Advances in Mathematics Education

Angelika Bikner-Ahsbahs
Susanne Prediger *Editors*

Networking of Theories as a Research Practice in Mathematics Education

Authored by the Networking Theories Group

Springer

Advances in Mathematics Education

For further volumes:
http://www.springer.com/series/8392

Angelika Bikner-Ahsbahs • Susanne Prediger
Editors

Networking of Theories as a Research Practice in Mathematics Education

Authored by the Networking Theories Group

Springer

Editors
Angelika Bikner-Ahsbahs
Fachbereich 03 für Mathematik und Informatik
AG Didaktik der Mathematik
University of Bremen
Bremen, Germany

Susanne Prediger
Institute for Development and Research in Mathematics Education
TU Dortmund University
Dortmund, Germany

ISSN 1869-4918 ISSN 1869-4926 (electronic)
ISBN 978-3-319-05388-2 ISBN 978-3-319-05389-9 (eBook)
DOI 10.1007/978-3-319-05389-9
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014942928

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Series Preface

The present volume of *Advances in Mathematics Education* examines a heavily debated topic in mathematics education, namely that of theories, theoretical frameworks and ways in which they are deployed in existing research. Given the heterogeneity of theoretical frameworks used in mathematics education today compared to the psychometric paradigm of the 1960s, which was firmly anchored in psychology, the current book examines how different theories can be made to network with each other and in particular inform researchers interested in analyzing their data from multiple perspectives.

The Networking Theories Group was initiated and coordinated by Angelika Bikner-Ahsbahs, with founding members Michèle Artigue, Ferdinando Arzarello, Marianna Bosch, Tommy Dreyfus, Ivy Kidron, Susanne Prediger, and Kenneth Ruthven in 2006. There were some forerunners to this group, such as the work of Hans-Georg Steiner in Germany and the PME research forum on Theories of Mathematics Education in Melbourne-2005, which led to the first volume in this series (Sriraman and English 2010). However in spite of these forerunners, the Networking Theories Group has been a consistent focus group in mathematics education, with intense work done on capturing the essence of data through the use of different theoretical lenses. The group formally established itself at CERME 2005 in Spain, and subsequently has held summer research meetings in the following years. A core group of researchers from the Networking Theories Group have also been involved in the working group on theories at the CERME congresses and has run various PME research forums on theories.

Given the substantial work of this group that was reported in a ZDM special issue on Comparing, Combining, Coordinating – Networking Strategies for Connecting Theoretical Approaches (Volume 40, Issue 2, 2008), based on a paper by Bikner-Ahsbahs and Prediger already in (2006), the mathematics education community has been eager to learn of newer developments within this group on how researchers can further utilize theories in advantageous ways. The present book may serve as basis for younger researchers who often indulge in bricolaging theories on an ad-hoc basis to construct theoretical frameworks that inform their work. Moreover the chapters in the book contain a diversity of perspectives that captures the current state of the art of networking theories in mathematics education. We are pleased to have this book in our series and thank the editors for producing what we hope will be a valuable resource for the community.

Hamburg, Germany Gabriele Kaiser
Missoula, MT, USA Bharath Sriraman

References

Bikner-Ahsbahs, A., & Prediger, S. (2006). Diversity of theories in mathematics education – How can we deal with it? *ZDM – Zentralblatt für Didaktik der Mathematik*, *38*(1), 52–57.

Sriraman, B., & English, L. (2010). *Theories of mathematics education*. Berlin: Springer.

Preface

How can we deal with the diversity of theories? This was the main question that led the authors of this book to found the Networking Theories Group with members from France, Germany, Israel, Italy, UK, and Spain. When the group first met at CERME 4 in 2005, the idea of networking theories arose: starting from the shared assumption that the existence of different theories is a resource for mathematics education research, we felt that the possibilities of connecting theories (without merging into one big theory) should be further explored. The group developed strategies for networking of theories and decided to investigate strands and issues of these networking practices empirically. From 2005 on, we met regularly at least once a year for commonly conducting empirical research and for reflecting the common practices on the level of theory and methodology. The Networking Theories Group was initiated and coordinated by Angelika Bikner-Ahsbahs, with founding members Michèle Artigue, Ferdinando Arzarello, Marianna Bosch, Tommy Dreyfus, Ivy Kidron, Susanne Prediger, and Kenneth Ruthven. Agnès Lenfant was a member during the first years, while further members joined the group in later years: Stefan Halverscheid, Mariam Haspekian, Cristina Sabena, Ingolf Schäfer, and, as latest member, Alexander Meyer. Meanwhile, Kenneth Ruthven changed his role to a critical friend of the group, Luis Radford also took over the role of critically accompanying this work, and Josep Gascón frequently contributed to our progression from outside in jointly working with Marianna Bosch.

This book is an outcome of these joint efforts in which we document one line of our work (other lines have led to further joint research projects, e.g., Kidron et al. 2008, 2011; Prediger and Ruthven 2007; Artigue et al. 2009, 2011; Bikner-Ahsbahs et al. 2010, 2011).

The book explains and illustrates what it means to network theories, and presents networking as a challenging but nevertheless fruitful research practice between five theoretical approaches: namely the approach of Action, Production, and Communication (APC), the Theory of Didactical Situations (TDS), the Anthropological Theory of the Didactic (ATD), the approach of Abstraction in Context (AiC), and the theory of Interest-Dense Situations (IDS). The book shows how the activity of networking generates questions at the theoretical and practical level and how these questions can be treated.

The structure and content of the book are organized around the most intense experience in these years of common work: starting with one set of video data, we wanted to explore how the analysis of the video differs when conducted with five different theoretical lenses. This raised the issue of the role of data and yielded to the collection of further data that from the theoretical perspectives were needed and led to deepening cooperation and additional research. On the basis of these experiences, the group undertook different case studies of networking while seeking further connections and differences. The methodology of networking of theories evolved while discussing these research practices on a meta-level and is documented in the subsequent chapters.

Although the book is organized systematically and can of course be best read linearly from beginning to end, we also wanted to allow the more spontaneous reader to use it flexibly to follow her or his main interests. Support for nonlinear reading is given by various links between chapters and the index that can help to clarify constructs if the reading includes a case study in which an unfamiliar theory appears. We hope to give the reader an idea not only of the process of networking of theories as a research practice, its strength and weaknesses, but also of the gains and difficulties we have met.

The work of the Networking Theories Group in the years 2006–2013 would not have been possible without financial support for the annual meetings. University Bremen in cooperation with Die Sparkasse Bremen and Nolting-Hauff-Stiftung financed the meetings in 2006, 2008, and 2011 at Bremen University. The meeting of 2007 in Barcelona at IQS – Universitat Ramon Llull was financed by Generalitat de Catalunya (ARCS 2007), and the meetings in Mariaspring in 2010 and 2012 were financed by the Georg-August-University Göttingen and the Ministry of Science and Culture of Lower Saxony, respectively. Finally, TU Dortmund University provided substantial personal resources for the editing process for this volume.

We thank Domingo Paola for sharing with us his interesting video episodes that took place in his classroom. Further, we are grateful to Luis Radford and Kenneth Ruthven for reading the whole book and writing comments from outside advancing the view on the networking of theories. And special thanks goes to Alexander Meyer, Frank Kuhardt and John Evans; without their thorough and constructively critical reading and editing, the book with its complex issues would be much less accessible and coherent.

Bremen, Germany Angelika Bikner-Ahsbahs
Dortmund, Germany Susanne Prediger

References

Artigue, M., Cerulli, M., Haspekian, M., & Maracci, M. (2009). Connecting and integrating theoretical frames: The TELMA contribution. *International Journal of Computers for Mathematical Learning*, *14*(3), 217–240.

Artigue, M., Bosch, M., & Gascón, J. (2011). Research praxeologies and networking theories. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh Congress of the European Society for Research in Mathematics Education* (pp. 2381–2390). Rzeszów: University of Rzeszow.

Bikner-Ahsbahs, A., Dreyfus, T., Kidron, I., Arzarello, F., Radford, L., Artigue, M., & Sabena, C. (2010). Networking of theories in mathematics education. In *Proceedings of the 34th conference of the International Group for the Psychology of Mathematics Education – International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 145–175), Belo Horizonte: PME.

Bikner-Ahsbahs, A., Kidron, I., & Dreyfus, T. (2011). *Epistemisch handeln können – aber wie?* [*Knowing to act epistemically – but how?*]. Invited lecture at the annual conference of the German Society of Mathematics Education. Beiträge zum Mathematikunterricht. http://www.mathematik.tu-dortmund.de/ieem/bzmu2011/_BzMU11_1_Einfuehrungen-Hauptvortraege/BzMU11_BIKNER_Angelika_Epistem.pdf. Accessed 30 Oct 2013.

Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M., & Dreyfus, T. (2008). Toward networking three theoretical approaches: the case of social interactions. *Zentralblatt für Didaktik der Mathematik*, *40*(2), 247–264.

Kidron, I., Bikner-Ahsbahs, A., & Dreyfus, T. (2011). How a general epistemic need leads to a need for a new construct – a case of networking two theoretical approaches. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh Congress of the European Society for Research in Mathematics Education* (pp. 2451–2461). Rzeszów: University of Rzeszow.

Prediger, S., & Ruthven, K. (2007). From teaching problems to research problems. Proposing a way of comparing theoretical approaches. In D. Pitta-Pantazi & G. Phillipou (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education (CERME 5)* (pp. 1745–1771). Cyprus: ERME.

Contents

Part III Case Studies of Networking

Part IV Reflections

Part I
Introduction

Chapter 1
Starting Points for Dealing with the Diversity of Theories

Angelika Bikner-Ahsbahs, Susanne Prediger, Michèle Artigue, Ferdinando Arzarello, Marianna Bosch, Tommy Dreyfus, Josep Gascón, Stefan Halverscheid, Mariam Haspekian, Ivy Kidron, Agnès Corblin-Lenfant, Alexander Meyer, Cristina Sabena, and Ingolf Schäfer

Abstract This chapter presents the main ideas and constructs of the book and uses the triplet (system of principles, methodologies, set of paradigmatic questions) for describing the theories involved. In Part II (Chaps. 3, 4, 5, 6, and 7), the diversity of five theoretical approaches is presented; these approaches are compared and systematically put into a dialogue throughout the book. In Part III (Chaps. 9, 10, 11, and 12), four case studies of networking practices between these approaches show how this dialogue can take place. Chapter 8 and Part IV (Chaps. 13, 14, 15, 16, and 17) provide methodological discussions and reflections on the presented networking practices.

Networking Theories Group: Angelika Bikner-Ahsbahs, Susanne Prediger, Michèle Artigue, Ferdinando Arzarello, Marianna Bosch, Tommy Dreyfus, Josep Gascón, Stefan Halverscheid, Mariam Haspekian, Ivy Kidron, Agnès Lenfant, Alexander Meyer, Cristina Sabena, and Ingolf Schäfer

A. Bikner-Ahsbahs (✉)
Fachbereich 03 für Mathematik und Informatik, AG Didaktik der Mathematik,
University of Bremen, Bibliothekstrasse 1, D-28359 Bremen, Germany
e-mail: bikner@math.uni-bremen.de; bikner@t-online.de

S. Prediger
Institute for Development and Research in Mathematics Education,
TU Dortmund University, D-44227 Dortmund, Germany
e-mail: prediger@math.uni-dortmund.de

M. Artigue
Laboratoire LDAR & IREM, Université Paris Diderot – Paris 7, Sorbonne Paris Cité,
Case 7018, 75205 Paris Cedex 13, France
e-mail: michele.artigue@univ-paris-diderot.fr

F. Arzarello
Dipartimento di Matematica, Università di Torino, Via Carlo Alberto, 10, 10123 Torino, Italy
e-mail: ferdinando.arzarello@unito.it

M. Bosch
IQS School of Management, Universitat Ramon Llull,
Via Augusta 390, 08017 Barcelona, Spain
e-mail: marianna.bosch@iqs.edu

A. Bikner-Ahsbahs and S. Prediger (eds.), *Networking of Theories as a Research Practice in Mathematics Education*, Advances in Mathematics Education,
DOI 10.1007/978-3-319-05389-9_1,

Keywords Theories • Networking • Methodology of networking

For about 15 years, the diversity of theories has been intensively discussed in the mathematics education research community (Ernest 1998; Steen 1999; Lerman 2006; Prediger et al. 2008a; Sriraman and English 2010; and many others). Our Networking Theories Group started in 2005. In this chapter, we make explicit

T. Dreyfus
Department of Mathematics Science and Technology Education, Joan and Jaime Constantiner School of Education, Tel Aviv University, P. O. B. 39040, Ramat Aviv, Tel Aviv 69978, Israel
e-mail: tommyd@post.tau.ac.il

J. Gascón
Departament de Matemàtiques, Facultat de Ciències, Universitat Autònoma de Barcelona, Edifici C – Campus de la UAB, 08193 Bellaterra (Cerdanyola del Vallès), Barcelona, Spain
e-mail: gascon@mat.uab.cat

S. Halverscheid
Mathematics Institute, Georg-August-University, Bunsenstr. 3-5, D-37073 Göttingen, Germany
e-mail: sth@uni-math.gwdg.de

M. Haspekian
Laboratoire EDA, Faculté des Sciences Humaines et Sociales, Université Paris Descartes, Sorbonne Paris Cité, 45, rue des Saints-Pères, 75270 Paris Cedex 06, France
e-mail: mariam.haspekian@parisdescartes.fr

I. Kidron
Department of Applied Mathematics, Jerusalem College of Technology, Havaad Haleumi Street, 21, Jerusalem 91160, Israel
e-mail: ivy@jct.ac.il

A. Corblin-Lenfant
IREM de REIMS, UFR Sciences Exactes et Naturelles, Moulin de la Housse, BP 1039, 51687 Reims Cedex 2, France
e-mail: Agnes.Corblin@ac-reims.fr

A. Meyer
TU Dortmund University, D-44227 Dortmund, Germany
e-mail: Alexander.Meyer@math.tu-dortmund.de

C. Sabena
Dipartimento di Filosofia e Scienze dell'educazione, Università di Torino, Via Gaudenzio Ferrari, 9/11, 10124 Torino, Italy
e-mail: cristina.sabena@unito.it

I. Schäfer
Fachbereich 3 for Mathematics and Information Technology, University of Bremen, Bibliothekstrasse 1, D-28359 Bremen, Germany
e-mail: ingolf.schaefer@uni-bremen.de

the starting points for our way of dealing with this diversity. We will refer to the following questions:

- Why do there exist so many theoretical approaches?
- What exactly do we mean by theories or theoretical approaches, and for what are they needed?
- How can we deal with the diversity of theoretical approaches?

Whereas the most important third question is treated throughout the whole book, this introduction starts with the first two questions.

1.1 Sources for the Diversity of Theoretical Approaches

The first question is easy: one important source for the diversity of theories in mathematics education is that they evolved independently in different regions of the world and different cultural circumstances, including traditions of typical classroom cultures, values, but also varying institutional settings (cf. English and Sriraman 2005, p.452). The (at least equally important) second reason for the existence of different theories and theoretical approaches is the complexity of the topic of research itself. Since mathematics learning and teaching is a multi-faceted phenomenon which cannot be described, understood, or explained by one monolithic theory alone, a variety of theories is necessary to grasp the complexity of the field (Bikner-Ahsbahs and Prediger 2010). A third reason has been outlined by Teppo (1998) in that there are various ways of knowing in the field of mathematics education which are situated in various paradigms and, thus, produce different kinds of theoretical views. Teppo takes the diversity of theories as a sign that the "field of mathematics education is alive and well" (1998, p. 5). We would add that the diversity is not only an *indicator* for the dynamic character of the field, but it is also an *outcome* of the dynamic of theories. This is the fourth source.

The work of the Networking Theories Group, which has grown from the CERME working groups on Theories since CERME 4, started from the claim of diversity as a resource (Artigue et al. 2006). In order to substantiate the claim of diversity as a resource for rich scientific progress, the second question is addressed in the following section (following Bikner-Ahsbahs and Prediger 2010).

1.2 Conceptualizations and Functions of Theoretical Approaches

There is *no shared unique definition* of theory or theoretical approach among mathematics education researchers (see Assude et al. 2008). The large diversity already starts with the heterogeneity of what is called a theoretical approach or a theory by various researchers and different scholarly traditions. Some refer to basic research paradigms (such as the interpretative approach within social constructivism), others to comprehensive general theories (such as the Theory of Didactical Situations), and others to local conceptual tools (such as the modeling cycle) (cf. Prediger 2014). Differences exist in the ways to conceptualize and question mathematical activities and educational processes, in the type of results they can provide, but also in their scopes and backgrounds.

Mason and Waywood distinguish between different characters of theories: *foreground theories* are local theories *in* mathematics education "about what does and can happen within and without educational institutions" (Mason and Waywood 1996, p. 1056). In contrast, a *background theory* is a (mostly) consistent philosophical stance *of* or *about* mathematics education which "plays an important role in discerning and defining what kind of objects are to be studied, indeed, theoretical constructs act to bring these objects into being" (ibid., p. 1058). The background theory can comprise implicit parts that refer to epistemological, ontological, or methodological ideas, for example about the nature and aim of education, the nature of mathematics, and the nature of mathematics education. Taking the notions of foreground and background theory as offering *relative distinctions* rather than an absolute classification, they can help to distinguish different views on theories.

The different understandings of "theory" cannot only be distinguished according to the focus on foreground or background theories, but also according to their general view on the relation between theory and research practices. For analytical reasons, we distinguish a more static and a more dynamic view on theories. A normative *more static view* regards theory as a human construction to present, organize, and systematize a set of results about a piece of the real world, which then becomes a tool to be used. In contrast, a *more dynamic view* regards a theory as a tool in use rooted in some kind of philosophical background which constantly has to be developed in a suitable way in order to answer a specific question about an object. In this sense the notion of theory is embedded in the practical work of researchers. It is not ready for use, but has to be developed in order to answer a given question. In this context, the term "theoretical approach" is sometimes preferred to "theory", and so do we in this volume. Even very well developed theories such as the Theory of Didactical Situations (see Chap. 4) or the Anthropological Theory of the Didactic (Chap. 5) are still in a state of flux and can better be described by a wider and less static view on theories.

Most conceptualizations of theoretical approaches define the *function* of theories as being "to explain a specific set of phenomena as in 'true in fact and theory'" and emphasize "sense-making […as] the subject of theorizing …" (Mason and Waywood 1996, p. 1056). This includes the function of (background) theories as perspectives which help to produce knowledge about *what*, *how*, and *why* things happen in a vague phenomenon of mathematics education. And hence Mason and Waywood conclude: "To understand the role of theory in a research program is to understand what are taken to be the things that can be questioned and what counts as an answer to that questioning" (Mason and Waywood 1996, p. 1056).

Silver and Herbst (2007) also approach the notion of theory in mathematics education in a dynamic way. Comparisons of different theories, with respect to their roles as instruments mediating between problems, practices, and research, show that *theories in mathematics education are mostly developed for certain purposes*. For example:

- theories which mediate practices and research can be understood as "a language of descriptions of an educational practice" or as "a system of best practices" (ibid., p. 56);
- theories which mediate problems and practices can be understood as a "proposed solution to a problem" or a "tool which can help design new practices" (ibid, p. 59);
- theories which mediate research and problems can be understood as "means to transform a commonsensical problem into a researchable problem" or as a "lens to analyze data and produce results of research on a problem" (ibid., p. 50).

Some theories are used to investigate problems or empirical phenomena in mathematics education; others provide the tools for design, and the language to observe, understand, describe, and even explain or predict, (conceptualized) phenomena.

If we approach the notion of theory in this way, from its role in research practices, theories can be understood as guiding research practices and at the same time being influenced by or being the aim of research practices. This dialectic between theory and research (Assude et al. 2008) has to be taken into account in many discourses about the notion of theory. For example, Radford (2008) takes this role into account by describing theories by means of a triplet of three components:

> A "theory can be seen as a way of producing understandings and ways of action based on:
>
> - A system, *P*, of *basic principles*, which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective.
> - A *methodology*, *M*, which includes techniques of data collection and data-interpretation as supported by P.
> - A set, *Q*, of paradigmatic *research questions* (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified)." (Radford 2008, p. 320)

Radford's conceptualization of theory as "a way of producing understandings and ways of action" again reflects that theories cannot be separated from the research practices in which they are grown and used. Radford considers this triplet as being a dynamic entity which evolves successively through the dialectic relationship of its components. Radford specifically names two ways of supporting the evolution of theories: through producing results, because "the results of a theory influence its components"; and also through the networking of theories (Radford 2012).

In this book, we work with five theoretical approaches, presented in Chaps. 3, 4, 5, 6, and 7. For presenting the theoretical approaches, we decided to follow Radford's triplet of Principles, Methodologies, and Questions. It was interesting to see that two decisions were necessary before this fitted for all five approaches: we had to extend the principles by Key Constructs; and we had to allow different orders among the four components Principles, Key Constructs, Questions and Methodology, since their mutual relationships are conceptualized differently in the five approaches.

1.3 A Journey on Networking Theories

Steen (1999) warned that the diversity of theoretical approaches in mathematics education research is an indicator of missing maturity of the discipline. In contrast, many researchers emphasize that the diversity is not a problem, but a necessity for grasping the complexity of the topic of research (Teppo 1998; Lerman 2006). However, accepting the co-existence of isolated, arbitrary theoretical approaches regularly can cause challenges for communication, for the integration of empirical results (e.g., for practical purposes in classrooms), and for scientific progress (Prediger et al. 2008b, p. 169). That is why we emphasize that the diversity of theoretical approaches can *only* become fruitful *if* connections between them are *actively established*.

During the years of common work in the CERME working groups (Artigue et al. 2006; Arzarello et al. 2008; Prediger et al. 2010; Kidron et al. 2011, 2013), many different strategies and methods for networking of theoretical approaches were developed (see Chap. 8 for an overview).

In this book, we report on the work of the Networking Theories Group (see Preface) on establishing connections among the following five theoretical approaches:

- *Action, Production, and Communication Approach* (introduced in Chap. 3): APC provides a frame for investigating semiotic resources in the classroom. It addresses the use of semiotic resources from a multimodal perspective including the analysis of gestures as a resource for expression and communication.
- *Theory of Didactical Situations* (introduced in Chap. 4): TDS provides a frame for developing and investigating didactical situations in mathematics from an epistemological and systemic perspective that includes a corpus of concepts relevant for addressing teaching and learning processes in mathematics classrooms and beyond.
- *Anthropological Theory of the Didactic* (introduced in Chap. 5): ATD provides a frame for investigating mathematical and didactical praxeologies on the institutional level of mathematics and its teaching and learning conditions. The main idea of the concept of praxeologies is that all human activities comprise and link two parts, a practice and a theory part.
- *Abstraction in Context* (introduced in Chap. 6): AiC provides a frame for investigating learning processes which lead to new concepts and how they are built through phases: the need for a new concept, the process of constructing the new concept, and its consolidation.
- *Theory of Interest-Dense Situations* (introduced in Chap. 7): IDS provides a frame for how interest-dense situations and their epistemic and interest-supporting character are shaped through social interactions in mathematics classes distinguishing three levels: the social interactions and how the participants are involved, the dynamic of the epistemic processes, and the attribution of mathematical value.

For establishing connections among these five approaches, we began by selecting a set of data as an empirical base. The original data provided by the APC team (see Chap. 2 for the presentation of the data) consisted of a video of two students' learning process on exponential functions in grade 10, namely Carlo and Giovanni.

Part II of the book (Chaps. 3, 4, 5, 6, and 7) presents the five theoretical approaches involved in the book. They describe their main principles, methodologies, and paradigmatic questions adding key constructs and – if necessary – additional results and show how these theories are used for analyzing the (for most approaches alien) set of data. Already these first presentations bear testimony of a strong experience recognized in this exercise, namely the need for different data: whereas for the APC team, their video together with the task and the written answers was completely sufficient for conducting an analysis, this data turned out to be insufficient for teams using other theoretical approaches because it does not address their relevant questions and it does not provide the data that is in the center of their methodologies.

That is why the initial set of data had to be appropriated for each approach and extended by background information about the intentions of the teacher, the curriculum of the class, students' previous knowledge, teacher's intentions etc. For making these specific needs for data transparent, the analysis for each theoretical approach in Chaps. 3, 4, 5, 6, and 7 is split into two parts: the first part with only the initial (alien) video and, wherever necessary, the second part with the extended and appropriated set of data.

A second issue was how it would be possible for the different groups to make sense of the given data. Besides the fact that all approaches needed a process of extension and appropriation of the data, they chose different subsets of data to be able methodically to work. This included differences in the focus on the mathematical task. For some theories, the character of the given task is important because they investigate specific questions that can be induced by the design of tasks. For others, the given mathematical learning situation is to be investigated and therefore the situation is taken as it is. Some approaches focus on learning in-depth, others include the teacher behavior or pose further questions to include institutional and societal conditions. These experiences with our home theories investigating alien data pointed to the function of theories as heuristics for research. Since data collection already belongs to the research practice that is specific for a certain approach, this attempt to analyze alien data is a networking endeavor on the theories' methodological level.

Whereas Part II of the book is mainly concerned with making the theoretical approaches understandable (also with respect to their research practices), *Part III* documents different ways of how to deepen the connection of theories. The introductory Chap. 8 presents different networking strategies and profiles on a general level and provides the language and some methodological considerations for networking.

The core of the book is the rest of Part III with four case studies of networking presented in Chaps. 9, 10, 11, and 12, all focused on the set of video data on Carlo, Giovanni, and the exponential function. These case studies not only show the development of new aspects of this research but also how alien and home theories can more deeply be understood by practices of networking:

- *Chapter 9* shows a case study of networking between APC and AiC. In the first case study, the role of gestures for the process of knowledge construction is considered empirically. APC and AiC are linked in a way that gesture studies are included into the frame of AiC through learning from research within the APC-space.

- *Chapter 10* shows a case study of networking between TDS, ATC, and AiC. The case of context, milieu, and the media-milieu dialectic contrasts and compares three complex key constructs and their status within each theory in order to learn how constructs which at a first glance seem to have a similar role in the understanding of teaching and learning can differ in each theory.
- *Chapter 11* shows a case study on networking involving only two theories, APC and IDS. It describes a networking case that starts from a situation of seeming contradiction and leads to a local integration of the new concept of the epistemological gap into both theories.
- *Chapter 12* shows a case study of networking between TDS and IDS. It investigates empirically two phenomena of two different theories and networks the theories by comparing and contrasting these phenomena. This process leads to deepening the understanding of the theories on the one hand and provides insight into the character of the phenomena and their common idea on the other. In addition, the two phenomena are contrasted with a third phenomenon from APC. A reflection from an ATD perspective as an outside view on this case further deepens the comprehension of the phenomena.

The lessons learnt from these different practices of bilateral and trilateral networking were on three levels:

- *On the empirical level*, we could gain deep and complex insights into the empirical and conceptualized phenomena in the videos and the role of data. These insights are reported in Chaps. 9, 10, 11, and 12.
- *On the theoretical level*, the networking gave many impulses for theory development by sharpening theoretical principles or constructs, extending theoretical approaches, building new concepts, posing new questions, or making explicit commonalities but always while keeping the theories' main identities. These developments are documented in Chaps. 9, 10, 11, and 12 and compared and systematized in Chaps. 14 and 15.
- *On a methodological level*, the case studies of networking also offered insights that can be transferred from the concrete cases to networking in principle. These experiences and reflections are made explicit in Part IV of the book, in Chaps. 13, 14, and 15.

Part IV of the book is dedicated to the reflection of networking practices from different perspectives:

- from an internal perspective considering individual and informal experiences (Chap. 13);
- from a bottom-up perspective that tries to systematize the experiences (Chap. 14), their gains and difficulties;
- from a top-down perspective in terms of research praxeologies (Chap. 15);
- and from two external perspectives adopted by our critical friends, Kenneth Ruthven and Luis Radford (Chaps. 16 and 17).

Since the journey of networking of theoretical approaches was very long and intense, this book is only partly able to capture and demonstrate our learning experiences. We started enthusiastically and continued being so, although we met difficulties for which we had to find ways to overcome. One typical difficulty is, for example, the limits arising from our common principle that the theories must not lose their specificity.

The challenge to be theoretically open-minded slowly changed our standpoints. Deep insights and interesting research results helped us carry on and further develop the view on theories, research practices, and their diversity, and to uncover the strengths and weaknesses of our networking enterprise.

In this way, the book intends to offer an opportunity for the readers to partly participate in this networking endeavor and form an opinion and critical standpoint on crucial methodological and meta-theoretical challenges that are as yet far from being completely clarified.

References

Artigue, M., Bartolini-Bussi, M., Dreyfus, T., Gray, E., & Prediger, S. (2006). Different theoretical perspectives and approaches in research in mathematics education. In M. Bosch (Ed.), *Proceedings of the 4th Congress of the European Society for Research in Mathematics Education* (pp. 1239–1244). Barcelona: Fundemi IQS.

Arzarello, F., Bosch, M., Lenfant, A., & Prediger, S. (2008). Different theoretical perspectives in research from teaching problems to research problems. In D. Pitta-Pantazi, G. Phillipou, et al. (Eds.), *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education (CERME 5)* (pp. 1618–1627). Cyprus: ERME.

Assude, T., Boero, P., Herbst, P., Lerman, S., & Radford, L. (2008). *The notions and roles of theory in mathematics education research.* Paper presented by the Survey Team at ICME 11, Mexico.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories – An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (Advances in mathematics education series, pp. 483–506). New York: Springer.

English, L., & Sriraman, B. (2005). Theories of mathematics education: A global survey of theoretical frameworks/trends in mathematics education research. *Zentralblatt für Didaktik der Mathematik, 37*(6), 450–456.

Ernest, P. (1998). A postmodern perspective on research in mathematics education. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (Vol. 1, pp. 71–85). Dordrecht: Kluwer.

Kidron, I., Bikner-Ahsbahs, A., Monaghan, J., Radford, L., & Sensevy, G. (2011). Different theoretical perspectives and approaches in research in mathematics education. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th conference of the European Society for Research in Mathematics Education* (pp. 2377–2380). Rzeszow: ERME/University of Rzeszow.

Kidron, I., Bosch, M., Monaghan, J., & Radford, L. (2013). Different theoretical perspectives and approaches in research in mathematics education. In B. Ubuz, C. Haser, & M. A Mariotti (Eds.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (CERME 8), Antalya.

Lerman, S. (2006). Theories of mathematics education: Is plurality a problem? *Zentralblatt für Didaktik der Mathematik, 38*(1), 8–13.

Mason, J., & Waywood, A. (1996). The role of theory in mathematics education and research. In A. J. Bishop et al. (Eds.), *International handbook of mathematics education* (pp. 1055–1089). Dordrecht: Kluwer.

Prediger, S., Arzarello, F., Bosch, M., & Lenfant, A. (Eds.). (2008a). Comparing, combining, coordinating – Networking strategies for connecting theoretical approaches. Thematic Issue of *ZDM – The International Journal on Mathematics Education*, *40*(2), 163–327.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008b). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40*(2), 165–178.

Prediger, S., Bosch, M., Kidron, I., Monaghan, J., & Sensevy, G. (2010). Different theoretical perspectives and approaches in mathematics education research – Strategies and difficulties when connecting theories. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 1529–1534). Lyon: Institut National de Recherche Pédagogique.

Prediger, S. (2014, in press). Rolle von Theorien. In R. Bruder, L. Hefendehl-Hebeker, B. Schmidt-Thieme, & H.-G. Weigand (Eds.), *Handbuch Mathematikdidaktik*. Heidelberg: Springer.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*(2), 317–327.

Radford, L. (2012, May 31–June 1). *On the growth and transformation of mathematics education theories.* Paper presented at the international colloquium: The didactics of mathematics: Approaches and issues. A homage to Michèle Artigue. http://www.laurentian.ca/educ/lradford. Accessed 29 Oct 2013.

Silver, E. A., & Herbst, P. G. (2007). Theory in mathematics education scholarship. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 39–68). Reston: NCTM.

Sriraman, B., & English, L. (Eds.). (2010). *Theories of mathematics education: Seeking new frontiers* (Advances in mathematics education series). New York: Springer.

Steen, L. A. (1999). Review of mathematics education as research domain. *Journal for Research in Mathematics Education, 30*(2), 235–241.

Teppo, A. (1998). Divers ways of knowing. In A. R. Teppo (Ed.), *Qualitative research methods in mathematics education* (pp. 1–16). Reston: NCTM.

Chapter 2
Description of the Data: Introducing the Session of Carlo, Giovanni, and the Exponential Function

Cristina Sabena

Abstract The chapter provides the basic information on the set of data that is used throughout the book. Data from a video recording show two students, Carlos and Giovanni, when investigating the exponential function in a dynamic geometry environment. An interview with the teacher gives background information.

Keyword Data

The common activity in the Networking Group started from considering a single set of data from different perspectives. The basis of the data is a video showing a session from the group-work of two students, Giovanni and Carlo, during a teaching experiment on the exponential function in secondary school. We analyzed the video from different theoretical perspectives.

This *initial set of data* was shared at the beginning of the networking activity. It consists of a video and its verbal transcript (translated into English), the students' written protocols, and some information on the research and didactical contexts. In Sect. 2.1 we present the data, specifying what was actually presented and used in our joint work. The complete transcript can be found in the Appendix A.1.

While this set of data was sufficient within the theoretical framework of the research project in which it was gathered (in an informal project following Paola 2006 and Arzarello et al. 2009), the researchers of other theoretical frameworks needed more data on students' backgrounds, teacher's perspectives and many other aspects. For gathering this *extended set of data*, an interview with the teacher was conducted (see Sect. 2.2.2).

C. Sabena (✉)
Dipartimento di Filosofia e Scienze dell'educazione, Università di Torino,
Via Gaudenzio Ferrari, 9/11, 10124 Torino, Italy
e-mail: cristina.sabena@unito.it

A. Bikner-Ahsbahs and S. Prediger (eds.), *Networking of Theories as a Research Practice in Mathematics Education*, Advances in Mathematics Education,
DOI 10.1007/978-3-319-05389-9_2, © Springer International Publishing Switzerland 2014

Further, a second video (here called "extra video on Task 3") was also considered during the networking process; the video is briefly introduced in Sect. 2.2.3 and its transcript fully presented in the Appendix A.2.

2.1 Initial Set of Data

2.1.1 Research Context

The data come from an Italian long-term teaching project on investigating changing quantities by ICT technologies in secondary schools. The project is supervised by Ferdinando Arzarello and is planned and realized with the active collaboration of the classroom teacher, Domingo Paola (2006).

Students have five hours of mathematics per week. Their teacher (Domingo) has been with them for 5 years. ICT technologies are used extensively in the classroom, in particular dynamic geometry software, spreadsheets and graphic-symbolic calculators. The teaching methods mainly alternate between group-work activities and classroom discussions.

At the time of the experiment (February 2004), students were in grade 10 (second year of secondary school), and already knew about dynamic discrete models of exponential and logistic growth, approached by using different software for graphic-symbolic manipulations. They knew that in a succession defined by recursion that represents an exponential growth, the ratio of two consecutive terms is constant. They had worked with first and second finite differences for functions described by numerical values for $(x, f(x))$ represented in tables. They usually described the features of increasing and decreasing functions using the words "it grows and grows more and more" and "it grows and grows less and less."

2.1.2 Professional Background of the Teacher

At the time of the project, Domingo was a 50-year-old teacher with long experience in mathematics education, developed through a long-lasting collaboration with many Italian researchers.[1] He was one of the most active Italian "teacher-researchers," and had published several papers in Italian and international journals and conference proceedings. He was engaged in pre-service and in-service teacher education programs, and took part in innovation projects funded by the Italian Ministry of Education.

[1] Teacher-researchers play a fundamental role in the Italian paradigm of "research for innovation" (see Arzarello and Bartolini Bussi 1998 for a full description). These teachers collaborate closely with researchers, and participate in all phases of classroom-based research, from planning to data analysis.

As a teacher, Domingo believes that the major goal of teaching and learning (in general, and of mathematics in particular) is to foster the formation and development of competences and knowledge essential for an informed, conscious, and critical citizenship. His didactical choices are aimed at this objective.

In his lessons he adopts an informal approach, and exploits different ICT tools (spreadsheets, symbolic-graphic calculators, devoted software for graphs of functions, …) in order to make the students visualize and reason on properties of functions starting from numerical data and a perceptive-descriptive approach. The formalization within the formal mathematical theory follows from the informal approach through technology. As a didactical technique, he poses problems through sheets and files that he prepares himself, with the students working on these in groups of two or three. During the group-work, Domingo supervises the work, resolving possible difficulties with the tools, and providing prompts with regard to the tasks. Classroom discussions follow the group-work sessions: in these lessons, the teacher guides the comparison between the students' productions, and introduces or refines the mathematical notions and methods, by enhancing an argumentative and theoretical approach to mathematics.

2.1.3 Activities and Tools

In the session that is investigated here, the students are involved in exploratory activities that are conducted in pairs using Cabri, a Dynamic Geometry Software (DGS) program. With three tasks presented in written worksheets and DGS files, they explore the graphs of exponential functions $y = a^x$ and of its tangent line[2] (a is a parameter whose value can be changed in a slider).

Carlo and Giovanni work together on a computer with files that the teacher has prepared for the exploration. Figures 2.1, 2.2, and 2.3 show the (translated) text of the worksheets and the configurations in the DGS (that was not on the worksheet but on the computer screen; some screenshots are added for easier reading).

The two students work on three tasks (to which we will sometimes refer as Episode 1, 2, and 3). Task 1 and Task 2 are presented on one worksheet and Task 3 on a second worksheet (since it required opening the new version of the software, Cabri II PLUS). Each task corresponds to a DGS file, which the students have to open and use in their work. The worksheets are translated below.

In Task 1, the students have first to explore the graph of $y = 2.7^x$ in the first DGS file; they can drag a point representing the abscissa x, and for every x, a number representing the ordinate $y = 2.7^x$ appears on the screen (Fig. 2.1). They can use the animation function of DGS to foster the observation of the different velocities of x and y. In Task 2, the students open another file and are asked to explore $y = a^x$ by changing the value of the base of the exponential (Fig. 2.2).

[2] The line is actually a secant line; the secant points are so near that the line appears on the screen as tangent to the graph. This issue had been discussed in the classroom in a previous lesson.

First Worksheet (Task 1)

Task 1

a. Open with Cabri II the file "y = (2.7)^x".

In this file you will see: the point x on the x-axis and the point y = 2.7^x, on the y-axis.

Move the point x on the x-axis and check what happens to the point y = 2.7^x on the y-axis; that is, observe how (2.7)^x varies as x is changing.

In order to make these observations, modify also the measure unit on the y-axis of your worksheet. After some trials, use animation. Move the point x towards the left until arriving nearly to the end of the field of variation of the negative x's, and then animate with a spring the point x so that it moves from the left towards the right.

Share all the observations that you think interesting on the coordinate movement of the two points, and describe briefly (but clearly) your argument on the sheet that has been given to you.

b. What trend do you think the function y = (2.7)^x has? Before drawing it on your sheet, agree about what you think are the most important features of the graph of y = (2.7)^x and then justify the graph that you draw.

Corresponding computer screen (not on Worksheet)

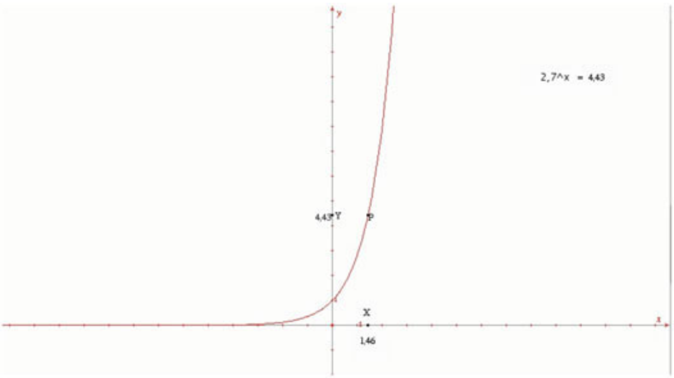

Fig. 2.1 Task 1 and corresponding DGS screen configurations

Task 3 (Fig. 2.3) is more structured than the previous tasks and proposes an exploration in order to highlight both local and global aspects of the exponential variation. It contains:

- the graph of $y = a^x$;
- the points P(x, a^x), H($x + \Delta x$, a^x);
- two sliders, one for Δx and another for a, whose variation allows the students to modify, respectively, the increment Δx and the base of the exponential.

The exploration carried out varying Δx has the didactical goal of highlighting local aspects relative to the value of the slope of the tangent line. The exploration carried

First Worksheet (Task 2)

Task 2

Open the file "a^x" with Cabri II. In it you see a point X on the x-axis, a point a^x, on the y-axis, a point P of coordinate (x, a^x) that, therefore, describes, as x is varying, the function's graphic y = a^x, and finally a ray, on which there is a point A, whose abscissa is the base of the exponential a^x.

That means that, by varying the position of A, you get exponentials with a different base (all bigger than 0: for this there is a precise reason that we will discuss). Then moving the point A changes the base of the exponential. Moving the point P, you run along the graph of an exponential function with a fixed base. Explore, share your impressions (is there something which is not clear and we were not expecting or that is clear and you were expecting). Describe briefly your exploration on the sheet.

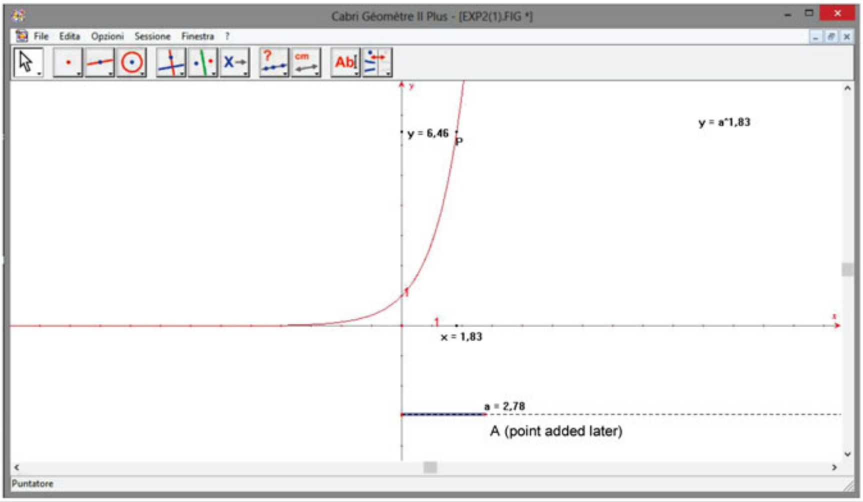

Fig. 2.2 Task 2 and corresponding DGS screen configurations

out dragging P has the goal of shedding light on global aspects of the exponential function, and in particular the variations related to its slope functions (according to the teacher's planning).

2.1.4 The Students Carlo and Giovanni

The video shows two male students, Carlo and Giovanni, who are used to working together during group-work activities in mathematics. We provide brief information about the students (the information has been provided by the teacher).

Carlo reveals good intuition in group-work and is very participative and motivated both in collective activities and in individual work. This attitude has not always been the case. At the beginning of grade 9 (first year of high school), he was

Second Worksheet (Task 3)

Task 3

a. Open the file "exp" with Cabri II PLUS.

Look carefully at the figure: you see that there are some objects that you can move (the points P, a and the segment Δx); observe also that the segments PH and Δx have the same length (they have been built so).

Describe briefly the figure, moving first P, then Δx (changing its length), then A; write briefly your observations on the sheet.

b. Look carefully at what happens when Δx tends to 0…
Does such a requirement suggest to you new observations with respect to those you have already discussed and written? Why?

c. Now we want to study how the slope of the linear function that approximates best the function $y = a$^x changes while x is moving. As already said, that means studying how the slope of the line tangent to the function $y = a$^x at the point of abscissa x changes while x moves.

You can use the Cabri worksheet to help you in order to answer the question. Explore and possibly use any other software on the PC you like, or you may also use no software.

Whatever is your decision concerning your (chosen) solving strategies, discuss briefly the features of the graph of the function $m = m(x)$, where m is the slope of the line tangent to the function $y = a$^x at the point of abscissa x.

You are required to say how the slope of the line tangent to the function $y = a$^x at the point of abscissa x changes as x changes.

After discussing your opinions, write briefly the features of the graph of $m=m(x)$ on the worksheet and draw also a sketch of the graph itself.

Corresponding Computer screen (not on Worksheet)

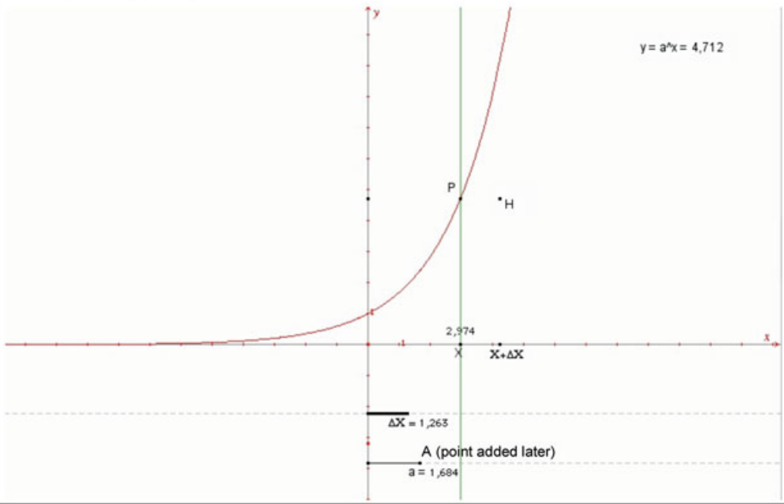

Fig. 2.3 Task 3 and corresponding DGS screen configuration

little involved in school, although his results were sufficient thanks to his capacity of using his possessed knowledge. In grade 10, together with the support of the family, Carlo's engagement in school has increased, arriving at brilliant results, especially in mathematics. In a short time the student has become one of the most