SYNTHESIZED
TRANSMISSION LINES
SYNTHESIZED TRANSMISSION LINES
DESIGN, CIRCUIT IMPLEMENTATION, AND PHASED ARRAY APPLICATIONS

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To our beloved families and motherland
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>xi</td>
</tr>
<tr>
<td><strong>1 Introduction to Synthesized Transmission Lines</strong></td>
<td>1</td>
</tr>
<tr>
<td>C. W. Wang and T. G. Ma</td>
<td></td>
</tr>
<tr>
<td>1.1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Propagation Characteristics of a TEM Transmission Line</td>
<td>2</td>
</tr>
<tr>
<td>1.2.1 Wave Equations</td>
<td>2</td>
</tr>
<tr>
<td>1.2.2 Keys to Miniaturization</td>
<td>5</td>
</tr>
<tr>
<td>1.3 Analysis of Synthesized Transmission Lines</td>
<td>7</td>
</tr>
<tr>
<td>1.3.1 Bloch Theorem and Characterization of a Periodic Synthesized Transmission Line</td>
<td>7</td>
</tr>
<tr>
<td>1.3.2 Characterization of a Non-Periodic Synthesized Transmission Line</td>
<td>9</td>
</tr>
<tr>
<td>1.3.3 Extraction of Line Parameters from S-Parameters</td>
<td>10</td>
</tr>
<tr>
<td>1.4 Lumped and Quasi-Lumped Approaches</td>
<td>11</td>
</tr>
<tr>
<td>1.4.1 Lumped Networks</td>
<td>11</td>
</tr>
<tr>
<td>1.4.2 Shunt-Stub Loaded Lines</td>
<td>14</td>
</tr>
<tr>
<td>1.5 One-Dimensional Periodic Structures</td>
<td>16</td>
</tr>
<tr>
<td>1.5.1 Complementary-Conducting-Strip Lines</td>
<td>19</td>
</tr>
<tr>
<td>1.6 Photonic Bandgap Structures</td>
<td>20</td>
</tr>
<tr>
<td>1.7 Left-Handed Structures</td>
<td>21</td>
</tr>
<tr>
<td>References</td>
<td>24</td>
</tr>
<tr>
<td><strong>2 Non-Periodic Synthesized Transmission Lines for Circuit Miniaturization</strong></td>
<td>26</td>
</tr>
<tr>
<td>C. W. Wang and T. G. Ma</td>
<td></td>
</tr>
<tr>
<td>2.1 Introduction</td>
<td>26</td>
</tr>
<tr>
<td>2.2 Non-Periodic Synthesized Microstrip Lines and Their Applications</td>
<td>27</td>
</tr>
<tr>
<td>2.2.1 Design Details and Propagation Characteristics</td>
<td>27</td>
</tr>
<tr>
<td>2.2.2 90° and 180° Hybrid Couplers</td>
<td>30</td>
</tr>
<tr>
<td>2.2.3 Application to Butler Matrix as Array Feeding Network</td>
<td>32</td>
</tr>
</tbody>
</table>
2.3 Non-Periodic Synthesized Coplanar Waveguides and Their Applications 34
2.3.1 Synthesis and Design 34
2.3.2 180° Hybrid Using Synthesized CPWs 37
2.3.3 Dual-Mode Ring Bandpass Filters 38
2.4 Non-Periodic Quasi-Lumped Synthesized Coupled Lines 42
2.4.1 Basics of Coupled Transmission Lines 42
2.4.2 Miniaturization of Coupled Lines and the Directional Couplers 44
2.4.3 Marchand Baluns Using Synthesized Coupled Lines 49
2.4.4 Lumped Directional Coupler and the Phase Shifter 53
2.5 Non-Periodic Synthesized Lines Using Vertical Inductors 55
References 60

3 Dual/Tri-Operational Mode Synthesized Transmission Lines: Design and Analysis 62
C. H. Lai and T. G. Ma

3.1 Introduction 62
3.2 Equivalent Circuit Models and Analysis 63
3.2.1 Ladder-Type Approximation in the Passband 63
3.2.2 Half-Circuit Model at Resonance 64
3.3 Dual-Operational Mode Synthesized Transmission Lines 65
3.3.1 Design Concept 65
3.3.2 Dual-Mode Synthesized Line Using a Series Resonator 66
3.3.3 Dual-Mode Synthesized Line Using Open-Circuited Stubs 70
3.3.4 Dual-Mode Synthesized Line Using Parallel Resonators 72
3.4 Tri-Operational Mode Synthesized Lines Using Series Resonators 74
3.4.1 Design Concept 74
3.4.2 Tri-Mode Synthesized Line as Category-1 Design 75
3.4.3 Tri-Mode Synthesized Line as Category-2 Design 79
3.4.4 Tri-Mode Synthesized Line as Category-3 Design 83
3.5 Multi-Operational Mode Synthesized Lines as Diplexer and Triplexer 87
3.5.1 Diplexer 87
3.5.2 Triplexer 89
References 94

4 Applications to Heterogeneous Integrated Phased Arrays 95
C. H. Lai and T. G. Ma

4.1 Introduction 95
4.2 Dual-Mode Retrodirective Array 96
4.2.1 Design Goal 96
4.2.2 System Architecture 97
4.2.3 Circuit Realization 98
4.2.4 Bistatic Radiation Patterns 102
4.2.5 Alternative Architecture 103
4.3 Dual-Mode Integrated Beam-Switching/Retrodirective Array 106
4.3.1 Design Goal 106
4.3.2 System Architecture 106
5 On-Chip Realization of Synthesized Transmission Lines Using IPD Processes
Y. C. Tseng and T. G. Ma
5.1 Introduction 126
5.2 Integrated Passive Device (IPD) Process 127
5.3 Tight Couplers Using Synthesized CPWs 128
  5.3.1 Quadrature Hybrid 128
  5.3.2 Wideband Rat-Race Coupler 129
  5.3.3 Dual-Band Rat-Race Coupler 132
  5.3.4 Coupled-Line Coupler 137
  5.3.5 Butler Matrix 139
5.4 Bandpass/Bandstop Filters Using Synthesized CPWs 142
  5.4.1 Bandpass Filter Using Synthesized Stepped-Impedance Resonators 143
  5.4.2 Transformer-Coupled Bandpass Filter 146
  5.4.3 Bridged T-Coils as Common-Mode Filter 147
5.5 Chip Designs Using Multi-Mode Synthesized CPWs 151
  5.5.1 Diplexer 151
  5.5.2 Dual-Mode Rat-Race Coupler 154
  5.5.3 Triplexer 157
  5.5.4 On-Chip Liquid Detector 161
References 166

6 Periodic Synthesized Transmission Lines with Two-Dimensional Routing
T. G. Ma
6.1 Introduction 168
6.2 Design of the Unit Cells 169
  6.2.1 Formulation 169
  6.2.2 Quarter-Wavelength Lines 172
6.3 Power Divider and Couplers 174
6.4 Broadside Directional Coupler 178
  6.4.1 Design Principle 178
  6.4.2 Circuit Realization 180
6.5 Common-Mode Rejection Filter 184
   6.5.1 Design Principle 184
   6.5.2 Circuit Realization 187
6.6 On-Chip Implementation 189
   6.6.1 Unit Cells and Quarter-Wavelength Lines 189
   6.6.2 Circuit Implementations and Compensation 192
References 194

Index 196
This book intends to provide comprehensive coverage of the recent progress in synthesized (or artificial) transmission lines for graduate students in electrical and telecommunication engineering. Synthesized transmission lines are microwave lumped or quasi-lumped networks that have similar electrical properties to a uniform transmission line, but are of a far more compact size. This unique feature makes this sort of microwave wave-guiding structure an ideal candidate for realizing miniaturized microwave passive components with comparable performances to their conventional counterparts. Add-on values such as harmonic suppression, non-integer ratio between passbands, multi-functional operation, and so on, are demonstrated through the years.

The first part of this book focuses on introducing basic synthesis techniques and analysis tools for developing synthesized transmission lines with or without periodicity. Classical approaches are introduced along with simple examples for easy understanding. The basic principles are followed by a variety of synthesized transmission lines in microstrip, coplanar waveguide, or stripline form, and their applications to miniaturized passive components including couplers, array feeding networks, filters, and phase shifters.

The second part of this book is devoted to providing a comprehensive introduction to a new sort of wave-guiding structure, termed multi-operational mode synthesized transmission lines. This is the result of 10 years of research work conducted by the authors at the National Taiwan University of Science and Technology, Taiwan. Multi-operational mode synthesized transmission lines, abbreviated to multi-mode synthesized lines, can provide distinct electrical properties at different frequencies or in different material media. Without using active switches, the synthesized line could be identical to a uniform transmission line in one band, but auto-configures as an open or short circuit in another band. A variety of applications not feasible with conventional microwave components, including multiplexers and multi-mode feeding networks for phased arrays, are introduced.

The third part of the book provides thorough coverage of recent on-chip development of synthesized transmission lines using an emerging fabrication technology, the integrated passive device (IPD) process. The IPD process is a competitive technology in the integration
of on-chip microwave passive and active components for system-in-package (SiP) applications. This book will be the first book dedicated to summarizing state-of-the-art on-chip components using synthesized transmission lines with IPD technology.

The final part of the book covers a new sort of one-dimensional periodic synthesized transmission line with two-dimensional routing capability. It is also an outcome of the research conducted by the author group. For the most part of this chapter, designs are disclosed to the public for the first time. The periodic synthesized transmission lines make the routing of a passive microwave component conformal to an arbitrary outline profile when integrated with other circuit modules in the same system.

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This book is dedicated to everyone who works hard over the years for our country. God bless Formosa.

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Taipei, Taiwan
1

Introduction to Synthesized Transmission Lines

C. W. Wang and T. G. Ma

1.1 Introduction

In modern communication systems, the rapid evolution of integrated circuit (IC) and packaging technologies have driven more and more function blocks to be integrated into a single chip/module. In the second decade of the twenty-first century, highly-integrated front-end modules such as microwave/millimeter-wave radar and image systems, [1–4], phased arrays [5, 6], and so on have hit the commercial markets. In general, the RF modules require a large number of transmission-line-based elements for vector signal processing in the analog domain. The transmission line elements, however, inevitably occupy a large circuit area. In the cost-driven market, area is the cost. It therefore leads to an enormous amount of research work focusing on developing various kinds of synthesized transmission lines for reducing the required circuit size. A synthesized transmission line is a lumped or quasi-lumped network that may function identically to a uniform transmission line within a given bandwidth.

Synthesized transmission lines can be developed with or without periodicity. In a broad sense, it could be either right-handed or left-handed depending on the forming blocks. To describe the general concept, in this chapter we will start from Maxwell’s equations and discuss the analog between plane wave propagation in a material media and the TEM mode in a parallel-plate waveguide. The parameters associated with the wave propagation and their corresponding circuit parameters in a transmission line are linked herein. Based on the fundamental principle, design formulae for periodic and non-periodic synthesized transmission lines are summarized. Classical design approaches are reviewed to demonstrate how synthesized transmission lines are realized practically. A brief review of left-handed synthesized lines, or metamaterial structures, is provided at the end of the chapter.
The formulae in this chapter form the basis of the non-periodic synthesized transmission lines in Chapter 2 for circuit miniaturization, and in Chapter 5 for chip implementation. The multi-operational mode synthesized transmission lines in Chapters 3 and 4, for phased array applications, are also derived using the same building blocks. The two-dimensional synthesized transmission lines in Chapter 6 also follow the periodic condition in Sec. 1.3.1.

1.2 Propagation Characteristics of a TEM Transmission Line

In this section, we start with the Maxwell’s equations to derive the governed equations in a wave-guiding structure under the assumption of a transverse electromagnetic (TEM) field distribution. The propagation characteristics are summarized and compared to a distributed transmission line having similar mathematical forms by using circuit parameters.

1.2.1 Wave Equations

As shown in Fig. 1.1, consider a parallel-plate waveguide operated in the TEM mode. The field distribution inside the wave-guiding structure is known to be identical to a uniform plane wave in free space with uniquely defined voltage and current in the transverse plane. Maxwell’s curl equations in a source-free region are:

\[
\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}, \quad (1.1)
\]

\[
\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}. \quad (1.2)
\]

Assuming that the wave propagates along the \(z\)-direction, the fields transverse to the direction of propagation in a parallel-plate waveguide, from (1.1) and (1.2), are:

\[
-\frac{\partial E_z}{\partial z} = \mu \frac{\partial H_y}{\partial t}, \quad (1.3)
\]

\[
-\frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_z}{\partial t}. \quad (1.4)
\]
Partially differentiate (1.3) with respect to $z$ and (1.4) with respect to $t$ to get,

$$\frac{-\partial^2 E_x}{\partial z^2} = \mu \frac{\partial^2 H_y}{\partial z \partial t},$$

and

$$\frac{-\partial^2 H_y}{\partial t \partial z} = \varepsilon \frac{\partial^2 E_x}{\partial t^2}.$$

Substitution of (1.6) into (1.5) yields

$$\frac{\partial^2 E_x}{\partial z^2} = \mu \varepsilon \frac{\partial^2 E_x}{\partial t^2}.$$

It is a second-order partial differential equation known as the one-dimensional wave equation, which can be applied to any wave-guiding structure supporting TEM wave propagation. The phase velocity of the TEM wave is simply

$$v_p = \frac{1}{\sqrt{\mu \varepsilon}}.$$

where $\mu$ and $\varepsilon$ are the permeability and permittivity of the medium filled within the wave-guiding structure.

Now, let us turn our attention to a lossless distributed uniform transmission line modeled by periodically loaded $LC$ sections, as shown in Fig. 1.2. Under the assumption that each lumped $LC$ segment is infinitesimal in length, the voltage and current along the line, from Kirchhoff’s laws, are related to each other by,

$$\frac{-\partial V}{\partial z} = L \frac{\partial I}{\partial t},$$

and

$$\frac{-\partial I}{\partial z} = C \frac{\partial V}{\partial t}.$$

$L$ and $C$ are the per-unit-length inductance and capacitance of the line. Equations (1.9) and (1.10) are known as the Telegrapher’s equations and actually take the same form as (1.3) and (1.4).

Figure 1.2  Equivalent lumped $LC$ model of a distributed uniform transmission line
Following the same mathematical procedure, the wave equation is derived in terms of the voltage \((V)\) or current \((I)\) as,

\[
\frac{\partial^2 V}{\partial z^2} = LC \frac{\partial^2 V}{\partial t^2}. \tag{1.11}
\]

The general solution of the voltage and current waves propagated along the lossless uniform transmission line, from (1.11), is

\[
V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z}, \tag{1.12}
\]

\[
I(z) = I^+ e^{-j\beta z} - I^- e^{+j\beta z}. \tag{1.13}
\]

\(\beta\) is known as the \textit{phase constant} or \textit{guided wavenumber},

\[
\beta = \frac{\omega}{v_p} = \omega \sqrt{LC}. \tag{1.14}
\]

The phase velocity of the voltage and current waves is therefore,

\[
v_p = \frac{1}{\sqrt{LC}}. \tag{1.15}
\]

Meanwhile, differentiating (1.12) with respect to \(z\), we have

\[
\frac{\partial V(z)}{\partial z} = -j\beta V^+ e^{-j\beta z} + j\beta V^- e^{+j\beta z} = -j\omega LI(z). \tag{1.16}
\]

Substitution of (1.13) into (1.16) yields

\[
-j\beta V^+ e^{-j\beta z} + j\beta V^- e^{+j\beta z} = -j\omega LI^+ e^{-j\beta z} + j\omega LI^- e^{+j\beta z}. \tag{1.17}
\]

From (1.17), the \textit{characteristic impedance} of a lossless transmission line is then defined as,

\[
Z_c = \frac{V^+}{I^+} = \frac{V^-}{I^-} = \frac{\omega L}{\beta} = \sqrt{\frac{L}{C}} = v_p L = \frac{1}{v_p C}. \tag{1.18}
\]

It is interesting to note that (1.3)–(1.8), (1.9)–(1.13), and (1.15) are actually in the same form, suggesting that under the TEM-mode operation, the electromagnetic (EM) parameters of a wave-guiding structure can be mapped one-to-one onto the circuit parameters of its transmission-line equivalence. The wave impedance of the parallel-plate waveguide in Fig. 1.1 is in the same form as the characteristic impedance in (1.18), as well.
Table 1.1 summarizes the analog between the EM parameters of a TEM wave-guiding structure and the circuit parameters of a lossless transmission line. The mapping holds exactly for TEM transmission lines and approximately for quasi-TEM ones. To simplify the design procedure, hereafter we will use the scalar circuit parameters \((V, I, L, C)\) to analyze the propagation characteristics of any kind of TEM/quasi-TEM transmission lines.

1.2.2 Keys to Miniaturization

In Sec. 1.2.1, the wave equation and general solution of a TEM transmission line are derived in terms of the field parameters \((E, H, \mu, \varepsilon)\) as well as circuit parameters \((V, I, L, C)\) at the same time. In this section, we further introduce the slow wave factor as a figure of merit for judging the circuit miniaturization capability of a given wave-guiding structure.

First of all, recall the guided wavenumber can be expressed in terms of both EM and circuit parameters as

\[
\beta_g = \frac{\omega}{v_p} = \omega \sqrt{LC} = \omega \sqrt{\mu \varepsilon} = \omega \sqrt{\mu, \varepsilon, \varepsilon_o}.
\]

The free space wavenumber, or the phase constant of a wave propagated in free space, is

\[
\beta_o = \omega \sqrt{\mu_o \varepsilon_o}.
\]

The slow wave factor is defined as the ratio of the guided wavenumber to free space wavenumber as

\[
SWF = \frac{\beta_g}{\beta_o} = \frac{\lambda_g}{\lambda_o} = \sqrt{\mu \varepsilon} = \frac{c}{\sqrt{LC}}.
\]

\(c\) is the speed of light in vacuum. The slow wave factor is a measure of how good a wave-guiding structure can be used for circuit miniaturization.

Meanwhile, a section of transmission line is commonly expressed in terms of its electrical length at the operating frequency as

\[
\theta = \beta_g l.
\]
$l$ is the physical length of the line section. From (1.22), for a given electrical length, increasing the guided wavenumber ($\beta_g$) becomes the key factor to reduce the required physical length of a transmission line. Choosing a material media with a higher $\varepsilon_r$ or $\mu_r$ is a possible way to reduce the physical length with a larger $\beta_g$. However, it is likely at the expense of higher fabrication cost. Alternatively, using synthesized transmission lines in accordance with (1.19) and (1.22) paves another way for circuit miniaturization by simultaneously increasing the per-unit-length inductance and capacitance of that line. Here, the *synthesized transmission line* is referred to as any microwave lumped/quasi-lumped network that can be electrically equivalent to a section of uniform transmission line over a frequency band of interest.

A further thought on developing a synthesized transmission line is: in a practical circuit, how can we fulfill the goal by simultaneously increasing the inductance and capacitance of a line? The answer is quite straightforward: “*just follow the fundamental physical rules.*” An extra current path always generates additional magnetic fields and, hence, the inductance. The charge accumulation between electrodes, in the meantime, results in extra capacitive loadings. Accordingly, using a meander or spiral high-impedance line is an effective way to provide more current paths or higher current density to increase the per-unit-length inductance in a real design. Meanwhile, adding parallel-plate or interdigital capacitor is a good way to boost the capacitance of the host transmission line. When one attempts to adjust the per-unit-length inductance and capacitance for raising the slow wave factor, it is interesting to note that the characteristic impedance of the line can be controlled at the same time using (1.18) within a reasonable range, say, 20–120 $\Omega$.

Quite a few synthesized transmission lines (or the so-call artificial transmission lines) are summarized and listed in [7]. Some of them are redrawn in Fig. 1.3 for easy reference [8–16]. They are all designed based on alternatively connected series inductance and shunt capacitance with or without periodicity. This sort of synthesized lines is right-handed in nature with lowpass responses. Readers are encouraged to find clues on how the line inductance and capacitance in the examples are boosted. Following the same rule, the readers can develop new and creative structures on their own. In fact, the number of layout patterns of a synthesized transmission line with given electrical properties can be extended to infinity!

Finally, although the aforementioned discussion is restricted to lossless transmission lines, similar statements hold true for a low-loss one. The only difference is the introduction of the

![Figure 1.3](typical_slow_wave_synthesized_transmission_lines_in_open_literature.png)
Introduction

attenuation constant \((\alpha)\), which results in a complex propagation constant \((\gamma = \alpha + j\beta)\) with a small real part representing the loss. The quality factor \((Q)\) is evaluated by

\[
Q = \frac{\beta}{2\alpha}.
\]  

In the following section, the common ways to analyze a periodic or non-periodic synthesized transmission line are introduced as the basis of the entire book.

1.3 Analysis of Synthesized Transmission Lines

Synthesized transmission lines can be realized with or without periodicity. While the non-periodic lines can be dealt with using simple transmission line equivalence, the periodic lines are in general analyzed by the Bloch theorem [17]. In this section, the general analysis procedures for periodic and non-periodic synthesized transmission lines are introduced in sequence as follows.

1.3.1 Bloch Theorem and Characterization of a Periodic Synthesized Transmission Line

Figure 1.4 shows a typical periodically-loaded synthesized transmission line. It comprises a uniform transmission line of \((Z_c)\) (the host line) periodically loaded by lumped or quasi-lumped shunt elements \((j\beta)\). The line is composed of infinite elements in cascade, with each element termed as a unit cell. The voltage and current waves on the \(n\)th and \((n + 1)\)th nodes are related by the \(ABCD\) matrix of the unit cell as

\[
\begin{bmatrix}
V_n \\
I_n
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
V_{n+1} \\
I_{n+1}
\end{bmatrix}.
\]  

Let the unit cell have a periodicity of \(p\) and the complex propagation constant equal to \(\gamma\), we have

\[
\begin{align*}
V_{n+1} &= V_n \cdot e^{\gamma p} \\
I_{n+1} &= I_n \cdot e^{\gamma p}.
\end{align*}
\]  

Figure 1.4 Typical periodically-loaded synthesized transmission line
Substitution of (1.25) into (1.24) yields

\[
\begin{bmatrix}
V_n \\
I_n
\end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_{n+1} \\
I_{n+1}
\end{bmatrix} = \begin{bmatrix} V_{n+1} e^{\gamma p} \\
I_{n+1} e^{\gamma p}
\end{bmatrix}.
\] (1.26)

or

\[
\begin{bmatrix} A - e^{\gamma p} & B \\ C & D - e^{\gamma p} \end{bmatrix} \begin{bmatrix} V_{n+1} \\
I_{n+1}
\end{bmatrix} = 0.
\] (1.27)

For a nontrivial solution of \((V_{n+1}, I_{n+1})\), the determinant of the matrix must be zero, which leads to

\[AD + e^{2\gamma p} - (A + D)e^{\gamma p} - BC = 0.\] (1.28)

Since the unit cell is a reciprocal network, we have \(AD - BC = 1\). Thus,

\[e^{2\gamma p} - (A + D)e^{\gamma p} + 1 = 0.\] (1.29)

From (1.29), the propagation constant can be expressed in terms of the \(ABCD\) matrix as

\[\cosh(\gamma p) = \frac{A + D}{2}.\] (1.30)

In lossless case with \(\alpha = 0\), (1.30) is reduced to

\[\cos(\beta p) = \frac{A + D}{2}.\] (1.31)

In a periodic synthesized transmission line, the phase constant of the line can be solved from the \(ABCD\) matrix of the unit cell with a given periodicity \(p\) (i.e., physical length) in accordance with (1.31). In the meantime, the characteristic impedance of the unit cell can be defined as

\[Z_B = \frac{V_{n+1}}{I_{n+1}}.\] (1.32)

This impedance is also called the Bloch impedance. From (1.27), we have

\[(A - e^{\gamma p})V_{n+1} + BI_{n+1} = 0.\] (1.33)

Substitution of (1.33) into (1.32), the Bloch impedance is

\[Z_B = \frac{-B}{A - e^{\gamma p}}.\] (1.34)