

# Continuous Bivariate Distributions

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Second Edition

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To my loving mother, Lakshmi,  
and lovely memories of my father,  
Narayanaswamy

N.B.

To Ai Ing, Joseph, Eugene, Serena,  
my brother Chin-Yii, and my parents  
C.D.L.

# Preface

This volume, which is completely dedicated to continuous bivariate distributions, describes in detail their forms, properties, dependence structures, computation, and applications. It is a comprehensive and thorough revision of an earlier edition of “Continuous Bivariate Distributions, Emphasizing Applications” by T.P. Hutchinson and C.D. Lai, published in 1990 by Rumsby Scientific Publishing, Adelaide, Australia.

It has been nearly two decades since the publication of that book, and much has changed in this area of research during this period. Generalizations have been considered for many known standard bivariate distributions. Skewed versions of different bivariate distributions have been proposed and applied to model data with skewness departures. By specifying the two conditional distributions, rather than the simple specification of one marginal and one conditional distribution, several general families of conditionally specified bivariate distributions have been derived and studied at great length. Finally, bivariate distributions generated by a variety of copulas and their flexibility (in terms of accommodating association/correlation) and structural properties have received considerable attention. All these developments and advances necessitated the present volume and have thus resulted in a substantially different version than the last edition, both in terms of coverage and topics of discussion.

In a volume of this size and wide coverage, there will inevitably be some mistakes and omissions of some important published results. We have made a sincere effort to minimize these, and what are left and left out are accidental and are certainly not due to nonscientific antipathy. We welcome the readers to write to us about the contents of this volume and inform us of any errors, misrepresentations, and omissions that you find. If ever there is a next edition, we will take your comments into account and make the necessary changes (keep in mind that our expected residual lives are probably not large enough to guarantee the next edition!).

We express first and foremost our sincere thanks and gratitude to Paul Hutchinson for his generosity in permitting us to use good portions from

the last edition that he was part of, and for his support and encouragement through out the course of this project. We also thank Ingram Olkin for proposing and initiating this revision through Springer-Verlag. Thanks are also due to John Kimmell (Editor, Springer-Verlag) for his interest in this book, and his support and immense patience during the long preparation period, and to Debbie Iscoe (McMaster University, Canada) for converting the not-so-presentable initial manuscript that we prepared into this fine-looking book that you hold in your hands. Our final special thanks go to our families who have endured all the countless hours we were away from them (it is quite possible, of course, that they enjoyed these times in our absence) just to make a bit of progress everytime.

We both enjoyed very much putting this book together and we sincerely hope that you, as reader, would enjoy it as much while using it!

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