STATISTICAL MODELING AND ANALYSIS FOR COMPLEX DATA PROBLEMS

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STATISTICAL MODELING AND ANALYSIS FOR COMPLEX DATA PROBLEMS

Edited by
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Foreword

GERAD celebrates this year its 25^{th} anniversary. The Center was created in 1980 by a small group of professors and researchers of HEC Montréal, McGill University and of the École Polytechnique de Montréal. GERAD's activities achieved sufficient scope to justify its conversion in June 1988 into a Joint Research Centre of HEC Montréal, the École Polytechnique de Montréal and McGill University. In 1996, the Université du Québec à Montréal joined these three institutions. GERAD has fifty members (professors), more than twenty research associates and post doctoral students and more than two hundreds master and Ph.D. students.

GERAD is a multi-university center and a vital forum for the development of operations research. Its mission is defined around the following four complementarily objectives:

- The original and expert contribution to all research fields in GERAD's area of expertise;
- The dissemination of research results in the best scientific outlets as well as in the society in general;
- The training of graduate students and post doctoral researchers;
- The contribution to the economic community by solving important problems and providing transferable tools.

GERAD's research thrusts and fields of expertise are as follows:

- Development of mathematical analysis tools and techniques to solve the complex problems that arise in management sciences and engineering;
- Development of algorithms to resolve such problems efficiently;
- Application of these techniques and tools to problems posed in related disciplines, such as statistics, financial engineering, game theory and artificial intelligence;
- Application of advanced tools to optimization and planning of large technical and economic systems, such as energy systems, transportation/communication networks, and production systems;
- Integration of scientific findings into software, expert systems and decision-support systems that can be used by industry.

One of the marking events of the celebrations of the 25th anniversary of GERAD is the publication of ten volumes covering most of the Center's research areas of expertise. The list follows: **Essays and Surveys in Global Optimization**, edited by C. Audet, P. Hansen and G. Savard; **Graph Theory and Combinatorial Optimization**, edited by D. Avis, A. Hertz and O. Marcotte; **Numerical Methods in Finance**, edited by H. Ben-Ameur and M. Breton; **Analysis, Control and Optimization of Complex Dynamic Systems**, edited by E.K. Boukas and R. Malhamé; **Column Generation**, edited by G. Desaulniers, J. Desrosiers and M.M. Solomon; **Statistical Modeling and Analysis for Complex Data Problems**, edited by P. Duchesne and B. Rémillard; **Performance Evaluation and Planning Methods for the Next Generation Internet**, edited by A. Girard, B. Sansò and F. Vázquez-Abad; **Dynamic Games: Theory and Applications**, edited by A. Haurie and G. Zaccour; **Logistics Systems: Design and Optimization**, edited by A. Langevin and D. Riopel; **Energy and Environment**, edited by R. Loulou, J.-P. Waaub and G. Zaccour.

I would like to express my gratitude to the Editors of the ten volumes, to the authors who accepted with great enthusiasm to submit their work and to the reviewers for their benevolent work and timely response. I would also like to thank Mrs. Nicole Paradis, Francine Benoît and Louise Letendre and Mr. André Montpetit for their excellent editing work.

The GERAD group has earned its reputation as a worldwide leader in its field. This is certainly due to the enthusiasm and motivation of GERAD's researchers and students, but also to the funding and the infrastructures available. I would like to seize the opportunity to thank the organizations that, from the beginning, believed in the potential and the value of GERAD and have supported it over the years. These are HEC Montréal, École Polytechnique de Montréal, McGill University, Université du Québec à Montréal and, of course, the Natural Sciences and Engineering Research Council of Canada (NSERC) and the Fonds québécois de la recherche sur la nature et les technologies (FQRNT).

Georges Zaccour
Director of GERAD

Avant-propos

Le Groupe d'études et de recherche en analyse des décisions (GERAD) fête cette année son vingt-cinquième anniversaire. Fondé en 1980 par une poignée de professeurs et chercheurs de HEC Montréal engagés dans des recherches en équipe avec des collègues de l'Université McGill et de l'École Polytechnique de Montréal, le Centre comporte maintenant une cinquantaine de membres, plus d'une vingtaine de professionnels de recherche et stagiaires post-doctoraux et plus de 200 étudiants des cycles supérieurs. Les activités du GERAD ont pris suffisamment d'ampleur pour justifier en juin 1988 sa transformation en un Centre de recherche conjoint de HEC Montréal, de l'École Polytechnique de Montréal et de l'Université McGill. En 1996, l'Université du Québec à Montréal s'est jointe à ces institutions pour parrainer le GERAD.

Le GERAD est un regroupement de chercheurs autour de la discipline de la recherche opérationnelle. Sa mission s'articule autour des objectifs complémentaires suivants :

- la contribution originale et experte dans tous les axes de recherche de ses champs de compétence;
- la diffusion des résultats dans les plus grandes revues du domaine ainsi qu'auprès des différents publics qui forment l'environnement du Centre;
- la formation d'étudiants des cycles supérieurs et de stagiaires post-doctoraux;
- la contribution à la communauté économique à travers la résolution de problèmes et le développement de coffres d'outils transférables.

Les principaux axes de recherche du GERAD, en allant du plus théorique au plus appliqué, sont les suivants :

- le développement d'outils et de techniques d'analyse mathématiques de la recherche opérationnelle pour la résolution de problèmes complexes qui se posent dans les sciences de la gestion et du génie;
- la confection d'algorithmes permettant la résolution efficace de ces problèmes;
- l'application de ces outils à des problèmes posés dans des disciplines connexes à la recherche opérationnelle telles que la statistique, l'ingénierie financière, la théorie des jeux et l'intelligence artificielle;
- l'application de ces outils à l'optimisation et à la planification de grands systèmes technico-économiques comme les systèmes énergé-

tiques, les réseaux de télécommunication et de transport, la logistique et la distributique dans les industries manufacturières et de service;

- l'intégration des résultats scientifiques dans des logiciels, des systèmes experts et dans des systèmes d'aide à la décision transférables à l'industrie.

Le fait marquant des célébrations du 25e du GERAD est la publication de dix volumes couvrant les champs d'expertise du Centre. La liste suit : **Essays and Surveys in Global Optimization**, édité par C. Audet, P. Hansen et G. Savard; **Graph Theory and Combinatorial Optimization**, édité par D. Avis, A. Hertz et O. Marcotte; **Numerical Methods in Finance**, édité par H. Ben-Ameur et M. Breton; **Analysis, Control and Optimization of Complex Dynamic Systems**, édité par E.K. Boukas et R. Malhamé; **Column Generation**, édité par G. Desaulniers, J. Desrosiers et M.M. Solomon; **Statistical Modeling and Analysis for Complex Data Problems**, édité par P. Duchesne et B. Rémillard; **Performance Evaluation and Planning Methods for the Next Generation Internet**, édité par A. Girard, B. Sansò et F. Vázquez-Abad; **Dynamic Games: Theory and Applications**, édité par A. Haurie et G. Zaccour; **Logistics Systems: Design and Optimization**, édité par A. Langevin et D. Riopel; **Energy and Environment**, édité par R. Loulou, J.-P. Waaub et G. Zaccour.

Je voudrais remercier très sincèrement les éditeurs de ces volumes, les nombreux auteurs qui ont très volontiers répondu à l'invitation des éditeurs à soumettre leurs travaux, et les évaluateurs pour leur bénévolat et ponctualité. Je voudrais aussi remercier Mmes Nicole Paradis, Francine Benoît et Louise Letendre ainsi que M. André Montpetit pour leur travail expert d'édition.

La place de premier plan qu'occupe le GERAD sur l'échiquier mondial est certes due à la passion qui anime ses chercheurs et ses étudiants, mais aussi au financement et à l'infrastructure disponibles. Je voudrais profiter de cette occasion pour remercier les organisations qui ont cru dès le départ au potentiel et la valeur du GERAD et nous ont soutenus durant ces années. Il s'agit de HEC Montréal, l'École Polytechnique de Montréal, l'Université McGill, l'Université du Québec à Montréal et, bien sûr, le Conseil de recherche en sciences naturelles et en génie du Canada (CRSNG) et le Fonds québécois de la recherche sur la nature et les technologies (FQRNT).

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Preface

The GERAD is a well-established multi-university center and a vital forum for the development of analysis tools required for solving complex problems. Traditionally, realizations of the GERAD concerned mainly Operations Research. In recent years, Statistical Sciences also emerged within GERAD, with members specializing in both theoretical and applied statistics, probability theory and stochastic processes. One objective of the present book is to present contributions to some important directions of research in the fields of interest of the statistics group at GERAD.

In Abdous, Genest and Rémillard, the authors study the properties of copulas, such as stochastic orders and dependence measure, in the context of elliptical distributions. Andrews and Feuerverger analyze complex survey data. They discuss the controversial result of the 2000 American election in Florida County. Functional estimation is the main topic of two articles. In Angers and MacGibbon, the authors develop Fourier expansion for the hazard rates in a Bayesian framework, while Berlinet and Rouvière consider the estimation of multivariate densities, with special attention to the resulting computational problems. Croteau, Cléroux and Léger present an example of the application of bootstrap interval estimation to parameters of a complex process in a periodic replacement problem. Three papers explore statistical testing in various contexts, and two of them deal with dependent data. Dabrowski proposes a statistical test of comparisons to detect difference between genes. Bellavance and Tardif study the validity of F-ratio tests in the case of dependent errors due to cross-over designs. In Larocque, the author establishes properties of a signed rank test of Wilcoxon for correlated cluster data. The analysis of time series is a common feature of four articles. Bou-Hamad and Duchesne study asymptotic properties of robust estimators to outliers, for autoregressive models with exogenous variables. Dufour and Jouini establish the asymptotic behavior for multivariate autoregressive moving-average time series models. In Francq and Zakoian, linear models with dependent but uncorrelated innovations are reviewed. Hallin and Lofti investigate the detection of periodicities in vectorial autoregressive models. Data mining is covered by one article, in which Bengio and Grandvalet consider the estimation of uncertainty for machine learning algorithms. Finally, interacting stochastic processes are covered in two papers. In Dawson and Del Moral, the authors prove a result of large deviation for interactive processes when the strong topology is used, and Gentil, Rémillard and Del Moral present an efficient non linear filtering algorithm for the position detection of multiple targets.

We would like to thank all authors of the present volume for their contribution to the successful realization of this project. We would also like to express our gratitude to Nicole Paradis for her efficient editorial coordination.

PIERRE DUCHESNE
BRUNO RÉMILLARD

Chapter 1

DEPENDENCE PROPERTIES OF META-ELLIPTICAL DISTRIBUTIONS

Belkacem Abdous
Christian Genest
Bruno Rémillard

Abstract A distribution is said to be meta-elliptical if its associated copula is elliptical. Various properties of these copulas are critically reviewed in terms of association measures, concepts, and stochastic orderings, including tail dependence. Most results pertain to the bivariate case.

1. Introduction

The study of meta-elliptical multivariate distributions was recently launched by Fang, Fang and Kotz (2002), and their extension of the meta-Gaussian family of distributions due to Krzysztofowicz and Kelly (1996) is sure to find its way gradually into statistical, actuarial, economic and financial applications, where elliptically contoured distributions are already in common use. A forerunner example is provided by the work of Frey, McNeil and Nyfeler (2001), who use multivariate Student and generalized hyperbolic distributions to model credit portfolio losses.

A vector $Z = (Z_1, \ldots, Z_p)$ is said to be elliptically contoured if it admits the stochastic representation $Z = \mu + RAU$, where $\mu \in \mathbb{R}^p$, R and U are independent random variables, R is non-negative, U is uniformly distributed on the unit sphere in $\mathbb{R}^p$, and A is a fixed $p \times p$ matrix such that $AA^\top = \Sigma$ is non-singular. In particular when $\mu = 0$ and R is absolutely continuous, the density of Z is of the form

$$h(z) = |\Sigma|^{-1/2} g\left(z^\top \Sigma^{-1} z\right), \quad z \in \mathbb{R}^p$$

where g is a scale function uniquely determined by the distribution of R. If σ_i^2 denotes the (i,i)th entry of Σ, the variables Z_i/σ_i are then identically distributed with density

$$f(z) = \frac{\pi^{(p-1)/2}}{\Gamma\{(p-1)/2\}} \int_{z^2}^{\infty} \left(t - z^2\right)^{(p-1)/2-1} g(t)dt, \quad z \in \mathbb{R}$$

and cumulative distribution function

$$F(z) = \frac{1}{2} + \frac{\pi^{(p-1)/2}}{\Gamma\{(p-1)/2\}} \int_0^z \int_{s^2}^{\infty} \left(t - s^2\right)^{(p-1)/2-1} g(t)dtds, \quad z \in \mathbb{R}.$$

The special case $g(t) \propto e^{-at}$ with $a \in \mathbb{R}_+$ corresponds to Z being multivariate Gaussian. Other examples include the multivariate Cauchy, Student and logistic, as well as generalized multivariate hyperbolic, Kotz and symmetric Pearson type–VII distributions. See Cambanis, Huang and Simons (1981) or Fang, Kotz and Ng (1987), among others, for theory and applications of elliptically contoured distributions.

Paraphrasing Fang, Fang and Kotz (2002), a random vector $X = (X_1, \ldots, X_p)$ with cumulative distribution function K and continuous marginals $K_i(x) = \mathrm{P}(X_i \leq x)$ is said to be meta-elliptically distributed if the joint distribution of the variables $Z_i = F^{-1}\{K_i(X_i)\}$ is elliptical with scale function g and matrix Σ whose main diagonal entries are equal to unity. In other words, the dependence structure of the vectors X and Z is characterized by the same copula. The latter is termed meta-elliptical, to avoid possible confusion with the elliptical copulas recently introduced and studied by Kurowicka and Cooke (2001), and Kurowicka, Misiewicz and Cooke (2001).

The purpose of this note is to review critically some of the elementary dependence properties of meta-elliptical distributions, mostly in the bivariate case. All measures, concepts and orders of dependence to be considered here are defined in terms of the distribution's underlying copula. However, as noted by Joe (1997), Nelsen (1999) or Drouet–Mari and Kotz (2001) and references therein, these various notions are invariant under monotone increasing transformations of the components. Thus when comparing two bivariate meta-elliptical copulas with the same scale function g, it will often prove more convenient to work directly from the associated elliptical vectors, whose distribution will only then differ by their value of r, the off-diagonal entry of Σ.

The most common association measures, concepts and stochastic orderings characterizing bivariate dependence are considered in turn in Sections 2–4. Recent results pertaining to tail dependence are also referenced in the concluding section.

Before proceeding, it should be noted that contrary to formulas (2.2)–(2.4) in Fang, Fang and Kotz (2002), the joint density and marginals of an elliptical vector (Z_1, Z_2) with scale function g and matrix

$$\Sigma = \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}$$

are given respectively by

$$h(x, y) = \frac{1}{\sqrt{1-r^2}}\, g\left(\frac{x^2 + y^2 - 2rxy}{1 - r^2}\right) \tag{1.1}$$

and

$$F(z) = \begin{cases} \displaystyle\int_{z^2}^{\infty} \arccos\left(-z/\sqrt{t}\right) g(t)dt & \text{for } z \leq 0, \\ \displaystyle 1 - \int_{z^2}^{\infty} \arccos\left(z/\sqrt{t}\right) g(t)dt & \text{for } z > 0. \end{cases} \tag{1.2}$$

Note also that $r = \text{corr}(Z_1, Z_2)$ whenever the latter exists, that is, when $\int_0^\infty tg(t)dt < \infty$.

2. Measures of association

The two most common nonparametric measures of dependence are Spearman's rho and Kendall's tau. In dimension $p = 2$, their respective values can either be expressed as

$$\rho = 12\,\mathrm{E}\{K_1(X_1)K_2(X_2)\} - 3 \quad \text{and} \quad \tau = 4\,\mathrm{E}\{K(X_1, X_2)\} - 1$$

in terms of expectations involving the pair (X_1, X_2) with joint distribution K and marginals K_1 and K_2, or equivalently as

$$\rho = 12\,\mathrm{E}\{F(Z_1)F(Z_2)\} - 3 \quad \text{and} \quad \tau = 4\,\mathrm{E}\{H(Z_1, Z_2)\} - 1$$

in terms of the transformed variables $Z_i = F^{-1}\{K_i(X_i)\}$ with joint cumulative distribution function H and common marginal F.

Fang, Fang and Kotz (2002) show that

$$\tau = \frac{2}{\pi}\arcsin(r)$$

is independent of g. This result is reported also by Lindskog, McNeil and Schmock (2001) and Frahm, Junker and Szimayer (2003). Curiously, however, these various authors omit to mention that

$$\tau = 4\,\mathrm{P}\,(Z_1 \leq 0, Z_2 \leq 0) - 1$$

also corresponds to Blomqvist's medial correlation coefficient, since the median of F is zero. These observations extend at once to any bivariate marginal from a multivariate meta-elliptical distribution.

In contrast, ρ generally depends on both g and r. Indeed, a simple change of variables yields

$$\rho = 12 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(x) F\left(xr + y\sqrt{1-r^2}\right) g\left(x^2 + y^2\right) dy dx - 3.$$

Note that while this formula does involve F, as stated by Fang, Fang and Kotz (2002), the dependence of ρ on this marginal distribution is only apparent, as the latter is entirely determined by g through (1.2).

Closed-form expressions for ρ rarely seem possible. A well-known exception is the bivariate normal distribution, for which Kruskal (1958) established that $\rho = 6 \arcsin(r/2)/\pi$. See Hult and Lindskog (2002) for another example of explicit calculation.

For fixed scale function g, it is obvious that τ is a continuous, strictly increasing function of r that ranges from -1 to 1. To show that ρ enjoys the same properties, one can call on the continuity of F and Lebesgue's Dominated Convergence Theorem; its strict monotonicity follows from Proposition 1.4, to be established in Section 4.

The extreme cases where $r = \pm 1$ describe situations of perfect functional dependence, namely Fréchet's upper and lower bound; one then has $\tau = \rho = \pm 1$. It is important, however, to realize that $r = 0$ does *not* correspond to stochastic independence, except in the meta-Gaussian case.

PROPOSITION 1.1 *If (X_1, X_2) is a meta-elliptical random pair with scale function g and dependence parameter r, then*

$$\rho(X_1, X_2) = \tau(X_1, X_2) = 0 \Leftrightarrow r = 0.$$

Furthermore if X_1 and X_2 are independent, then $g(t) = e^{-t/2}/(2\pi)$.

Proof. That $\tau(X_1, X_2) = 0 \Leftrightarrow r = 0$ is immediate from the formula for Kendall's tau. Furthermore, stochastic independence between X_1 and X_2 implies $\tau = r = 0$, in which case it follows from Lemma 5 of Kelker (1970) that the variables $Z_i = F^{-1}\{K_i(X_i)\}$ are Gaussian.

Now when $r = 0$, the pairs $(\pm Z_1, \pm Z_2)$ all have the same joint distribution, whose common marginal F is also symmetric with respect to the origin. Thus if Z_1', Z_2' are independent observations from F, one has

$$\mathrm{P}\left(Z_1' \leq Z_1, Z_2' \leq Z_2\right) = \mathrm{P}\left(\epsilon_1 Z_1' \leq \epsilon_1 Z_1, \epsilon_2 Z_2' \leq \epsilon_2 Z_2\right)$$

with $\epsilon_1, \epsilon_2 \in \{-1, 1\}$. Since the probabilities of the four possible events on the right sum up to one, it follows that

$$\mathrm{E}\{F(Z_1)F(Z_2)\} = \mathrm{P}\left(Z_1' \leq Z_1, Z_2' \leq Z_2\right) = 1/4$$

and hence $\rho = 0$.

To establish the reverse implication, observe that if $\rho(r_0) = 0$ held true for some $r_0 > 0$, say, one would then have $\rho(r) \equiv 0$ on $[0, r_0]$, contradicting the strictly increasing nature of Spearman's rho as a function of r, for fixed g. □

Because it is based on a symmetry property of elliptically contoured vectors with $r = 0$, the chain of implications in Proposition 1.1 can be verified for other copula-based measures of association such as Gini's coefficient or the asymmetric and symmetric versions of Blest's index of association, respectively studied by Blest (2000) and Genest and Plante (2003). Although these additional nonparametric measures of association cannot be expressed explicitly, they too are increasing in r and sweep all degrees of dependence between -1 and 1.

3. Concepts of dependence

In the classical work of Lehmann (1966), the weakest notion of association between the components of a random pair (X_1, X_2) with distribution K and marginals K_1 and K_2 is that of positive quadrant dependence. This condition, which is met whenever $K(x, y) \geq K_1(x)K_2(y)$ for all $x, y \in \mathbb{R}$, guarantees that ρ, τ and the other above-mentioned nonparametric measures of dependence are non-negative.

A stronger notion, alternatively called monotone regression dependence or stochastic monotonicity of X_2 in X_1, requires $\mathrm{P}(X_2 > y | X_1 = x)$ to be increasing in x for every fixed value of $y \in \mathbb{R}$. However, the most stringent condition considered by Lehmann (1966) is likelihood ratio dependence, which is verified if K is absolutely continuous and if its density k is TP_2 in the sense of Karlin (1968), namely

$$k(x, y')\, k(x', y) \leq k(x, y)\, k(x', y') \tag{1.3}$$

for all $x \leq x'$ and $y \leq y'$. The latter requirement implies monotone regression dependence, but also several other notions of dependence, such as association, left-tail decreasingness, right-tail increasingness, etc.

In their paper, Fang, Fang and Kotz (2002) claim that every pair of meta-elliptically contoured random variables with $r \geq 0$ is likelihood ratio dependent. As the property is really an attribute of the copula associated with K, their Theorem 3.3 amounts to saying that the density (1.1) of any absolutely continuous elliptical vector (Z_1, Z_2) should satisfy condition (1.3). This is well known to hold true in the bivariate Gaussian case; see Rüschendorf (1981) for a multivariate extension of this result. Unfortunately, the claim turns out to be *false* in general, as stated next.

PROPOSITION 1.2 *Assume that an elliptically contoured distribution has a scale function g such that $\phi(t) = \log\{g(t)\}$ is twice differentiable. In* *order that this elliptically contoured distribution be likelihood ratio dependent for a given $r \in [0,1)$, it is necessary and sufficient that $\phi''(t) = 0$ whenever $\phi'(t) = 0$ and that*

$$-\frac{r}{1+r} \leq \inf_{t \in T} \frac{t\phi''(t)}{\phi'(t)} \leq \sup_{t \in T} \frac{t\phi''(t)}{\phi'(t)} \leq \frac{r}{1-r},$$

where $T = \{t \in \mathbb{R}_+ : \phi'(t) < 0\}$. In particular, if this elliptically contoured distribution is likelihood ratio dependent for $r = 0$, then it is Gaussian.

Proof. If h denotes the density of the elliptically contoured distribution with scale function g and dependence parameter $r \geq 0$, the assumed conditions on ϕ imply that property (1.3) holds true if and only if

$$\frac{\partial^2}{\partial x \partial y} \log\{h(x,y)\} \geq 0$$

everywhere on $\mathbb{R}^2$. Introduce the change of variables

$$x = \sqrt{t}\sin(\theta), \quad \frac{y - rx}{\sqrt{1-r^2}} = \sqrt{t}\cos(\theta),$$

so that $(x^2+y^2-2rxy)/(1-r^2) = t$. The condition on h is then satisfied if and only if

$$2\cos\theta\left(\sqrt{1-r^2}\sin\theta - r\cos\theta\right)t\phi''(t) \geq r\phi'(t) \tag{1.4}$$

for every $t \in \mathbb{R}_+$ and $\theta \in [0, 2\pi]$. Letting $r = \sin\alpha$ for some $\alpha \in [0, \pi/2)$, and in view of the trigonometric identity

$$\cos\theta\left(\sqrt{1-r^2}\sin\theta - r\cos\theta\right) = \cos\theta\sin(\theta - \alpha),$$

one may then reexpress (1.4) in the form

$$2\cos\theta\sin(\theta-\alpha)t\phi''(t) \geq r\phi'(t), \tag{1.5}$$

which again must be valid for every $t \in \mathbb{R}_+$ and $\theta \in [0, 2\pi]$.

Now it is easy to check that $A_r(\theta) \equiv 2\cos\theta\sin(\theta-\alpha)$ takes all possible values in the interval $[-1-r, 1-r]$ as θ varies in $[0, 2\pi]$. In particular if $r = \alpha = 0$, it follows at once that $\phi''(t) = 0$ for all $t > 0$, because this is the only way that the left-hand side of (1.5) can be non-negative for all choices of θ. Accordingly, one then has $\phi(t) = -at + b$ and hence $g(t) \propto e^{-at}$ for some $a \in \mathbb{R}_+$, which entails that h is Gaussian.

Next, for fixed $r \in (0,1)$, letting $\theta = \alpha$ in (1.5) shows that $\phi'(t)$ is necessarily non-positive on its entire domain. Consequently, the condition on ϕ is met if and only if $\phi''(t) = 0$ whenever $\phi'(t) = 0$ and, in addition,

$$A_r(\theta)\,\frac{t\phi''(t)}{\phi'(t)} \leq r$$

for all $t \in T$, which may be reformulated as in the statement of the proposition. □

As an illustration, take $g(t) \propto \exp(-\beta t^\alpha)$ with α, $\beta \in \mathbb{R}_+$. Then $t\phi''(t)/\phi'(t) = \alpha - 1$. The elliptically contoured distribution generated by g is thus likelihood ratio dependent whenever $r \geq \max(1-1/\alpha, 1/\alpha - 1)$ and $r \neq 0$, which can happen only when $\alpha > 1/2$. As another example, note that when $g(t) \propto t^{\gamma-1}\exp(-\beta t^\alpha)$ with α, β, $\gamma \in \mathbb{R}_+$, and $\gamma \neq 1$, the corresponding Kotz-type elliptically contoured distribution is not likelihood ratio dependent for any $r \in [0,1)$, since $\lim_{t\to 0} t\phi''(t)/\phi'(t) = -1 < -r/(1+r)$ for any such value of r.

The following result represents a strengthening of Proposition 1.2 in the special case where $r = 0$.

PROPOSITION 1.3 *If an elliptically contoured distribution is positive quadrant dependent at $r = 0$, then it is Gaussian.*

Proof. It was observed earlier (see the proof of Proposition 1.1) that when $r = 0$, the four pairs $(\pm Z_1, \pm Z_2)$ have the same distribution. Thus in particular

$$\begin{aligned} H(x,y) &= \mathrm{P}\,(Z_1 \leq x, Z_2 \leq y) \\ &= \mathrm{P}\,(-Z_1 \leq x, Z_2 \leq y) = F(y) - H(-x,y) \end{aligned}$$

for arbitrary $x, y \in \mathbb{R}$, so that if the pair $(Z_1,\ Z_2)$ is positive quadrant dependent, then

$$\begin{aligned} F(x)F(y) &\leq H(x,y) = F(y) - H(-x,y) \\ &\leq F(y) - F(-x)F(y) = F(x)F(y), \end{aligned}$$

whence $H(x,y) = F(x)F(y)$ for all $x, y \in \mathbb{R}$. In other words, Z_1 and Z_2 are independent and hence Gaussian by Lemma 5 of Kelker (1970). □

The identification of general conditions under which meta-elliptical copulas are positive quadrant dependent poses an interesting challenge. Until this open problem has been solved, caution should be exerted in modelling association with structures of this sort. For fixed scale function g, it will nevertheless be seen below that the association parameter

r orders meta-elliptical distributions by their relative degree of dependence, as characterized by various stochastic orderings. These facts may serve to justify the use of meta-elliptical distributions in robustness and power studies.

4. Stochastic orderings

Two random vectors X and X^* with respective joint distributions K and K^* and the same univariate marginals are said to be ordered by a dependence ordering $\prec$ whenever the relation $X \prec X^*$ implies that the degree of association among the components of X^* is higher than between the components of X. Generally, $\prec$ reduces to a concept of positive dependence when K is the product of the marginals.

In dimension $p = 2$, the standard extension of Lehmann's notion of monotone regression dependence is the ordering $\prec_1$ of Yanagimoto and Okamoto (1969). Let $K_x(y) = \mathrm{P}(X_2 \le y | X_1 = x)$ and write $K_{x',x} = K_{x'} \circ K_x^{-1}$. Define K_x^* and $K_{x',x}^*$ *mutatis mutandis* for another pair (X_1^*, X_2^*) with distribution K^* having the same marginals as K. Following Capéraà and Genest (1990), $(X_1, X_2) \prec_1 (X_1^*, X_2^*)$ if and only if the implication

$$x \le x' \Rightarrow K_{x',x}^*(u) \le K_{x',x}(u) \tag{1.6}$$

is valid for all $u \in (0, 1)$. The choice of marginals is immaterial, so long as they are the same for both distributions. As it is actually the copulas that are being compared through $\prec_1$, the relation $X \prec_1 X^*$ implies that $\rho \le \rho^*$ and $\tau \le \tau^*$, and likewise for the indices of Gini, Blest (2000) or Genest and Plante (2003), among others.

It is a simple matter to see that if two meta-elliptical distributions correspond to the same scale function g, they are then ordered by their parameter r in Yanagimoto and Okamoto's relation $\prec_1$. Equivalently, one has the following result.

PROPOSITION 1.4 *Let (Z_1, Z_2), (Z_1^*, Z_2^*) be observations from two elliptically contoured distributions with the same scale function g but different parameters r and r^*, respectively. Then $(Z_1, Z_2) \prec_1 (Z_1^*, Z_2^*) \Leftrightarrow r \le r^*$.*

Proof. Introduce

$$L_x(y) = \int_{-\infty}^{y} g\left(x^2 + t^2\right) dt \Big/ \int_{-\infty}^{\infty} g\left(x^2 + t^2\right) dt,$$

which is nothing but the cumulative distribution function of Z_2 given $Z_1 = x$ when $r = 0$. Note that more generally,

$$H_x(y) = \mathrm{P}\left(Z_2 \le y | Z_1 = x\right) = L_x\left(\frac{y - rx}{\sqrt{1 - r^2}}\right),$$

so that

$$H_{x',x}(u) = H_{x'}\left\{H_x^{-1}(u)\right\} = L_{x'}\left\{L_x^{-1}(u) + (x - x')q(r)\right\}$$

with $q(r) = r/\sqrt{1-r^2}$. Implication (1.6) is then immediate from the fact that $q(r)$ is monotone increasing on $(0, 1)$. □

As a consequence of this proposition, meta-elliptical distributions H_r generated by the same scale function g are ordered by their values of r in the positive quadrant dependence ordering $\prec_0$, defined by $H \prec_0 H^*$ if and only $H(x, y) \le H^*(x, y)$ for all $x, y \in \mathbb{R}$. Thus if $r \le r^*$, one has $H_r \le H_{r^*}$ and hence by Hoeffding's identity,

$$\rho(r^*) - \rho(r) = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{H_{r^*}(x, y) - H_r(x, y)\}\, dF(y) dF(x) \ge 0,$$

with equality if and only if $r = r^*$, which shows the strict increasingness of ρ as a function of r.

In Capéraà and Genest (1990), a stronger ordering than $\prec_1$ was proposed which extends the notion of likelihood ratio dependence to the comparison of two arbitrary copulas. Specifically, write $(Z_1, Z_2) \prec_3 (Z_1^*, Z_2^*)$ if and only if

$$J(u) = (x' - x)\, \frac{d}{du}\, H^*_{x',x}(u) \Big/ \frac{d}{du}\, H_{x',x}(u)$$

is increasing in u for all choices of $x, x' \in \mathbb{R}$. These authors showed, among others, that Gaussian distributions are ordered in $\prec_3$ by their correlation coefficient. The following result extends their finding to meta-elliptical distributions under mild conditions on the scale function g.

PROPOSITION 1.5 *Let (Z_1, Z_2), (Z_1^*, Z_2^*) be observations from two elliptically contoured distributions with the same scale function g but different parameters r and r^*, respectively. Further assume that g is twice differentiable, decreasing and log-concave. Then $(Z_1, Z_2) \prec_3 (Z_1^*, Z_2^*) \Leftrightarrow r \le r^*$.*

Proof. It follows at once from the definitions of $H_{x',x}$ and $H^*_{x',x}$ that

$$J(u) = (x' - x)\, \frac{g\left[(x')^2 + \left\{(x - x')\, q(r^*) + L_x^{-1}(u)\right\}^2\right]}{g\left[(x')^2 + \left\{(x - x')\, q(r) + L_x^{-1}(u)\right\}^2\right]}.$$

Thus if $\Delta = x' - x$, $s = x'$, $t = L_x^{-1}(u) - \Delta q(r^*)$ and $q = q(r^*) - q(r) \geq 0$, what needs to be shown is that

$$\Delta \frac{g\left(s^2 + t^2\right)}{g\left\{s^2 + (t + \Delta q)^2\right\}}$$

is increasing in t for every Δ and s. Equivalently, one must check that

$$\Delta(t + \Delta q) \frac{g'\{s^2 + (t + \Delta q)^2\}}{g\{s^2 + (t + \Delta q)^2\}} \leq \Delta t \frac{g'(s^2 + t^2)}{g(s^2 + t^2)}$$

for fixed $q \geq 0$ and arbitrary s, t, Δ. Since $(t + \Delta q) - t$ is of the same sign as Δ, the conclusion clearly obtains if

$$t \frac{g'(s^2 + t^2)}{g(s^2 + t^2)}$$

is decreasing in t for arbitrary s. As this is guaranteed by the log-concavity and decreasingness of g, the argument is complete. □

The conditions of Proposition 1.5 seem close to minimal. It is easy to check, for example, that the bivariate Cauchy distributions, which are generated by the log-convex function $g(t) \propto (1 + t)^{-3/2}$, are not ordered by their values of r in the $\prec_3$ relation.

Naturally, the assumptions that g is twice differentiable, decreasing and log-concave also imply that as r increases, an observation from (Z_1, Z_2) from a meta-elliptical distribution becomes more and more left-tail decreasing and right-tail increasing, in the sense given to those orderings by Avérous and Dortet–Bernadet (2000). This is because their orders, like those denoted $\prec_{2-}$ and $\prec_{2+}$ by Capéraà and Genest (1990), are intermediate between $\prec_1$ and $\prec_3$.

Recently, Genest and Verret (2002) showed that bivariate Gaussian distributions are ordered by an even stronger relation than $\prec_3$ due to Kimeldorf and Sampson (1987). It would be interesting to know whether this property is shared by other meta-elliptical families of distributions.

5. Discussion

In years past, elliptically contoured distributions have proved a useful alternative to the multivariate Gaussian paradigm, not only in statistics, but in several areas of applications such as actuarial science, economics and finance. They have played a prominent role, notably in robustness studies and in modelling multidimensional populations with heavy tails and dependent extreme values.

It is only natural that similar applications should jump to mind for the class of meta-elliptical distributions introduced by Fang, Fang and

Kotz (2002). Among their advantages are the facts that for fixed scale function g: (i) they can have arbitrary marginals; (ii) they cover all possible degrees of association (as measured by Kendall's tau, say); and (iii) variations in the parameter r are consistent with the monotone regression dependence relation $\prec_1$ and (subject to appropriate conditions on g) with the stronger ordering $\prec_3$.

There are, however, two serious limitations to modelling association with meta-elliptical copulas. First, no value of r corresponds to independence, unless the copula is actually that of the normal. Second, except in that special meta-Gaussian case or in the conditions of Proposition 1.2, it is not entirely clear under which circumstances elliptically contoured structures of association meet the concept of positive quadrant dependence. Yet, the latter represents a bare minimum in many applications.

To illustrate this point, consider a vector $X = (X_1, \dots, X_p)$ of insurance claim amounts and their total $S = X_1 + \cdots + X_p$. One common measure of the portfolio's riskiness is given by the stop-loss premium

$$\pi(S, d) = \mathrm{E}\{\max(0, S - d)\}, \quad d \geq 0.$$

Given marginal distributions $K_1, \dots, K_p$ and two possible dependence structures C and C^* for the vector X, Müller (1997) showed that the stop-loss premiums may be ordered for all retention amounts $d \geq 0$, provided that the two copulas are ordered in the supermodular ordering $\prec_{\mathrm{SM}}$. Specifically, one has $C \prec_{\mathrm{SM}} C^*$ if and only if

$$\mathrm{E}_C\{\varphi(X)\} \leq \mathrm{E}_{C^*}\{\varphi(X)\}$$

holds true for all supermodular functions φ for which both expectations exist (for twice differentiable functions φ, supermodularity is equivalent to the condition that $\partial^2\varphi(x_1, \dots, x_p)/\partial x_i \partial x_j \geq 0$ for all $i \neq j$). However if C stands for the independence copula and $p = 2$, $C \prec_{\mathrm{SM}} C^*$ could not possibly hold true unless C^* is at least positive quadrant dependent. Thus if actuaries chose to model the dependence between insurance contracts using a meta-elliptical distribution, they could not tell offhand whether the total loss associated with this portfolio is smaller or larger than under mutual independence of the risks.

As a second example, suppose that amounts $\alpha_1, \dots, \alpha_p$ have been invested in $p \geq 2$ dependent assets. Let $X_1, \dots, X_p$ and $X_1^*, \dots, X_p^*$ represent their returns under two stochastic models with possibly different marginals but the same underlying copula structure C. Müller and Scarsini (2001) show that if C is conditionally increasing and $\pi(X_i, d) \leq \pi(X_i^*, d)$ for every $d \geq 0$ and $i \in \{1, \dots, p\}$, then

$$\pi\left(\sum_{i=1}^{p}\alpha_i X_i, d\right) \le \pi\left(\sum_{i=1}^{p}\alpha_i X_i^*, d\right)$$

uniformly in $d \ge 0$. Equivalently, if $\mathrm{E}\{\varphi(X_i)\} \le \mathrm{E}\{\varphi(X_i^*)\}$ for every $i \in \{1, \ldots, p\}$ and every convex function φ for which these expectations exist, then also

$$\mathrm{E}\left\{\varphi\left(\sum_{i=1}^{p}\alpha_i X_i\right)\right\} \le \mathrm{E}\left\{\varphi\left(\sum_{i=1}^{p}\alpha_i X_i^*\right)\right\}$$

with φ any convex utility function.

In view of the work of Karlin and Rinott (1980), conditional increasingness of a $p-$variate copula C is verified as soon as it is absolutely continuous with density c such that

$$c(x)c(y) \le c(x \wedge y)c(x \vee y)$$

for all $x, y \in \mathbb{R}^p$. Here, $\wedge$ and $\vee$ refer to the componentwise minimum and maximum operators. The latter concept, termed multivariate total positivity of order two, reduces to (1.3) in dimension $p = 2$. As mentioned before, it holds true for the Gaussian distribution but *contra* Theorem 3.3 of Fang, Fang and Kotz (2002), it is not generally verified for other meta-elliptical distributions. Proposition 1.2 delineates conditions under which it is.

Another attractive feature of the class of meta-elliptical distributions is the fact that some of its members exhibit tail dependence, in the sense given to that term in Chapter 2 of the book by Joe (1997). It is this property of the multivariate Student and generalized hyperbolic distributions that motivated Frey, McNeil and Nyfeler (2001) to use them in modelling credit portfolio losses.

In dimension $p = 2$, a pair (X_1, X_2) with marginals K_1 and K_2 is said to exhibit (upper) tail dependence whenever the index

$$\lambda = \lim_{u \to 1} \mathrm{P}\left\{X_2 > K_2^{-1}(u) | X_1 > K_1^{-1}(u)\right\}$$

is well defined and strictly positive. For meta-elliptical vectors, this copula-based property is equivalent to the condition

$$\lambda = \lim_{z \to \infty} \mathrm{P}\left(Z_2 > z | Z_1 > z\right) > 0$$

where as before, $Z_i = F^{-1}\{K_i(X_i)\}$, $i = 1, 2$. Schmidt (2002) recently showed that this behavior is induced by a property of regular variation (at infinity) of the scale function g. Namely if there exists $\beta > 0$ such that

$$\lim_{t\to\infty} \frac{g(xt)}{g(t)} = x^{-1-\beta/2}$$

for all $x > 0$, then

$$\lambda = \int_0^{\sqrt{(1+r)/2}} \frac{t^\beta}{\sqrt{1-t^2}}\, dt \Big/ \int_0^1 \frac{t^\beta}{\sqrt{1-t^2}}\, dt.$$

See Schmidt (2002) for a formulation and proof of this result in arbitrary dimension p. Note that a typographical error in his final expression (he writes $t^2 - 1$ instead of $1 - t^2$ under the two square roots) is repeated by Frahm, Junker and Szimayer (2003), who give alternative expressions and illustrations in special cases. Refer also to Abdous, Fougères and Ghoudi (2005) for a more general study of the asymptotic behavior of the conditional distribution of one component on the other in a bivariate meta-elliptical distribution.

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Chapter 2

THE STATISTICAL SIGNIFICANCE OF PALM BEACH COUNTY

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Abstract This paper emphasizes certain issues and problems that arise when a statistical analysis must be undertaken on complex and evolving data, under tight constraints of time. In such circumstances, it typically is not possible to develop extensive or problem-specific methodology, yet an answer may be required almost immediately, and must be correct, defensible, understandable, and carry impact. It must also be able to withstand the test of comparison with analyses yet to come.

We illustrate these points by presenting the background to, and an analysis of, the State of Florida results in the 7 November, 2000 U.S. Presidential elections with emphasis on Palm Beach County. The analysis we discuss was carried out in the days immediately following that election. The statistical evidence strongly suggested that the use of the 'butterfly' ballot in Palm Beach County had resulted in a significant number of votes having been counted for presidential candidate Pat Buchanan which had not so been intended. The design of the 'butterfly' ballot suggests that many of these votes had likely been intended for the Democratic candidate Al Gore. This confusion was sufficient to affect the overall outcome of the 2000 U.S. Presidential election, conferring the office to George W. Bush, and this result is statistically significant.

1. Mise en scène

On the evening of Tuesday November 7, 2000, the United States of America, along with much of the world, found itself in a state of suspended animation as a consequence of an inconclusive outcome to the U.S. federal election. While history will record the remarkable circumstances of that day, and its subsequent consequences, it should be borne in mind that at the core of these events and ensuing controversies, no discipline played a more substantive role than Statistics.