may your forces be conservative,
your constraints holonomic, your coordinates ignorable,
and your principal function separable
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Preface

In his great work, *Mecanique Analytique* (1788) Lagrange used the term “analytical” to mean “non-geometrical.” Indeed, Lagrange made the following boast:

“No diagrams will be found in this work. The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field.”

This was in marked contrast to Newton’s *Philosophiae Naturalis Principia Mathematica* (1687) which is full of elaborate geometrical constructions. It has been remarked that the classical Greeks would have understood some of the *Principia* but none of the *Mecanique Analytique*.

The term analytical dynamics has now come to mean the developments in dynamics from just after Newton to just before the advent of relativity theory and quantum mechanics, and it is this meaning of the term that is meant here. Frequent use will be made of diagrams to illustrate the theory and its applications, although it will be noted that as the book progresses and the material gets “more analytical”, the number of figures per chapter tends to decrease, although not monotonically.

Dynamics is the oldest of the mathematical theories of physics. Its basic principles are few in number and relatively easily understood, and its consequences are very rich. It was of great interest to the Greeks in the classical period. Although the Greeks had some of the concepts of statics correct, their knowledge of dynamics was seriously flawed. It was not until Kepler, Galileo, Descartes, Huyghens, and others in the seventeenth century that the principles of dynamics were first understood. Then Newton united both terrestrial and celestial dynamics into one comprehensive theory in the *Principia*. To this day, the “proof” of a new result in classical dynamics consists in showing that it is consistent with Newton’s three “Laws of Motion”. Being the oldest and best established, dynamics has become the prototype of the branches of mathematical physics.

Among the branches of physics that have adapted the techniques of dynamics are the various theories of deformable continuous media, thermodynamics, electricity and magnetism, relativity theory, and quantum
mechanics. The concept of a dynamic system has been abstracted and applied to fields as diverse as economics and biology. Branches of mathematics that have benefited from concepts that first arose in dynamics are algebra, differential geometry, non-Euclidean geometry, functional analysis, theory of groups and fields, and, especially, differential equations.

A few decades ago, the dynamics of interest in engineering applications was extremely simple, being mostly concerned with two-dimensional motion of rigid bodies in simple machines. This situation has now changed. The dynamics problems that arise in the fields of robotics, biomechanics, and space flight, to name just a few, are usually quite complicated, involving typically three-dimensional motion of collections of inter-connected bodies subject to constraints of various kinds. These problems require careful, sophisticated analysis, and this has sparked a renewed interest in the methods of analytical dynamics.

Essentially, the development of analytical dynamics begins with d'Alembert's *Traite de dynamique* of 1743 and ends with Appell's *Traite de Mecanique Rationelle* of 1896. Thus the time span is the 150 years covering the last half of the eighteenth and the entire nineteenth century. Many authors contributed to the theory, but two works stand out: First, the *Mecanique Analytique* of Lagrange, and second, the *Second Essay on a General Method in Dynamics* (1835) by Hamilton.

Over the years, the body of knowledge called analytical dynamics has coalesced into two parts, the first called Lagrangian and the second Hamiltonian dynamics. Although the division is somewhat artificial, it is a useful one. Both subjects are covered in this book.

There are two principal sources for this book. These are the books by Rosenberg (*Analytical Dynamics of Discrete Systems*, 1977) and by Pars (*A Treatise on Analytical Dynamics*, 1965). The scope of the first of these is more limited than the present book, being confined to Lagrangian Dynamics, and the second is much broader, covering much material of limited interest to engineering analysts. For the Lagrangian portion of the book, the excellent treatment of Rosenberg is the primary source. It should be remarked that one of the primary sources for Rosenberg is the book by Pars. Pars' book is a monumental work, covering the subject in a single comprehensive and rigorous volume. Disadvantages of these books are that Rosenberg is out of print and Pars has no student exercises. No specific references will be made to material adapted from these two books because to do so would make the manuscript cumbersome; the only exceptions are references to related material not covered in this book.
As would be expected of a subject as old and well-established as classical dynamics, there are hundreds of available books. I have listed in the Bibliography only those books familiar to me. Also listed in the Bibliography are the principal original sources and two books relating to the history of the subject.

There are two types of examples used in this book. The first type is intended to illustrate key results of the theoretical development, and these are deliberately kept as simple as possible. Thus frequent use is made of, for example, the simple pendulum, the one-dimensional harmonic oscillator (linear spring-mass system), and central gravitational attraction. The other type of example is included to show the application of the theoretical results to complex, real-life problems. These examples are often quite lengthy, comprising an entire chapter in some cases. Features involved include three-dimensional motion, rigid bodies, multi-body systems, and nonholonomic constraints.

Throughout the book there are historical footnotes and longer historical remarks describing the origins of the key concepts and the people who first discovered them. For readers less interested in the history of dynamics than I, this historical information may be skipped with no loss in continuity.

Because most dynamics problems may be solved by Newton’s laws, alternative methods must have relative advantages to warrant interest. In the case of Lagrangian dynamics this justification is easy, since, as will be shown in this book, many specific dynamics problems are easier to solve by the Lagrangian than by the Newtonian method. Because the Hamiltonian formalism generally requires the same effort to solve dynamics problems as the Lagrangian, the study of Hamiltonian dynamics is more difficult to justify.

One of the aims of Hamiltonian dynamics is to obtain not just the equations of motion of a dynamic system, but their solution; however, it must be confessed that the usefulness of the techniques developed is often limited. Another advantage lies in the mathematical elegance of the presentation, although this may be of limited importance to engineering analysts. The Hamiltonian approach to dynamics has had, and continues to have, a far-reaching impact on many fields of mathematical physics, and this is an important reason for its study. Perhaps Gauss said it best: “It is always interesting and instructive to regard the laws of nature from a new and advantageous point of view, so as to solve this or that problem more simply, or to obtain a more precise presentation”.

One approach to dynamics is to develop it by the axiomatic method
familiar from Euclidean geometry, and this has been done by many authors. It seems to me, however, that this method is inappropriate for a subject that is experimentally based. Today we use the principles of classical dynamics because they give a sufficiently accurate model of physical phenomena. This is in marked contrast to the metaphysical view in Newton's time, which held that Newton's laws described how nature actually behaves.

A summary of the organization of the book is as follows. Chapter 1 is a review of Newtonian dynamics. This is not meant to be comprehensive but rather covers only concepts that are needed later. Chapters 2 - 4 cover the foundations of analytical dynamics that will be used throughout the rest of the book — constraints, virtual displacements, virtual work, and variational principles. (It may be somewhat frustrating to some students to spend so much time on preliminary material, but this effort will pay off in the long run.) Lagrangian dynamics is contained in Chapters 5 - 11. This includes the derivation of Lagrange's equations as well as numerous applications. The next three Chapters — on stability, impulsive motion, and the Gibbs-Appell equations — are outside the main development, and, although important topics, are not necessary to subsequent developments. The remaining Chapters, 15 - 18, concern the development of Hamiltonian dynamics and its applications.

Some comments on function notation used in the book are required. The symbol $F(x)$ will be used to mean both "$F$ is a function of $x$" and "the value of the function $F$ for a specific value of $x$"; when there is a chance of confusion, the distinction will be made. I will write $F = F(x_s)$ to mean that $F$ is a function of the variables $x_1, \ldots, x_n$, at most, and no others. Similarly, $F \neq F(x_s)$ will mean that $F$ is not (allowed to be) a function of the variables $x_1, \ldots, x_n$. The same symbol will denote sometimes two different functions. For example, if $F(x_s)$ and $x_s = f(q_r)$ then $F(q_r)$ will mean $F(f_1(q_1, \ldots, q_n), \ldots, f_n(q_1, \ldots, q_n)$, when it is clear what is meant. The symbolism $F(x_s) \in C^n$ will mean that $F(\cdot)$ is of class $n$, that is that it is continuous with continuous derivatives up to order $n$ in all of its arguments $x_1, \ldots, x_n$.

This book is intended both as an advanced undergraduate or graduate text, and as a reference for engineering analysts. In my own graduate course, the material is covered in forty fifty-minute lectures. The background expected is an undergraduate understanding of Newtonian dynamics and of mathematics, especially differential equations.

Finally, it is with great pleasure that I acknowledge the faculty at the University of California at Berkeley who first imparted to me the knowl-
edge and appreciation of dynamics – Professors Rosenberg, Leitmann, and Goldsmith. I am especially indebted to the late Professor Rosenberg who granted me permission to use freely material from his book.

Notes

1 Specific works referenced here, and in the rest of the book, are listed in the Bibliography.

2 Lagrange himself was deeply interested in the history of dynamics, devoting much space in *Mechanique Analytique* to the subject.

3 This was first clearly recognized by Carnot.
Chapter 1

Review of Newtonian Dynamics

1.1 Basic Concepts

Assumptions. Classical mechanics rests on three basic assumptions:

1. The physical world is a three dimensional Euclidean space. This implies that the Pythagorean theorem, vector addition by parallelograms, and all elementary geometry and trigonometry are valid.

2. There exist inertial (Galilean) reference frames in this space. An inertial frame is one in which Newton's three laws hold to a sufficient degree of accuracy. We generally will take reference frames fixed relative to the surface of the earth to be inertial.

3. The quantities mass and time are invariant, that is, they are measured as the same by all observers.

4. Physical objects are particles or collections of particles constituting rigid bodies.

Assumptions (1) – (3) were regarded as laws of physics at one time; now they are regarded as engineering approximations. Assumption (4) is clearly an approximation; all known materials deform under forces, but this deformation is frequently negligible.

Newton's Laws. Let $\sum F$ be the resultant (vector sum) of all the forces acting on a mass particle of mass $m$. Then Newton's Second Law
Analytical Dynamics

states that:\(^2\)

\[
\sum F = ma
\]  
(1.1)

where \( a = d^2 \vec{r}/dt^2 = \ddot{\vec{r}} \) is the acceleration of the mass particle, and where \( \vec{r} \) is the position vector of the mass particle in an inertial frame of reference (Fig. 1-1) and \( d(\ )/dt = (\cdot) \) is the time derivative in that frame. That is, force is proportional to acceleration with proportionality constant \( m \).

Newton’s Third Law states that given any two particles \( p_1 \) and \( p_2 \) with masses \( m_1 \) and \( m_2 \), the force exerted by \( p_1 \) on \( p_2 \), say \( F_{21} \), is equal and opposite to that exerted by \( p_2 \) on \( p_1 \), \( F_{12} \), and these forces act on a line adjoining the two particles (Fig. 1-2):

\[
F_{12} = |F_{12}| \hat{e} = -F_{21}
\]  
(1.2)

where \( \hat{e} \) is a unit vector in direction \( F_{12} \).

Newton’s First Law states that if \( \sum F = 0 \) then \( v(t) = \text{constant} \). This “Law” is therefore a consequence of the Second Law.

Newton stated these laws for a single particle, as we have just done; L. Euler and others generalized them to a rigid body, that is a collection of particles whose relative positions are fixed.

**Definitions.** The subject of mechanics is conveniently divided into branches as follows:
1. Statics is that branch of mechanics concerned with the special case $v(t) = 0$. This implies that $a(t) = 0$ and consequently $\sum F = 0$.

2. Dynamics is that branch of mechanics concerned with $v(t) \neq \text{constant}$.

3. Kinematics is that branch of dynamics concerned with motion independent of the forces that produce the motion.

4. Kinetics is that branch of dynamics concerned with the connection between forces and motion, as defined by Newton's three laws.

**Basic Problems in Kinetics.** It is clear that there are two basic problems:

1. Given the forces, find the motion (that is, the position, velocity, and acceleration as a function of time). This is sometimes called the "forward" or "dynamics" problem.

2. Given the motion, find the forces that produced it (actually, one can usually only find the resultant force). Statics is a special case of this. This is sometimes called the "backward" or "controls" problem.

In practice, mixed problems frequently arise; for example, given the motion and some of the forces, what is the resultant of the remaining force(s)?

**Reasons for Reviewing Newtonian Dynamics.** In the rest of this chapter, we will review elementary Newtonian dynamics, for the following reasons:

1. Some of this material will be needed later.

2. It allows a chance in a familiar setting to get used to the approach and notation used throughout this book.

3. It gives insight that leads to other approaches to dynamics.

4. It provides a benchmark with which to measure the worth of these new approaches.

Many of the following results will be presented without proof.
1.2 Kinematics and Newtonian Particle Dynamics

**Motion of a Point.** Consider a point $P$ moving along a curve $C$ relative to a reference frame $\{i, j, k\}$. Denote the position vector of $P$ at time $t$ by $r(t)$. Then the velocity of $P$ is defined as (Fig. 1-3):

$$v(t) = \frac{dr}{dt} = \dot{r} = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} \quad (1.3)$$

Similarly the acceleration of the point is defined as:

$$a(t) = \frac{d\dot{r}}{dt} = \ddot{r} = \lim_{\Delta t \to 0} \frac{\dot{v}(t + \Delta t) - \dot{v}(t)}{\Delta t} \quad (1.4)$$

Note that $v(t)$ is tangent to the curve $C$. The magnitude of the velocity vector, $v(t) = |\dot{r}(t)|$, is called the speed of the point.

**Rectangular Components.** To obtain scalar equations of motion, the vectors of interest are written in components. In rectangular components (Fig. 1-4), the position vector is given by:

$$r = xi + yj + zk \quad (1.5)$$

From Eqns. (1.3) and (1.4) $v$ and $a$ are:

$$v = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \quad (1.6)$$

$$a = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} \quad (1.7)$$
We call \( (x, y, z) \) the rectangular components of position (or the rectangular coordinates) of point \( P \). Similarly, \( (\dot{x}, \dot{y}, \dot{z}) \) and \( (\ddot{x}, \ddot{y}, \ddot{z}) \) are the rectangular components of velocity and acceleration, respectively. The distance of \( P \) from the origin and the speed of \( P \) are given by:

\[
\begin{align*}
    r &= |r| = \sqrt{x^2 + y^2 + z^2}, \\
    v &= |v| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}
\end{align*}
\] (1.8)

Expressing the resultant force on the mass \( m \) at point \( P \) in rectangular components

\[
\sum F = \sum F_x \hat{i} + \sum F_y \hat{j} + \sum F_z \hat{k}
\] (1.9)

and combining this with Eqns. (1.1) and (1.7) gives three scalar equations of motion:

\[
\begin{align*}
    \sum F_x &= m\ddot{x}, \\
    \sum F_y &= m\ddot{y}, \\
    \sum F_z &= m\ddot{z}
\end{align*}
\] (1.10)

This is a sixth order system of ordinary differential equations.

Any vector in three dimensions can be written as a linear combination of any three linearly independent vectors, called basis vectors. In this book, all basis vectors will be triads of mutually-orthogonal, right-handed unit vectors and will be denoted by "hats".

**Normal – Tangential Components.** It is possible, and frequently desirable, to express \( r, v, \) and \( a \) in components along directions other than \( \{\hat{i}, \hat{j}, \hat{k}\} \). For planar motion (take this to be in the \( (x, y) \) plane), normal-tangential components are frequently useful. Introduce unit vectors tangent and normal to \( v \) as shown in Fig. 1-5.
The velocity and acceleration vectors expressed in these components are:

\[ \mathbf{v} = \mathbf{v} \hat{e}_t \]
\[ \mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{v} \hat{e}_t + \mathbf{v} \frac{d\hat{e}_t}{dt} \]

It is clear that in general \( \dot{e}_t \) will vary with time and thus \( d\hat{e}_t/dt \neq 0 \). We must consider the two cases shown on Fig. 1-5 separately. First, for \( \dot{\theta} > 0 \) (Fig. 1-6):

\[ \hat{e}_t = \cos \dot{\theta} \hat{i} + \sin \dot{\theta} \hat{j} \]
\[ \hat{e}_n = -\sin \dot{\theta} \hat{i} + \cos \dot{\theta} \hat{j} \]
\[ \frac{d\hat{e}_t}{dt} = -\dot{\theta} \sin \hat{i} + \dot{\theta} \cos \hat{j} = \dot{\theta} \hat{e}_n \]
Next for $\dot{\theta} < 0$ (Fig. 1-7):

\[ \ddot{e}_t = \cos \theta \dot{i} + \sin \theta \dot{j} \]
\[ \ddot{e}_n = \sin \theta \dot{i} - \cos \theta \dot{j} \]
\[ \frac{d\ddot{e}_t}{dt} = -\dot{\theta} \sin \theta \dot{i} + \dot{\theta} \cos \theta \dot{j} = -\dot{\theta} \dot{e}_n \]

Thus for both cases:

\[ \frac{d\ddot{e}_t}{dt} = |\dot{\theta}| \dot{e}_n \tag{1.13} \]

so that, from Eqn. (1.12),

\[ \ddot{a} = \dot{v} \dot{e}_t + v |\dot{\theta}| \dot{e}_n \tag{1.14} \]

We call $(\dot{v}, v |\dot{\theta}|)$ the tangential and normal components of acceleration.

Now let $\rho = \frac{v}{|\dot{\theta}|} \geq 0$. Suppose the motion is on a circle of radius $R$ (Fig. 1-8); then

\[ S = R\theta \iff \dot{S} = R \dot{\theta} = v \iff R = \frac{v}{\theta} \]

Thus we call in general $\rho$ the radius of curvature. Hence we may write

\[ \ddot{a} = \dot{v} \dot{e}_t + \frac{v^2}{\rho} \dot{e}_n \tag{1.15} \]

Note that $\dot{e}_n$ is undefined and $\rho = \infty$ for $\dot{\theta} = 0$, i.e. for rectilinear motion or at a point of inflection (Fig. 1-9). If the resultant force acting on a particle is expressed in normal-tangential components, $\sum F = \sum F_t \ddot{e}_t + \sum F_n \dot{e}_n$, then Eqns. (1.1) and (1.15) give the scalar equations of motion:

\[ \sum F_t = m \dot{v} , \quad \sum F_n = m \frac{v^2}{\rho} \tag{1.16} \]
Cylindrical and Spherical Coordinates and Components. In three-dimensional (3-D) motion it is often advantageous to resolve the velocity and acceleration into cylindrical or spherical components; only the velocity components will be given here. The cylindrical coordinates \((r, \phi, z)\) are shown on Fig. 1-10. From the geometry, the cylindrical and rectangular coordinates are related by

\[
\begin{align*}
x &= r \cos \phi \\
y &= r \sin \phi \\
z &= z
\end{align*}
\]  

The velocity expressed in cylindrical components is

\[
v = \dot{r} \hat{r} + r \dot{\phi} \hat{\phi} + \dot{z} \hat{k}
\]  

Fig. 1-10

Note that if \(z = \) constant, \((r, \theta)\) are just the familiar plane polar coordinates.

The spherical coordinates \((r, \theta, \phi)\) are shown on Fig. 1-11. The relation to rectangular coordinates is given by

\[
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta
\end{align*}
\]  

(1.19)
and the velocity in spherical components is

\[ \mathbf{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\phi}\sin\theta\hat{e}_\phi \] (1.20)

Spherical coordinates and components are particularly advantageous for central force motion (Chapter 10).

**Relative Velocity.** It is sometimes necessary to relate the motion of a point as measured in one reference frame to the motion of the same point as measured in another frame moving with respect to (w.r.t.) the first one. First consider two reference frames moving with respect to each other such that one axis, say \( z \), is always aligned (Fig. 1-12). Define the angular velocity and angular acceleration of frame \( \{i,j\} \) w.r.t. frame \( \{\hat{i},\hat{j}\} \) by:

\[ \omega = \dot{\theta}\hat{k} = \ddot{\theta}\hat{K} \] (1.21)
\[ \alpha = \ddot{\omega} = \dddot{k} = \dddot{\theta}\hat{K} \] (1.22)

where we have used the fact that \( \dot{k} = \dot{\theta}\hat{K} \) is a constant vector.
Figure 1-13 shows a point $A$ moving in a plane w.r.t. to two frames which are also moving w.r.t. each other.

Let

$$\frac{D}{Dt} = \text{time derivative w.r.t.}\{\hat{i}, \hat{j}\}$$

$$\frac{d}{dt} = \text{time derivative w.r.t.}\{i, j\}$$

For a scalar $Q$, $DQ/Dt = dQ/dt$, but for a vector $Q$, $DQ/Dt \neq dQ/dt$, in general. The relation between the two is given by the basic kinematic equation

$$\frac{DQ}{Dt} = \frac{dQ}{dt} + \omega \times Q$$

which holds for any vector $Q$. We are now ready to derive the relative velocity equation. From Fig. 1-13:

$$\tau_A = \tau_B + \tau$$  \hspace{1cm} (1.24)

Differentiating and applying Eqn. (1.23):

$$\frac{Dr_A}{Dt} = \frac{Dr_B}{Dt} + \frac{Dr}{Dt}$$

$$v_A = v_B + \frac{dr}{dt} + \omega \times \tau$$

$$v_A = v_B + v_r + \omega \times \tau$$  \hspace{1cm} (1.25)

where

$$v_A = D\tau_A/Dt = \text{velocity of } A \text{ w.r.t. } \{\hat{i}, \hat{j}\}$$

$$v_B = D\tau_B/Dt = \text{velocity of } B \text{ w.r.t. } \{\hat{i}, \hat{j}\}$$

$$v_r = dr/dt = \text{velocity of } A \text{ w.r.t. } \{i, j\}$$
These results also apply to general 3-D motion provided that the angular velocity is suitably defined. This is most conveniently done using Euler’s Theorem. This theorem states that any displacement of one reference frame relative to another may be replaced by a simple rotation about some line. The motion of the one frame w.r.t. the other may then be thought of as a sequence of such rotations. At any instant, $\omega$ is defined as the vector whose direction is the axis of rotation and whose magnitude is the rotation rate. With this definition of $\omega$, Eqns. (1.23) and (1.25) are valid for 3-D motion. For a full discussion of 3-D kinematics see Ardema, *Newton-Euler Dynamics*.

**Example.** Car $B$ is rounding a curve of radius $R$ with speed $v_B$ (Fig. 1-14). Car $A$ is traveling toward car $B$ at speed $v_A$ and is distance $x$ from car $B$ at the instant shown. We want the velocity of car $A$ as seen by car $B$. The cars are modelled as points.

Introduce reference frames:

- $\{\hat{i}, \hat{j}\}$ fixed in ground (the data is given in this frame)
- $\{i, j\}$ fixed in car $B$ (the answer is required in this frame)

Applying Eqn. (1.25):

$$v_A = v_B + v_r + \omega \times r$$
$$v_B = v_B \hat{j}$$
$$v_r = v_A - v_B - \omega \times r$$
$$r = -x \hat{i}$$
$$\omega = -\frac{v_B}{R} \hat{k}$$
$$v_r = v_A \hat{i} - \left(v_B + \frac{v_B x}{R}\right) \hat{j}$$
1.3 Work and Energy

Definitions. Suppose a force $F$ acts on a particle of mass $m$ as it moves along curve $C$ (Fig. 1-15). Define the work done by $F$ during the displacement of $m$ from $r_0$ to $r_1$ along $C$ by

$$U_{0,1} = \int_{r_0}^{r_1} F \cdot dr$$

(1.26)

Since $v = \frac{dr}{dt}$, this may be written as

$$U_{0,1} = \int_{t_0}^{t_1} F \cdot v \, dt = \int_{t_0}^{t_1} P \, dt$$

(1.27)

where $P = F \cdot v$ is called the power.

![Fig. 1-15](image)

Now suppose $F$ is the resultant of all forces and $\{\hat{i}, \hat{j}\}$ is an inertial frame; then, Newton's Second Law, Eqn. (1.1), holds:

$$m \frac{dv}{dt} = F$$

Taking the scalar product of both sides with $v$ and inserting the result in Eqn. (1.27):

$$m \frac{dv}{dt} \cdot v = F \cdot v$$
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\[
U_{0,1} = \int_{t_0}^{t_1} m \frac{dv}{dt} \cdot v \, dt = m \int_{t_0}^{t_1} v \cdot dv = \frac{1}{2} m \left( v_1^2 - v_0^2 \right)
\]

where the **kinetic energy** of the particle is defined as:

\[
T = \frac{1}{2} mv^2
\]

In words, Eqn. (1.28) states that the change in kinetic energy from position 0 to position 1 is equal to the work done by the resultant force from 0 to 1.

**Potential Energy.** In rectangular coordinates,

\[
\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} ;
\]

\[
\mathbf{d}r = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} ;
\]

\[
\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}
\]

and if \( \mathbf{F} \) is a function only of position \( \mathbf{r} \), Eqn. (1.26) gives

\[
U_{0,1} = \int_{L_0}^{L_1} \mathbf{F} \cdot d\mathbf{r} = \int_{x_0}^{x_1} F_x dx + \int_{y_0}^{y_1} F_y dy + \int_{z_0}^{z_1} F_z dz
\]

Generally, this integration will depend on path \( C \), and not just the end points.

Recall that the **gradient** of a scalar function of a vector argument \( V(\mathbf{r}) \) in rectangular coordinates is

\[
\text{grad } V(\mathbf{r}) = \frac{\partial V}{\partial x} \mathbf{i} + \frac{\partial V}{\partial y} \mathbf{j} + \frac{\partial V}{\partial z} \mathbf{k}
\]

Suppose that \( \mathbf{F}(\mathbf{r}) \) is such that there exists a function \( V(\mathbf{r}) \) such that

\[
\mathbf{F}(\mathbf{r}) = -\text{grad } V(\mathbf{r}) = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}
\]

Then, comparing Eqns. (1.31) and (1.32), and writing \( V = V(x, y, z) \),

\[
F_x = -\frac{\partial V}{\partial x} , \quad F_y = -\frac{\partial V}{\partial y} , \quad F_z = -\frac{\partial V}{\partial z}
\]

so that

\[
\mathbf{F} \cdot d\mathbf{r} = -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz = -dV
\]
Therefore, from Eqn. (1.26),

\[ U_{0,1} = \int_{V_0}^{V_1} (-dV) = -(V_1 - V_0) = -\Delta V_{0,1} \]  

(1.35)

This shows that now the work done by \( F \) depends only on the endpoints and not on the path \( C \).

\( V(r) \) is called a potential energy function and \( F(r) \) with this property is called a conservative force.

**Gravitation.** Consider two masses with the only force acting on them being their mutual gravitation (Fig. 1-16). If \( m_e \) (the earth, for example) \( \gg m \) (an earth satellite, for example), we may take \( m_e \) as fixed in an inertial frame. If the two bodies are spherically symmetric they can be regarded as particles for the purpose of determining the gravitational force.

\[ \vec{F} = -\frac{K m_e m}{r^2} \hat{\vec{r}} \]  

(1.36)

where \( K = 6.673 \times 10^{-11} \text{ m}^3/(\text{Kg} \cdot \text{sec}^2) \) is the universal gravitational constant.

Because

\[ r = r \hat{\vec{r}} \]

\[ r = x \hat{i} + y \hat{j} + z \hat{k} \]

\[ r = (x^2 + y^2 + z^2)^{1/2} \]

we have

\[ F = -\frac{K m_e m}{r^3} (x \hat{i} + y \hat{j} + z \hat{k}) \]  

(1.37)
so that
\[ F_x = -Km_em \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \]
\[ F_y = -Km_em \frac{y}{(x^2 + y^2 + z^2)^{3/2}} \]  \hspace{1cm} (1.38)
\[ F_z = -Km_em \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \]

Thus the gravitational force is conservative with potential energy function given by
\[ V = -\frac{Km_em}{(x^2 + y^2 + z^2)^{1/2}} \]  \hspace{1cm} (1.39)

This is verified by observing that Eqns. (1.33) are satisfied for this function.

In central force motion, as mentioned earlier, it is usually best to use spherical coordinates. Writing \( V = V(r, \theta, \phi) \), the gradient of \( V \) in spherical components is
\[
\text{grad } V(r) = \frac{\partial V}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{e}_\phi \]  \hspace{1cm} (1.40)

Thus the gravitational potential function is
\[ V = -\frac{Km_em}{r} \]  \hspace{1cm} (1.41)

which could have been obtained directly from Eqn. (1.39).

For motion over short distances on or near the surface of the earth it is usually sufficient to take the gravitational force as a constant in both magnitude and direction (Fig. 1-17). The force acting on the particle is
\[ F = F_x \hat{i} = -mg \hat{i} \]

Fig. 1-17
Therefore the gravitational potential energy function is

$$V(x) = mgx$$  \hspace{1cm} (1.42)$$

because $$-\frac{\partial V}{\partial x} = -mg = F_x$$.

**Energy Equation.** Suppose a number of forces act on \( m \), some conservative and some not. Then

$$F_{c_i} = -\text{grad} \; V_i$$  \hspace{1cm} (1.43)$$

for each conservative force. The resultant force is

$$F = \sum_{i=1}^{n_c} F_{c_i} + \sum_{j=1}^{n_{nc}} F_{nc}^j = \sum_i (-\text{grad}_i V_i) + \sum_j F_{nc}^j$$

The work done is

$$U_{0,1} = \int_{r_0}^{r_1} F \cdot dr = -\sum_i \left[ V_i(r_1) - V_i(r_0) \right] + \int_{r_0}^{r_1} \sum_j F_{nc}^j \cdot dr$$

Using \( U_{0,1} = \Delta T_{0,1} \), this becomes

$$\Delta T_{0,1} = -\Delta V_{0,1} + U_{0,1}^{nc}$$  \hspace{1cm} (1.44)$$

where \( V \) is the sum of all potential energies and \( U_{0,1}^{nc} \) is the work done by all nonconservative forces.

Let the *total mechanical energy* be defined by:

$$E = T + V$$  \hspace{1cm} (1.45)$$

Then Eqn. (1.44) may be written

$$\Delta E_{0,1} = U_{0,1}^{nc}$$  \hspace{1cm} (1.46)$$

and, in particular if all forces are conservative and accounted for in \( V \),

$$\Delta E_{0,1} = 0$$  \hspace{1cm} (1.47)$$

that is, energy is conserved.

Remarks:

1. \( U \) is defined over an interval of motion but \( T \), \( V \), and \( E \) are defined at an instant.
2. $U$, $T$, $V$, and $E$ are all scalars. Therefore, the energy equation gives only one piece of information.

3. The energy equation is a once-integrated form of Newton’s Second Law; it is a relation among speeds, not accelerations.

4. The energy equation is most useful when a combination of the following factors is present: the problem is of low dimension, forces are not needed to be determined, and energy is conserved.

5. The energy equation involves only changes in $T$ and $V$ between two positions; thus adding a constant to either one does not change the equation.

### 1.4 Eulerian Rigid Body Dynamics

**Kinetics of Particle System.** First consider a collection of particles (not necessarily rigid), Fig. 1-18. Let $\{i, j\}$ be an inertial reference frame and:

- $F^e_i = \text{sum of all external forces on particle } i.$
- $F_{ij} = \text{(internal) force exerted by particle } j \text{ on particle } i.$

The center of mass is a position, labeled $G$, whose position vector is given by

$$\vec{r} = \frac{1}{m} \sum_i m_i \vec{r}_i$$  \hspace{1cm} (1.48)

where $m = \sum_i m_i$ is the total mass.

![Fig. 1-18](image)