity, Young’s modulus and shear modulus of the composite adherends depending on the laminate lay-up sequence and different lamina types (e.g., glass/epoxy versus carbon/epoxy). For an example analysis, the three adhesive stress component profiles (two shear stress components, one normal stress) for joints having \([0/45]_s\) and \([45/0]_s\) woven glass/epoxy adherends are compared with FEA predictions. Yield criterion based on von Mises effective stress is applied using the analytically predicted adhesive stress solutions to establish elastic limit loading envelopes. Carbon/epoxy composite adherends and glass/epoxy composite adherends with four different lay-ups are used to compare the effects of bending rigidity and modulus on the yield envelope.

1.2 Equations and Solution

1.2.1 Model Description

A general single lap joint with in-plane tension load (per unit width) \(N_x\) and the in-plane shear (per unit width) \(N_{xy}\) is shown in Figure 1.1. The following assumptions are made for the single lap joint:

- adherends and adhesive have uniform thickness
- adhesive carries shear and peel stresses only
- uniform shear and peel stress profiles through the adhesive thickness (z-direction)
- adherends do not deform due to transverse shear
- linear elastic material behavior

The multi-axially loaded joint can be considered as a combination of two independent problems since the material behavior is assumed elastic and the in-plane tension load \(N_x\) and the in-plane
shear load $N_{xy}$ are independent of each other. For the tension loading (which includes edge moments), two adhesive strain components $\gamma_{xz}$ and $\varepsilon_{zz}$ are developed and, therefore, needed to be considered [17]. For the shear loading, only one adhesive strain component $\gamma_{yz}$ exists [21] and is independent of the strains produced from tension loading. The governing equations, written in terms of these three independent adhesive strain components, are based on the in-plane $x$-direction ($u_1, u_2$), in-plane $y$-direction ($v_1, v_2$) and transverse $z$-direction ($w_1, w_2$) displacements at the upper and lower adherend-adhesive interfaces, where the index 1 refers to (upper) Adherend 1, and the index 2 refers to (lower) Adherend 2, as shown in Figure 1.1.

1.2.2 Governing Equations for Tension Loading $N_x$

The $x$-direction displacement $u_i$ and transverse $z$-direction displacement $w_i$ (where $i = 1, 2$) are used to compute the adhesive shear strain $\gamma_{xz}$ and peel strain $\varepsilon_{zz}$ for the in-plane tension loading $N_x$. These displacements at the adhesive-adherend interface are functions of the in-plane normal stress resultants ($N_1$ and $N_2$) and the moment resultants ($M_1$ and $M_2$) from the in-plane normal stress component $\sigma_{xx}$, as well as the joint geometric and material parameters: thickness $t_i$ and effective Young’s modulus $E_i$ and bending rigidity $D_i$ of the adherends in the $x$-direction. These resultants are depicted in Figure 1.2 which shows a differential slice of the joint with local coordinates $z_1$ and $z_2$ for each adherend defined such that the adhesive-adherend interface is
located at \( z_1 = 0 \) and \( z_2 = t_2 \). The bending rigidities \( D_1 \) and \( D_2 \) for the composite adherends are calculated from classical laminated plate theory bending rigidity \([D]\) matrix. Specifically, these are the \( D_{11} \) matrix terms representing the x-direction bending rigidity of each adherend. The adhesive shear strain \( \gamma_{xz}^a \) is defined in terms of the interface-adjacent horizontal x-displacements \( u_1 \) and \( u_2 \) and thickness \( t_a \) of the adhesive.

\[
\gamma_{xz}^a = \frac{1}{t_a} (u_1 - u_2)
\]  

(1.1)

Figure 1.2 Differential segment of single lap joint under tension loading.
Differentiating Eq. 1.1 with respect to $x$ yields

$$\frac{d\gamma_{xz}^a}{dx} = \frac{1}{t_a}(\varepsilon_{xx1} - \varepsilon_{xx2})$$ (1.2)

$\varepsilon_{xx1}$ and $\varepsilon_{xx2}$ are the $x$-directional normal strains in the adherends at the adhesive interface. These can be determined from the in-plane normal stress resultants ($N_1$ and $N_2$) and the internal moment resultants ($M_1$ and $M_2$) based on simple beam theory [17].

$$\varepsilon_{xx1} = \frac{N_1}{t_1} + \frac{M_1 t_1}{2D_1} = N_1 \left(\frac{1}{E_1 t_1} + \frac{t_1^2}{4D_1}\right)$$ (1.3)

$$\varepsilon_{xx2} = \frac{N_2}{t_2} - \frac{M_2 t_2}{2D_2} = N_2 \left(\frac{1}{E_2 t_2} + \frac{t_2^2}{4D_2}\right)$$ (1.4)

where the moment resultants $M_1 = \frac{1}{2}N_1 t_1$ and $M_2 = -\frac{1}{2}N_2 t_2$ are calculated [17] based on summing moments at the adhesive-adherend interface (where shear stress acts on each adherend) as shown in Figure 1.2. Inserting Eqs. 1.3 and 1.4 into Eq. 1.2 and differentiating with respect to $x$ once more yields the relationship

$$\frac{d^2\gamma_{xz}^a}{dx^2} = \frac{1}{t_a}\left[\left(\frac{1}{E_1 t_1} + \frac{1}{E_2 t_2}\right) + \frac{1}{4}\left(\frac{t_1^2}{D_1} + \frac{t_2^2}{D_2}\right)\right] \tau_{xz}^a$$

$$+ \frac{1}{2t_a}\left(\frac{t_1}{D_1} Q_1 + \frac{t_2}{D_2} Q_2\right)$$ (1.5)

where $\tau_{xz}^a$ is the adhesive shear stress which can be shown to relate $N_1$, $M_1$ and the transverse shear resultants $Q_i$ via force and moment equilibrium applied to the differential slices shown in Figure 1.2 [17].

The adhesive peel strain $\varepsilon_{zz}^a$ is defined in terms of the interface-adjacent $z$-direction displacements $w_1$ and $w_2$ and thickness $t_a$ of the adhesive.

$$\varepsilon_{zz}^a = \frac{1}{t_a}(w_1 - w_2)$$ (1.6)

The adhesive peel stress $\sigma_{zz}^a$ is determined from a beam on elastic foundation model by considering the two adherends as beams
connected by a deformable interface. The relative transverse dis-
placements \( \bar{w} (= w_1 - w_2) \) of the adherends are related as [17]

\[
\frac{d^4 \bar{w}}{dx^4} = -\left( \frac{1}{D_1} + \frac{1}{D_2} \right) \sigma_{zz}^a
\] (1.7)

where \( \sigma_{zz}^a \) is the adhesive peel stress. Eq. 1.7 can be written as a
function of adhesive peel strain \( \varepsilon_{zz}^a \) via the relationship in Eq. 1.6.

\[
\frac{d^4 \varepsilon_{zz}^a}{dx^4} = -\frac{1}{t_a} \left( \frac{1}{D_1} + \frac{1}{D_2} \right) \sigma_{zz}^a
\] (1.8)

### 1.2.3 Governing Equation for In-Plane Shear Loading \( N_{xy} \)

The in-plane shear loading \( N_{xy} \) produces an adhesive shear strain
\( \gamma_{yz}^a \) which is defined in terms of the interface-adjacent y-direction
displacements \( v_1 \) and \( v_2 \) in adherends 1 and 2, respectively, and
thickness \( t_a \) of the adhesive.

\[
\gamma_{yz}^a = \frac{1}{t_a} (v_1 - v_2)
\] (1.9)

Differentiating Eq. 1.9 with respect to \( x \) and assuming very small
(negligible) variation of the displacements with respect to \( y \) yields

\[
\frac{d \gamma_{yz}^a}{dx} = \frac{1}{t_a} (\gamma_{xy1} - \gamma_{xy2}) = \frac{1}{t_a} \left( \frac{\tau_{xy1}}{G_1} - \frac{\tau_{xy2}}{G_2} \right)
\] (1.10)

where \( \gamma_{xy1}, \gamma_{xy2} \) and \( \tau_{xy1}, \tau_{xy2} \) are the in-plane (x-y plane) shear strain
and average shear stress components in adherends 1 and 2, respectively. \( G_1 \) and \( G_2 \) are the in-plane (x-y) effective shear moduli of
adherends 1 and 2.

In Figures 1.1 and 1.3, the applied in-plane shear load \( N_{xy} \) is
shown to be continuous through the overlap region and at any
point it must be equal to the sum of the product of each adherend’s
in-plane shear stress with its respective thickness \( t_1 \) and \( t_2 \).

\[
N_{xy} = \tau_{xy1} t_1 + \tau_{xy2} t_2
\] (1.11)

From Eq. 1.11, the shear stress in the adherend 2 can be written as,

\[
\tau_{xy2} = \frac{N_{xy} - \tau_{xy1} t_1}{t_2}
\] (1.12)
Substituting Eq. 1.12 into Eq. 1.10 yields

\[
\frac{d\gamma_{yz}^a}{dx} = \frac{t_1}{t_a} \left( \frac{\tau_{xy1}}{G_1t_1} + \frac{\tau_{xy1}}{G_2t_2} \right) - \frac{N_{xy}}{t_aG_2t_2}
\]

(1.13)

Force equilibrium performed on a differential element of the adherend 1, shown in Figure 1.4, results in relationship between the adhesive shear stress components \(\tau_{yz}^a\) and the adherend 1 in-plane shear stress \(\tau_{xy1}\).

\[
\tau_{yz}^a = t_1 \frac{\partial \tau_{xy1}}{\partial x}
\]

(1.14)

Differentiating Eq. 1.13 with respect to \(x\) one more time yields

\[
\frac{d^2\gamma_{yz}^a}{dx^2} = \frac{t_1}{t_a} \left( \frac{1}{G_1t_1} + \frac{1}{G_2t_2} \right) \frac{\partial \tau_{xy1}}{\partial x}
\]

(1.15)

Substituting Eq. 1.14 into Eq. 1.15 yields the relationship

\[
\frac{d^2\gamma_{yz}^a}{dx^2} = \frac{1}{t_a} \left( \frac{1}{G_1t_1} + \frac{1}{G_2t_2} \right) \tau_{yz}^a = \frac{G_a}{t_a} \left( \frac{1}{G_1t_1} + \frac{1}{G_2t_2} \right) \gamma_{yz}^a
\]

(1.16)

where \(G_a\) is the adhesive shear modulus.

Eqs. 1.5, 1.8 and 1.16 are the adhesive strain governing equations for a generally asymmetric joint, i.e., one with different adherends. The case of a symmetric joint is now considered for design purposes since symmetric joints are generally more used in practice. Due to the geometry and material properties of adherends 1 and 2
being the same for a symmetric joint, Eqs. 1.5, 1.8 and 1.16 can be further simplified to Eqs. 1.17 to 1.19, respectively.

\[
\frac{d^2 \gamma_{xz}^a}{dx^2} = \lambda_1^2 \gamma_{xz}^a
\]  

(1.17)

\[
\frac{d^4 \varepsilon_{zz}^a}{dx^4} = -4 \beta^4 \varepsilon_{zz}^a
\]  

(1.18)

\[
\frac{d^2 \gamma_{yz}^a}{dx^2} = \lambda_2^2 \gamma_{yz}^a
\]  

(1.19)

where

\[
\lambda_1 = \left[ \frac{G_a}{t_a} \left( \frac{2}{E_1 t_1} + \frac{t_1^2}{2D_1} \right) \right]^{\frac{1}{2}}
\]  

(1.20)

\[
\beta = \frac{1}{\sqrt{2}} \left( \frac{2E_a}{t_a D_1} \right)^{\frac{1}{2}}
\]  

(1.21)

\[
\lambda_2 = \left[ \frac{2G_a}{t_a G_1 t_1} \right]^{\frac{1}{2}}
\]  

(1.22)
1.2.4 Solutions

In order to find the closed-form solutions for the second-order ordinary differential equations (Eqs. 1.17 and 1.19) for shear strains, two boundary conditions are required for each equation. Four boundary conditions are needed to solve the fourth-order ordinary differential equation (Eq. 1.18) for peel strain. With the condition of the adherends being identical, symmetry with respect to the overlap center location at $x = 0$ can be used to reduce by half the number of boundary conditions needed. The boundary conditions for the multi-axially loaded joint are considered as the superposition of the boundary conditions applied separately to the tension loaded joint and to the shear loaded. These boundary conditions are shown in Figures 1.1 and 1.5 and were discussed as the governing equations were derived.

For the adhesive peel governing equation (Eq. 1.18), the assumption is made that the left hand side of the joint is fixed for all degrees of freedom at $x = -c$, and the right hand side at $x = c$ can translate in the $z$-direction but not rotate with respect to $x$-axis. Real structures with significant unbonded length, e.g., the thin skin of an aircraft in a single lap splice joint would not have the transverse displacement constraint which exists when testing a lap joint in a test machine. Therefore the transverse displacement was not confined. However, to preserve consistency of the loading direction, the condition of no rotation with respect to $x$-axis was enforced. This free translation with enforcement of no rotation results in a considerable moment reaction producing larger shear and peel stresses in the joint than the case of fixed translation with free rotation (typically assumed in other works). Thus the internal moment resultant $M_1$ which is induced through the adherends from the geometric eccentricity of

![Figure 1.5 Boundary conditions for tension loaded joint (D.O.F. = Degrees of Freedom).](image-url)