Solid Mechanics and Its Applications

Uwe Mühlich

Fundamentals of Tensor Calculus for Engineers with a Primer on Smooth Manifolds

Springer

Solid Mechanics and Its Applications

Volume 230

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Preface

Continuum mechanics and its respective subtopics such as strength of materials, theory of elasticity, and plasticity are of utmost importance for mechanical and civil engineers. Tensors of different types, such as vectors and forms, appear most naturally in this context. Ultimately, when it comes to large inelastic deformations, operations like push-forward, pull-back, covariant derivative, and Lie derivative become inevitable. The latter form a part of modern differential geometry, also known as tensor calculus on differentiable manifolds.

Unfortunately, in many academic institutions, an engineering education still relies on conventional vector calculus and concepts like dual vector space, and exterior algebra are successfully ignored. The expression "manifold" arises more or less as a fancy but rather diffuse technical term. Analysis on manifolds is only mastered and applied by a very limited number of engineers. However, the manifold concept has been established now for decades, not only in physics but, at least in parts, also in certain disciplines of structural mechanics like theory of shells. Over the years, this has caused a large gap between the knowledge provided to engineering students and the knowledge required to master the challenges, continuum mechanics faces today.

The objective of this book is to decrease this gap. But, as the title already indicates, it does not aim to give a comprehensive introduction to smooth manifolds. On the contrary, at most it opens the door by presenting fundamental concepts of analysis in Euclidean space in a way which makes the transition to smooth manifolds as natural as possible.

The book is based on the lecture notes of an elective course on tensor calculus taught at TU-Bergakademie Freiberg. The audience consisted of master students in Mechanical Engineering and Computational Materials Science, as well as doctoral students of the Faculty of Mechanical, Process and Energy Engineering. This introductory text has a special focus on those aspects of which a thorough understanding is crucial for applying tensor calculus safely in Euclidean space, particularly for understanding the very essence of the manifold concept. Mathematical proofs are omitted not only because they are beyond the scope of the book but also because the author is an engineer and not a mathematician. Only in some particular cases are proofs sketched in order to raise awareness of the effort made by mathematicians to work out the tools we are using today. In most cases, however, the interested reader is referred to corresponding literature. Furthermore, invariants, isotropic tensor functions, etc., are not discussed, since these subjects can be found in many standard textbooks on continuum mechanics or tensor calculus.

Prior knowledge in real analysis, i.e., analysis in $\mathbb{R}$, is assumed. Furthermore, students should have a prior education in undergraduate engineering mechanics, including statics and strength of materials. The latter is surely helpful for understanding the differences between the traditional and modern version of tensor calculus.

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I would like to thank TU-Bergakademie Freiberg and especially Meinhard Kuna for the opportunity to teach a course about tensor calculus and for providing the freedom to teach a continuum mechanics lecture during the summer term of 2014 which deviated significantly from the approach commonly applied in engineering education. In addition, I would like to express my appreciation to all students and colleagues who attended the courses mentioned above for their interest and academic curiosity, but especially for critical remarks which helped to uncover mistakes and to clarify doubts.

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Contents

Selected Symbols

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integer numbers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\cap$	Intersection of sets: $A \cap B = \{x \vert x \in A \text{ and } x \in B\}$
$\cup$	Union of sets: $A \cup B = \{x \vert x \in A \text{ or } x \in B\}$
$\subset$	Subset $A \subset B$
A^{c}	Complement of a set A
$A \rightarrow B$	Mapping from set A to set B
$\mathbf{u}, \underset{\sim}{\boldsymbol{\omega}}$	Vector, dual vector
$\oplus, \boxplus, +$	Symbols used to indicate addition of elements of a vector space
$\odot, \boxdot$	Symbols used to indicate multiplication of elements of a vector space by a real number
✛	Hybrid addition between points and vectors
$\wedge$	Exterior product
$*\mathbf{u}$	Star operator applied to a vector
$g_{\star}, g^{\star}$	Push forward and pull back operation under a mapping g
$\top$	General tensor
δ_i^k	Kronecker symbol, see (3.9)
$\underset{\sim}{\boldsymbol{d}}\vert_p f(\mathbf{u})$	Directional derivative of the function f at point p in direction $\mathbf{u}$
$\boldsymbol{\partial}_i\vert_p$	Tangent vector at a point p on a smooth manifold
$\underset{\sim}{\boldsymbol{\partial}}^i\vert_p$	Cotangent vector at a point p on a smooth manifold

$\left.\frac{\partial}{\partial \boldsymbol{x}^i}\right\vert_{\mu(p)}$	Tangent vector at the image of point p in a chart generated by a mapping μ with coordinates x^i
$\underset{\sim}{\boldsymbol{d}}\big\vert_{\mu(p)} x^i$	Cotangent vector at the image of point p in a chart generated by a mapping μ with coordinates x^i

Chapter 1
Introduction

Abstract The introduction aims to remind the reader that engineering mechanics is derived from classical mechanics, which is a discipline of general physics. Therefore, engineering mechanics also relies on a proper model for space, and the relations between space and geometry are discussed briefly. The idea of expressing geometrical concepts by means of linear algebra is sketched together with the concept of vectors as geometrical objects. Although this book provides only the very first steps of the manifold concept, this chapter intends to make its importance for modern continuum mechanics clear by raising a number of questions which cannot be answered by the conventional approach. Furthermore, aspects regarding mathematical notation used in subsequent chapters are discussed briefly.

1.1 Space, Geometry, and Linear Algebra

One of the most important tools for mechanical and civil engineers is certainly mechanics, and in particular continuum mechanics, together with its subtopics, linear and nonlinear elasticity, plasticity, etc. Mechanics is about the motion of bodies in space. Therefore, a theory of mechanics first requires concepts of space and time, particularly space–time, and, in addition, a concept to make the notion of a material body more precise. Traditionally, the scientific discipline concerned with the formulation of models for space is geometry. Starting from a continuum perspective of space–time, the simplest model one can think of is obtained by assuming euclidean geometry for space and by treating time separately as a simple scalar parameter.

In two-dimensional euclidean geometry, points and straight lines are primitive objects and the theory of euclidean space is based on a series of axioms. The most crucial one is the fifth postulate, which distinguishes euclidean geometry from other possible geometries. It introduces a particular notion of parallelism, and one way to express this in two dimensions is as follows. Given a straight line L, there is only one straight line through a point P not on L which never meets L.

It took mankind centuries to understand that euclidean geometry is only one of many possible geometries, see e.g., Holme [2] and BBC [1]. But, once this fundamental fact had been discovered, it became apparent rather quickly that the question

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as to which space or space–time model is appropriate for developing a particular theory of physics can only be answered through experiments and not just through logical reasoning.

Historically, mechanics has been developed based on the assumption that euclidean space is an appropriate model for our physical space, i.e., the space we live in performing observations and measurements. The idea of force as an abstract concept of what causes a change in the motion of a body was a cornerstone of this development. However, at the beginning of the twentieth century, it became apparent to theoretical physicists that physical space is actually curved, and gravitation is seen nowadays as the cause for space curvature. Models for space, other than euclidean geometry, had to be employed in order to develop the corresponding theories.

In the course of this development, it became clear as well that new mathematical tools are required in order to encode new physical ideas properly. Tensor analysis using Ricci calculus was elaborated, Grassmann algebra was rediscovered, the differentiable manifold concept gained more and more use in theoretical physics, etc. Nowadays, these tools are matured and rather well established. However, the process of using them for reformulating existing theories in order to gain deeper understanding, is still going on.

Differentiable manifolds are by definition objects which are at least locally euclidean. Therefore, affine and euclidean geometry are crucial for the treatment of more general differentiable manifolds. Although linear algebra is not the same as euclidean geometry, the former provides concepts which can be used to formulate euclidean geometry in an elegant way by employing the vector space concept. A straight line in the language of linear algebra is a point combined with a the scalar multiple of a vector. Euclidean parallelism is expressed by the notion of linear dependence of vectors, etc.

1.2 Vectors as Geometrical Objects

Most of us were confronted with vectors for the first time in the following way. Starting with the definition that a vector **a** is an object which possesses direction and magnitude, this object is visualized by taking, e.g., a sheet of paper, drawing a straight line of some length, related to the magnitude, and indicating the direction by an arrow. Afterwards, it is agreed upon that two vectors are equal if they have same length and direction. A further step consists in the definition of two operations, namely the addition of vectors and multiplication of a vector with a real number. At this stage, these operations are defined only graphically, which can be expressed by the following definitions.

Definition 1.1 (*Addition of two vectors (graphically)*) In order to determine the sum **c** of two vectors **a** and **b** graphically, written as $\mathbf{c} = \mathbf{a} \oplus \mathbf{b}$

1. displace **a** and **b** in parallel such that the head of **a** meets the tail of **b** and
2. **c** is the directed line from the tail of **a** to the head of **b**,