Teaching Mathematical Reasoning in Secondary School Classrooms

Karin Brodie

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With Contributions by

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Foreword

The Road to Reasoning

The teachers in this book share a worthy and courageous mission. They have all set out to provide children with one of the most important educational experiences it is possible to have – a form of mathematics teaching that is based upon sense making and discussion, rather than submission and silence. Mathematical "reasoning" is what mathematicians do – it involves forming and communicating a path between one idea or concept and the next. When students form these paths they come to enjoy mathematics, understand the reasons why ideas work, and develop a connected and powerful form of knowledge. When students do not engage in reasoning, they often do not know that there are paths between different ideas in mathematics and they come to believe, dangerously, that mathematics is a set of isolated facts and methods that need to be remembered. I have visited hundreds of classrooms across the world in which students have been required to work in silence on maths questions, never talking about the ideas or forming links and connections between ideas; most of these students come to dislike mathematics and drop the subject as soon as they can. Such students are not only being denied the opportunity to learn in the most helpful way, but they are denied access to real, living mathematics.

The teachers in this book, through their work with Karin Brodie, the author, learned about the value of mathematical reasoning and set out to teach students to engage in this valuable act. This book shares their important journey and provides the world with new lenses for considering the teaching acts that were involved, as well as the challenges and obstacles that stood in their way. For whilst we know the importance of reasoning to children's mathematical futures it would be dishonest to pretend that teaching approaches that invite students to communicate their mathematical thoughts and make connections between ideas are easy or well understood. We have reached an advanced stage in the development of education and yet, incredibly, we are still relatively uninformed about the ways teachers of mathematics can teach students to reason, which is part of the reason this book is so valuable and could be a wonderful resource for many.

When Deborah Ball, in the United States, then an elementary teacher of mathematics, now a university dean, released a videotape of her teaching 7- and 8-year olds to reason about odd and even numbers, the world was shocked to witness a boy named Shea propose a new way of classifying odd numbers. His numbers – those that can be grouped into even numbers of pairs of twos – came to be known as “Shea numbers”. The rich conversations in which the young children engaged in the mathematics class that appeared on tape, seemed to unfold effortlessly, although in reality they were expertly choreographed by the teacher. Deborah Ball has offered records of her teaching decisions and actions, which have been read by scores of people worldwide, including the teachers who write in this book. She was one of the first teachers to offer such valuable records and analyses. This book adds to the small but important collection of teachers who have engaged students in mathematical reasoning and documented and unpacked the important teaching acts that took place.

But what makes a record of teaching useful and worthwhile? Every act of teaching, with a classroom full of children and their many thoughts and actions, is extremely complex, and descriptions of a class in action can remain highly contextualized and difficult for others to learn from. A teacher may record thoughts and moves without communicating them in such a way that they are useful for other teachers, educators, and analysts. The art in producing a record that is powerful and valuable for others comes partly from having important teaching experiences to talk about and partly from having a way of raising the individual acts to a higher and more generalizable level that other teachers can learn from. This is where the combination of the reports of the teachers who engaged students in reasoning, and the theoretical lenses applied by Karin, are so generative and fruitful for the rest of the world to learn from. When a new idea and teaching act is connected with a theory of learning, the result can be very powerful indeed.

An example of the way a teaching act can be named and made more general is the case of a set of interactions that has become known as IRE. These describe a common teaching situation when a teacher initiates something (I), elicits a response from a student (R), and then evaluates the response (E). Researchers found that the majority of the interactions that take place in classrooms follow the IRE response pattern and they gave it a particular classification. Since that initial classification IRE has been used by scores of researchers and analysts over many years and has proved extremely useful in the advancement of teaching. Yet teaching classifications such as IRE are rare and the field of mathematics education has not benefitted from a similar mapping and classification of the teaching interactions that take place when students are taught to reason about mathematics. This book provides such a mapping.

Karin notes that a reasoning approach to mathematics involves a change in authority. Students no longer need to look to teachers or textbooks to know if they are moving in the right directions in mathematics, as they have learned a set of reasons and connections that they can refer back to, evaluating their own thoughts and ideas. This may seem as though the authority is shifting from the teacher to the students and this is partly true, but it is important to note that the authority is also shifting from the teacher to the domain of mathematics itself. Students no longer need to refer to teachers to evaluate their mathematical thoughts, because they can refer to the domain of mathematics, to consider whether they have followed the correct connections and paths. This is just one way in which reasoning as an act brings classrooms closer to real and living mathematics. In addition, we now have evidence that when students receive opportunities to discuss mathematics and express their own thoughts, they become more open-minded as they learn to be appreciative and respectful of other people's ideas. Mathematical reasoning encourages respect, responsibility, and a personal empowerment that has long been missing in mathematics classrooms. Karin starts this book by quoting the goals of the new South African curriculum – to heal the divisions of the past and build a human rights culture. Mathematics, the subject so many believe to be abstract and removed from such responsibilities, has a key role to play in promoting such a culture, in South Africa and beyond. This book communicates the way that mathematics can provide this valuable contribution and the important work of teachers in doing so. I hope you enjoy it and use it as both inspiration and resource.

Jo Boaler The University of Sussex

Contents

List of Tables

Introduction to Part 1

Over the past 10 years, South Africa has introduced a new curriculum in all subjects at all levels of schooling. The curriculum was inspired by the end of apartheid and informed by curriculum developments and visions of reform in many other countries. The new curriculum has an impressive set of goals for individual learners and society, including healing the divisions of the past, building a human rights culture, and developing skilled and knowledgeable citizens who can contribute to and benefit from a growing economy and a participatory democracy (Department of Education 2003).

The new curriculum posits a very different view of mathematical knowledge from that of previous curricula. Mathematics is seen as both conceptual and practical; abstract and applied. The curriculum argues for conceptual understanding of mathematical ideas, skill in performing mathematical calculations, and the ability to relate mathematical concepts to other subjects and to real-world applications. Mathematical concepts and skills are developed and linked by "creative and logical reasoning" and "rigorous logical thinking" (Department of Education 2003, p. 9). The new curriculum puts mathematical reasoning firmly on the agenda arguing "competence in mathematical process skills such as investigating, generalizing and proving is more important than the acquisition of content knowledge for its own sake" (Department of Education 2003, p. 9). A view of mathematics as a combination of conceptual depth, flexible skills, and mathematical reasoning resonates with curriculum developments elsewhere in the world. Some proponents of reform mathematics argue that reasoning should be taught alongside many of the basic facts and concepts of mathematics and some even argue that mathematical reasoning is in itself a "basic" mathematical skill (Ball and Bass 2003).

Worldwide, too many learners struggle with mathematics, fail mathematics, and hate mathematics. Moreover, when the factors of race and class are considered, it is clear that a disproportionate number of black and economically disadvantaged learners do not achieve success in mathematics and do not believe that they can do mathematics (Association for Mathematics Education of South Africa 2000; Department of Education 2001; Moses and Cobb 2001; Secada 1992). The proponents of reform curricula propose that new approaches to mathematics and to teaching mathematics will make mathematics more accessible, enjoyable, and inspiring for many more learners, enable more learners to be successful in mathematics and will begin to close achievement gaps between rich and poor and black and white learners.

There has been much debate as to whether current mathematics reforms can be a mechanism for ensuring more equitable participation and achievement in mathematics (see Brodie 2006 for a summary of these debates). Empirical evidence in well-resourced countries is beginning to show that curriculum reforms do mitigate achievement gaps between marginalized and other learners (Boaler 1997; Hayes et al. 2006; Kitchen et al. 2007; Schoenfeld 2002). However, the evidence also suggests an important caveat. The implementation of the new curriculum ideas is not widespread and is inequitably distributed, tending to be found in more well-resourced schools and countries, hence further disadvantaging poor learners (Kitchen et al. 2007). There are two important implications of these findings. First, in many contexts, lack of resources, including big classes and few materials, teacher confidence and knowledge, and support for teachers, can be major barriers to developing new ways of teaching (Tabulawa 1998; Tatto 1999). If reforms are successful in promoting equity and if they are not taken up in less-resourced countries and schools, then existing divides between rich and poor will be exacerbated. Second, it is not only a lack of resources that creates barriers to reform practice. In fact, research in many well-resourced contexts suggests that most teachers struggle to take up reform practices in substantive ways (Fraivillig et al. 1999; Hayes et al. 2006; Hufferd-Ackles et al. 2004; Lavi and Shriki 2008; Nolan 2008). This suggests that reasons for the difficulties that teachers experience with reform curricula cannot be found in resources alone. Something deeper is at play here.

The research on teacher change in the context of reform is based on the assumption, shared in this book, that teachers are key to achieving the visions of better mathematics learning for more learners. This research can be divided into two main categories. The first describes models of exemplary reform teaching, making the claim that such teaching is possible, albeit with many challenges, illuminating different approaches to reform teaching and showing how the challenges can be overcome. Such cases come mainly from well-resourced contexts, further adding to the concern that such teaching is only possible in these contexts (Boaler 1997; Boaler and Humphreys 2005; Chazan and Ball 1999; Hayes et al. 2006; Heaton 2000; Lampert 2001; Staples 2007). The second set of research, which includes some of my own work, argues that teachers do not substantially shift their practices, even after extensive pre-service or in-service courses (Brodie et al. 2002; Fraivillig et al. 1999; Lavi and Shriki 2008; Nolan 2008; Tatto 1999). These two sets of findings tend to dichotomize the field, suggesting an ideal "reform vision", which unfortunately only a few can attain. A disturbing consequence of dichotomizing the field in this way is that teachers are often blamed for not being able to implement visions of reform because they do not live up to the ideals.

A far more promising line of research takes the middle road, presenting more textured descriptions of points of difficulty for teaching and when, how and why teaching in reform-oriented ways breaks down (Gamoran Sherin 2002). In my own work with colleagues and students, we have shown that some aspects of reform practice are easier for teachers to work with, for example selecting tasks of higher cognitive demand (Modau and Brodie 2008) whereas others are more difficult, for example interacting with students while maintaining the level of task demand (Jina and Brodie 2008; Modau and Brodie 2008; Stein et al. 1996, 2000). We have also argued that adopting reforms requires teachers to coordinate a range of new practices and to think about their current practices in new ways. Such coordination is an immense task and means that teachers' taken-for-granted practices might break down in the face of the new curriculum practices (Slonimsky and Brodie 2006). It is thus highly likely that teachers attempting to work with reforms may resort to traditional practices, more or less deliberately (Brodie 2007c).

This book aims to contribute to this third emerging strand of research and to give substance to a number of claims that such research can make. The first is that teachers' difficulties in working with new curricula need to be taken seriously, because, as we show in this book, mathematical reasoning is challenging to learn and to teach. For teachers who learned mathematics and learned to teach mathematics in traditional ways, the challenges are enormous. However, this does not mean that teachers cannot begin to work towards teaching mathematical reasoning in ways suggested by the reforms. But it does mean that researchers and teacher educators need to find ways to capture teachers' successes and challenges in ways that can help teachers to move forward. The successes might be small and the challenges might be large, but we need to find ways to show where and how teachers are shifting and what the next steps for progress might be. It is our experience, some of which we hope to share in this book, that teachers who do take risks and embark on the journey of learning to teach in new ways, have experienced both the exhilaration of success, when learners actually do begin to reason with each other and their teachers, and the extreme challenges of sustaining the practice with all learners, particularly given overcrowded curricula and high stakes tests and examinations.

So this book explores some of the successes and challenges faced by a group of South African teachers who worked to develop mathematical reasoning among their learners. In doing so, it explores what it means to teach mathematical reasoning in secondary school mathematics classrooms, addressing some important questions like what mathematical reasoning means; how can mathematical reasoning be taught; how teaching mathematical reasoning differs from more conventional mathematics teaching; and the demands that teaching mathematical reasoning makes on teachers and learners. A number of chapters address these questions from the perspective of the teachers analysing their own practice, and others address the questions from the perspective of an academic researcher, analysing the teachers' practices.

The book is the result of an ongoing collaboration between five teachers and an academic. We came together with a joint interest in promoting mathematical reasoning in South African classrooms. We share a passion for improving the experiences of learners in mathematics classrooms, and a belief that working in reform-oriented ways can do this. At the same time, we experience the real constraints of classrooms and are inspired by the need to find ways to work with contextual realities to support mathematical reasoning. Although this book is set in a

South African context, it has strong implications for other contexts. This work was conducted in a range of differently resourced classrooms, from very poorly resourced to very well resourced, and so can illuminate the teaching of mathematical reasoning in relation to contextual differences and speak to readers working in a wide range of countries and schools.

Each teacher conducted her/his study as the research component of an Honours programme in mathematics education, in which they were all enrolled. At the time, I was engaged in my doctoral research, and worked with the teachers as subjects of my research and as advisor on their research projects. Each teacher worked on a subset of her/his data from the larger data corpus, which I collected. We worked closely together as a group and the results of each study informed the others. The dilemmas, strengths, and challenges of such a collaboration have been discussed in detail elsewhere (Brodie 2005; Brodie et al. 2005).

Outline of the Book

This book comprises three parts. Part 1, consisting of Chaps. 1 and 2, sets the context of the work within the literature on teaching and learning mathematical reasoning and describes the school contexts in which the research took place. In Chap. 1, I review the literature and develop the concepts of mathematical reasoning, learning mathematical reasoning and teaching mathematical reasoning, that informed the work of the project and the subsequent chapters of this book. In Chap. 2, I discuss the differently resourced contexts in which the teachers worked. I argue that the learners' knowledge in the different classrooms forms a substantive part of the teaching context. Through the development of this work, we came to see learners' knowledge as a key resource for teaching mathematical reasoning. I also discuss the tasks that the teachers developed to teach mathematical reasoning and which they refer to in their chapters.

Part 2, consisting of Chaps. 3–7, describes the studies that the teachers conducted. Each teacher researched an aspect of her/his practice, trying to understand more deeply the challenges and successes that she/he experienced. These chapters were informed by some of the literature discussed in Chap. 1 and each teacher chose particular parts of the literature to work with as a conceptual framework. In Chap. 3, we take a close look at how a set of tasks supported learners' reasoning and in Chap. 4 we explore how collaborative, whole-class discussion supported both individual and group learning. In Chap. 5, we describe a set of practices, which the teacher developed to enable learners to reason with each other and show how his learners appropriated these practices to help with their thinking and reasoning. In Chap. 6, we show how the teacher supported the development of mathematical proficiency among her learners and in Chap. 7, we focus on the development of justification among learners. Taken together, these chapters provide a rich account of challenges and successes in teaching mathematical reasoning, and what is possible to achieve even in very difficult circumstances. They deal with a number of key aspects in teaching mathematical reasoning, namely, what tasks might be useful to elicit learners' reasoning and how are they best implemented in the classroom, how different tasks support learners' mathematical reasoning in different ways, and how classroom interaction helps to support the development of mathematical reasoning.

Part 3, consisting of Chaps. 8–11, comprises an overview of the practices in the five classrooms, drawing on my doctoral research. Chapters 8 and 9 look across all five classrooms and develop categories for talking about learner contributions and teacher moves, as the beginnings of a language of description for reform-oriented teaching. These two chapters together suggest a trajectory for the emergence of learner contributions and teachers' responses that promote mathematical reasoning. The argument is that by finding more specific ways to talk about how teachers and learners interact in classrooms, we can find ways to help teachers move forward in engaging learners' mathematical reasoning. Chapter 10 focuses in more depth on two teachers and the dilemmas they experienced in teaching mathematical reasoning. Chapter 11 focuses even more closely on one classroom and explores the resistance of the learners to their new experience of learning mathematical reasoning. Such resistance is often reported by teachers but not often explored. In this chapter I suggest that resistance is an important aspect of the new methods of teaching and suggest ways of managing it. The overarching argument of Part 3 is that every success in reform pedagogy produces new challenges, a range of learner contributions to respond to, new dilemmas in relation to these contributions, and possible learner resistance. Teachers, teacher–educators, and researchers cannot ignore these challenges; we have to find ways to talk about them as a normal part of learning to teach in new ways, and the challenges that change brings.

Because teachers have been involved in the work of this book and in writing parts of it, we hope that the work will speak to teachers, and to teacher educators and researchers who are trying to work in and with new curriculum developments. We have written this book for both teachers and researchers because we strongly believe that teachers and researchers can and should speak to each other in many ways, including through books such as this one. At the same time, we also realize that teaching and research are distinct practices, with their own discourses. This has caused some discomfort in the writing of this book in that we have had to continually consider two different audiences. We have resolved this by writing some chapters in a more "academic" tone and others somewhat more colloquially. In doing this, we have made sure to keep our research focus strong and rigorous throughout the book. In particular Chap. 1 sets out the academic field of teaching and learning mathematical reasoning and may not be the best part of the book for teachers to start reading. The case studies in Chaps. 3–8 are structured to form part of the ongoing narrative of the book as a whole but can also be read individually. In these chapters, key parts of the literature have been revisited for the purposes of the particular case study, and although this has meant some repetition of key ideas, we believe it will help readers to see these ideas working in different contexts. Chapters 8 and 9 develop a language of description, which might be more appealing to researchers than teachers. Chapters 10 and 11 deal with particular issues in two of the classrooms, and although building on the language of Chaps. 8 and 9, can also be read on their own.

In our work together, practice has spoken closely to research, and so we hope that the research described here will find ways to speak to practice. We believe that the work in this book will provide useful models for other teachers wanting to teach mathematical reasoning, for teachers wanting to research their own practice, and for teacher–educators and researchers wanting to develop and analyse the teaching of mathematical reasoning. We undertook a journey together in which we learned a tremendous amount. We hope to convey some of it in this book and also inspire others to embark on similar journeys.

Curriculum reform has become a global movement over the past 30 years. Similarities among the South African and other mathematics curricula will be discussed further in this chapter and the book.

Terminology used across contexts to refer to new curricula is different. In this book we use interchangeably the terms "new curriculum" which applies to South Africa and other countries, which have national curricula, and "reform" which is used predominantly in the United States. In both cases we include the enacted curriculum, i.e. teaching and learning in classrooms.

In the South African Higher Education system an Honours degree follows a 3-year undergraduate degree and a professional teaching qualification and is necessary for entry into Masters.

Chapter 1
Teaching Mathematical Reasoning: A Challenging Task

The Centrality of Mathematical Reasoning in Mathematics Education

When we "reason", we develop lines of thinking or argument, which might serve a number of purposes – to convince others or ourselves of a particular claim; to solve a problem; or to integrate a number of ideas into a more coherent whole. Two processes are important to reasoning – first, that the different steps or moves in the line of reasoning are connected with each other (not necessarily analytically or deductively); and second, that these links are somehow "reasoned", there are reasons why one move follows another and how a number of moves come together to form an argument or to solve a problem (Ball and Bass 2003). Brousseau and Gibel (2005) point out that these reasons are only considered to be reasonable when they relate to the constraints of the problem or the knowledge under consideration. An appeal to authority, for example to what a teacher or textbook says, does not count as a reason for a productive argument.

The product of a reasoning process is a text, either spoken or written (Douek 2005), which presents warrants for a conclusion that is acceptable within the community that is producing the argument (Krummheuer 1995). An individual can reason, or a group of people can reason together, co-producing the line of argument[1]. Mathematical reasoning assumes mathematical communication (Ball and Bass 2003; Douek 2005; Krummheuer 1995). Communication is an integral part of the process of reasoning, both for an individual working with previously produced texts to produce a new one, and for groups working together to produce an argument. The texts or products of reasoning have, as their main purpose, to communicate reasoning.

Mathematical reasoning is reasoning about and with the objects of mathematics. However, the relationship between mathematical reasoning and mathematics is not obvious (Steen 1999), and the processes involved in mathematical reasoning need

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[1] Social perspectives on learning and thinking would argue that even an individual reasoning, seemingly on her own, is in fact in dialogue with others, co-producing an argument, with an imagined audience, with ideas from others, and in a social and historical context (see below).

K. Brodie, *Teaching Mathematical Reasoning in Secondary School Classrooms*,
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some elaboration. For Ball and Bass (2003) reasoning is a “basic skill” (p. 28) of mathematics and is necessary for a number of purposes – to understand mathematical concepts, to use mathematical ideas and procedures flexibly, and to reconstruct once understood, but forgotten mathematical knowledge. Kilpatrick et al. (2001) define a notion of mathematical proficiency which requires five intertwined and mutually influential strands – conceptual understanding, which entails comprehension of mathematical concepts, operations, and relations; procedural fluency, involving skill in carrying out procedures flexibly, accurately, efficiently, and appropriately; strategic competence, which is the ability to formulate, represent, and solve mathematical problems; adaptive reasoning, which is the capacity for logical thought, reflection, explanation, and justification; and productive disposition, an orientation to seeing mathematics as sensible, useful, worthwhile, and reasonable, and that anyone can reason to make sense of mathematical ideas[2]. For Kilpatrick et al. (2001), although all the strands are important and mutually influential, “adaptive reasoning is the glue that holds everything together” (p. 129) in that it allows for concepts and procedures to connect together in sensible ways, suggests possibilities for problem solving, and allows for disagreements to be settled in reasoned ways. Central to adaptive reasoning is the justification of claims and development of arguments.

This view of mathematical proficiency has informed all of the work in this book. Most directly, in Chap. 6 we reflect on one teacher’s attempt to teach the five strands in a holistic way. The teacher found that she devoted most of the time to conceptual understanding rather than procedural fluency, which is traditionally the norm in mathematics classrooms (Kilpatrick et al. 2001; Schoenfeld 1988; Stigler and Hiebert 1999). However, she was concerned that she devoted less time to strategic competence and adaptive reasoning. She also found that more than half of the learners in her class showed evidence of all five strands in their written work. In Chaps. 5 and 7 we focus on the strand of adaptive reasoning and show how two teachers supported learners to reason adaptively.

Justifying and Generalizing

The literature suggests that there are two key practices involved in mathematical reasoning – justifying and generalizing – and other mathematical practices such as symbolizing, representing, and communicating, are key in supporting these (Ball 2003; Ball and Bass 2003; Davis and Maher 1997; Triandafillidis and Potari 2005). For Kilpatrick et al. justifying is a key element of adaptive reasoning and to justify means “to provide sufficient reason for” (p. 130). They argue “students need to be able to justify and explain ideas in order to make their reasoning clear, hone their

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[2] I note here that Kilpatrick et al.’s work is an extension of the more usual distinctions of conceptual and procedural understandings of mathematics (Hiebert and Lefevre, 1986).

reasoning skills and improve their conceptual understanding" (p. 130). For Ball and Bass, "unjustified knowledge is unreasoned and, hence, easily becomes unreasonable" (p. 29). Justification is a key mathematical practice that allows mathematicians and mathematics teachers and learners to make connections between different ideas and parts of an argument, to provide warrant for claims and conjectures, to settle disputes, and to develop new mathematical ideas.

For Russell (1999), mathematical reasoning is "essentially about the development, justification and use of mathematical generalizations" (p. 1). These generalizations create an interconnected web of mathematical knowledge – conceptual understanding in Kilpatrick et al.'s terms. For Russell, "seeing mathematics as a web of interrelated ideas is both a result of an emphasis on mathematical reasoning and a foundation for reasoning further" (p. 5). Creating generalizations also enables problem solving, as generalizations support learners to see the underlying structure of the problem and the bigger class of problems or ideas that it instantiates (Brousseau and Gibel 2005; Kilpatrick et al. 2001; Russell 1999). Russell also introduces a notion of "mathematical memory", which is a memory of fundamental mathematical relationships, rather than of isolated facts. This kind of memory is what allows mathematical knowers to reconstruct, in a reasoned way, mathematical concepts, procedures, and principles that they might have forgotten (Ball and Bass 2003; Brousseau and Gibel 2005). It also supports sense making and insight in mathematics, and creates the conditions for solving problems.

In Chap. 7, we directly address the challenges that a teacher faced in supporting his learners to justify their thinking. The vast majority of learners in his class were not able to answer the question: "can x^2+1 be less than zero, when x is a real number", with appropriate justifications. We show how the teacher worked through a number of different contributions from learners, ranging from incorrect justifications through those that were partially correct, to one that was completely correct, asking them to discuss and communicate their reasoning. Even though his learners had very weak mathematical knowledge, they were, with a lot of help from the teacher, able to contribute and to help each other develop better justifications. In each of the other teachers' chapters, we see examples of learners' successes and challenges as they work to justify, explain, and generalize their ideas.

The Role of Proof in Mathematical Reasoning

Justification and generalization are closely related to proof in mathematics. In fact, for many mathematicians and in many mathematics curricula, mathematical reasoning is equated with proof. In this book we take the view, together with others (Ball and Bass 2003; Davis and Hersh 1981; Hanna and Jahnke 1996; Kilpatrick et al. 2001; Kline 1980; Krummheuer 1995), that whereas proof is one form of argument and justification, not all arguments and justifications are proofs, and a formal proof is not always an adequate justification or explanation of mathematical ideas. Although formal proof has long been thought to produce infallibility in mathematical knowledge, in fact it does not do so (Davis and Hersh 1981; Ernest 1991; Hanna and Jahnke 1996). Standards of rigour are socially constructed (Ernest 1991; Volmink 1990) and “there has never been a single set of universally accepted criteria for the validity of a mathematical proof” (Hanna and Jahnke 1996, p. 884). For example, most mathematics teachers are convinced by the standard one-page presentation of the proof of Pythagoras’ theorem; however, a completely logically rigorous proof would take about eighty pages (De Villiers 1990).

Just as in other disciplines, communities of practice (Wenger 1998) exist in the various domains of mathematics, which review new mathematical proofs in accordance with the current questions, objects of study, ways of thinking, methods, and results of the specific mathematical domain. The nature of the discipline of mathematics, founded and built on fundamental, shared concepts means that there is more agreed upon knowledge in mathematics than in other disciplines, such as psychology or sociology. However, this does not mean that mathematical knowledge is not socially constructed or contested. Proof does not shield us from the uncertainty of our knowledge (Hanna and Jahnke 1996; Kline 1980). At the same time, proof is an important embodiment of mathematical reasoning and needs to be taught as a particular form of reasoning, justification, and generalization within the discipline of mathematics (Hanna and Jahnke 1996).

Creativity and Reasoning

A strong rebuttal to the hegemony of proof in mathematics comes from practising mathematicians, who often work intuitively and creatively, searching for understanding and meaning, rather than rigour and formality. Sternberg and his colleagues distinguish between creative and analytical thinking (Sternberg 1999; Sternberg et al. 1998), arguing that “analytical tasks involve analysing, judging, evaluating, comparing and contrasting, and critiquing; creative tasks involve creating, inventing, discovering, imagining and supposing” (1998, p. 374). Although creative and analytical thinking are often posed as dichotomous, they actually support each other in mathematical problem solving and reasoning, for example imagining would require some form of comparing and supposing usually requires some analysing. Comparing alternative solutions, ideas, and imaginings all require reasoning and justification; creative thinking can support links between previously unconnected ideas; and leaps of imagination are often necessary to see a problem from a different perspective.

Intuition has also been studied as an important part of mathematical problem solving, creating mathematical arguments, and proving mathematical theorems (Fischbein 1987). Intuition might precede more formal arguments, justifications, and proof, and in some instances, might replace it. A mathematician who intuitively feels that something is wrong in a proof, will search to find the mistake, doubting the proof rather than her intuition (Hanna and Jahnke 1996). Crucial to notions of creativity and intuition is a sense that conviction and understanding do not necessarily come from formal, deductive, or analytic proofs. Although these have their place, they are certainly not sufficient to solve mathematical problems and communicate mathematical justifications and generalizations. If practices in the mathematics classroom are to be authentic to the discipline of mathematics (Brown et al. 1989), then a broader range of reasoning should be acknowledged and developed in mathematics classrooms.

Empirical and inductive reasoning play an important part in the reasoning practices of mathematicians and mathematics learners, often complementary to theoretical and deductive reasoning. Simon (1996) argues for a notion of "transformational" reasoning, where dynamic transformations of objects are visualised and which provide the reasoner(s) with a sense of conviction and understanding of how and why something is the case. Transformational reasoning supports and is supported by both inductive and deductive reasoning. Drawing on Toulmin, Krummheuer (1995) argues for substantive arguments, rather than merely analytic ones. Substantive arguments show relationships between the main objects and premises, rather than merely drawing deductive conclusions based on previously proved results or axioms. This distinction is similar to Hanna's characterization of proofs that prove and proofs that explain (Hanna and Jahnke 1996). De Villiers (1990) argues for five key functions for proof – verification, explanation, systematization, discovery, and communication. It is useful to see these as functions of mathematical reasoning as well. Verification establishes that something is the case, i.e. sufficient justification has been produced to confirm that a claim is true. Explanation establishes why something is the case, showing what are the key properties that are necessary for the truth of a claim. Explanatory proofs, or substantive arguments are more satisfying to both mathematicians and mathematics learners (Hanna and Jahnke 1996; Krummheuer 1995). Systematization organizes disparate mathematical concepts that are already established into a coherent mathematical system. As argued above, mathematical reasoning is a key part of mathematical discovery and mathematical reasoning also functions to help communicate our ideas and their warrants to others.

The idea that mathematical reasoning involves creativity, discovery, and communication is central to the work of this book. In Chap. 4, we show how a collaborative conversation among learners supported the development of the mathematical concept of function. Communication was the key in enabling learners to make creative, reasoned conceptual leaps. In Chap. 5, we show how the teacher's practices supported the learners' mathematical reasoning by encouraging them to question and challenge each other and himself. Again, we see reasoned creativity among his learners.

In this section, I have argued that mathematical reasoning is a key element of mathematics and thus is central to learning mathematics in school. I have argued for a broader notion of mathematical reasoning, in which intuition, creativity, imagination, explanation, and communication all play an important role. Fundamental to all forms of mathematical reasoning is the practice of justification and creating adequate arguments in defence of claims. Throughout this section, I have drawn on the notions of mathematical practices, communities of practice, and that mathematics is fundamentally a social practice. In the next section, I explore these ideas further.

Theories of Learning and Mathematical Reasoning

The work in this book is informed by a number of theories of learning, in particular constructivist, socio-cultural, and situated theories. Following Sfard (1998, 2001), I argue that none of the above theories is sufficient on their own to explain the learning and teaching of mathematical reasoning, and in this project I use them in careful combination. Although some scholars argue that since the fundamental mechanisms that generate learning posed by the theories are so different (biological equilibration for constructivists and social relations for socio-cultural and situated theories), the theories may be incommensurable, my argument is that the different mechanisms operate at different levels and in combination with each other and as long as the differences are acknowledged and specified, we can use these theories together to inform teaching and account for learning in mathematics classrooms (see also Sfard 2001).

Constructivism

Constructivism, in its many varieties, is centrally concerned with how knowledge is constructed and restructured in order to make sense of ever-increasing complexity, both in one's knowledge and in the outside world. Constructivism has had an important influence on theories of mathematics learning and mathematical reasoning (Confrey and Kazak 2006; Hanna and Jahnke 1996), and on the new curriculum in South Africa (Department of Education 2000). However, just as there are many varieties of constructivism, there are many ways in which constructivism can be misconstrued (Moll 2000).

The version of constructivism that informs the work in this book is derived from Piagetian constructivism (Piaget 1964, 1968, 1975), informed by the interpretations of Hatano (1996) and Rowell (1989). Two key principles of this version are first, that what people learn is constrained and afforded by what they know; and second, that there is an integrity to learners' thinking – what learners think, say, and do makes sense to them in relation to what they know. The role of current knowledge is very particular in that current knowledge is not merely built upon (as in behaviourist theories); rather it is restructured and reorganized into richer, more connected, and more powerful knowledge (Hatano 1996). Just as new knowledge is transformed in relation to prior knowledge, so prior knowledge is transformed in relation to new knowledge. From constructivist perspectives a deepening or transforming of thinking involves a deepening or transforming of cognitive structures, either integrating previously separate structures into more general and powerful