Methods for Constructing Exact Solutions of Partial Differential Equations

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METHODS FOR CONSTRUCTING EXACT SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

Mathematical and Analytical Techniques with Applications to Engineering

S. V. MELESHKO
## Contents

Preface xi

1 Notes to the reader xii

2 Organization of the book xiii

3 Acknowledgments xv

1. EQUATIONS WITH ONE DEPENDENT FUNCTION 1

1 Basic definitions and examples 2
   1.1 Replacement of the independent variables 3
   1.2 Functional dependence. 4

2 The Cauchy method 5

3 Complete and singular integrals 9

4 Systems of linear equations 14

5 Tangent transformations 18
   5.1 The Legendre transformation 19
   5.2 The Darboux equation 20
   5.3 The Hopf–Cole transformation 21
   5.4 The Bäcklund transformation 22

6 A linear hyperbolic equation 23

7 Construction of particular solutions 26
   7.1 Separation of variables 26
   7.2 Self–similar solutions 27
   7.3 Travelling waves 28
   7.4 Partial representation 30

8 Functionally invariant solutions 32
   8.1 Erugin’s method 35
   8.2 Generalized functionally invariant solutions 37

9 Intermediate integrals 39
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.1</td>
<td>Application to a hyperbolic second order equation</td>
<td>39</td>
</tr>
<tr>
<td>9.2</td>
<td>Application to the gas dynamic equations</td>
<td>41</td>
</tr>
<tr>
<td>2.1</td>
<td>The problem of stretching an elastic–plastic bar</td>
<td>49</td>
</tr>
<tr>
<td>3</td>
<td>Hodograph method</td>
<td>50</td>
</tr>
<tr>
<td>4</td>
<td>Self–similar solutions</td>
<td>52</td>
</tr>
<tr>
<td>4.1</td>
<td>Definitions and basic properties</td>
<td>52</td>
</tr>
<tr>
<td>4.2</td>
<td>Self–similar solutions in an inviscid gas</td>
<td>57</td>
</tr>
<tr>
<td>4.3</td>
<td>An intense explosion in a gas</td>
<td>58</td>
</tr>
<tr>
<td>5</td>
<td>Solutions with a linear profile of velocity</td>
<td>60</td>
</tr>
<tr>
<td>6</td>
<td>Travelling waves</td>
<td>61</td>
</tr>
<tr>
<td>7</td>
<td>Completely integrable systems</td>
<td>63</td>
</tr>
<tr>
<td>3.1</td>
<td>General theory</td>
<td>71</td>
</tr>
<tr>
<td>3.2</td>
<td>Isentropic flows of a gas</td>
<td>73</td>
</tr>
<tr>
<td>4</td>
<td>Double waves</td>
<td>76</td>
</tr>
<tr>
<td>4.1</td>
<td>Homogeneous $2n - 1$ equations</td>
<td>76</td>
</tr>
<tr>
<td>4.2</td>
<td>Four quasilinear homogeneous equations</td>
<td>79</td>
</tr>
<tr>
<td>4.2.1</td>
<td>Equivalence transformations</td>
<td>80</td>
</tr>
<tr>
<td>4.2.2</td>
<td>Solution of system (3.35)</td>
<td>82</td>
</tr>
<tr>
<td>4.2.3</td>
<td>Solutions of system (3.36)</td>
<td>82</td>
</tr>
<tr>
<td>4.2.4</td>
<td>Classification of plane isentropic double waves of gas flows</td>
<td>88</td>
</tr>
<tr>
<td>4.3</td>
<td>Unsteady space nonisentropic double waves of a gas</td>
<td>93</td>
</tr>
<tr>
<td>4.3.1</td>
<td>The case $H \neq 0$</td>
<td>94</td>
</tr>
<tr>
<td>4.3.2</td>
<td>The case $H = 0$</td>
<td>101</td>
</tr>
<tr>
<td>5</td>
<td>Double waves in a rigid plastic body</td>
<td>109</td>
</tr>
<tr>
<td>5.1</td>
<td>Unsteady plane waves</td>
<td>109</td>
</tr>
<tr>
<td>5.1.1</td>
<td>Double waves</td>
<td>109</td>
</tr>
<tr>
<td>5.2</td>
<td>Steady three-dimensional double waves</td>
<td>114</td>
</tr>
<tr>
<td>5.2.1</td>
<td>Functionally independent $v_1$ and $v_2$</td>
<td>115</td>
</tr>
<tr>
<td>5.2.2</td>
<td>The case $v_i = v_i(v_1)$, $(i = 2, 3)$</td>
<td>124</td>
</tr>
</tbody>
</table>
## Contents

6 Triple waves of isentropic potential gas flows 125

4. METHOD OF DIFFERENTIAL CONSTRAINTS 131

1 Theory of compatibility 132
2 Method formulation 135
3 Quasilinear systems with two independent variables 136
   3.1 Involutive conditions 137
   3.2 Theorems of Existence 140
   3.3 Characteristic curves 143
   3.4 Generalized simple waves 145
   3.4.1 Compatibility conditions 146
   3.4.2 Integration method 148
   3.4.3 Centered rarefaction waves 149

4 Generalized simple waves in gas dynamics 150
   4.1 One-dimensional gas dynamics equations 151
   4.2 Two-dimensional gas dynamic equations 155
   4.3 Example of differential constraint of higher order 156

5 Multidimensional quasilinear systems 157
   5.1 Involutive conditions 157
   5.2 Differential constraints admitted by the gas
dynamics equations 159
      5.2.1 Irrotational gas flows 159
      5.2.2 One differential constraint 160

6 One-parameter Lie-Bäcklund group of transformations 161

7 One class of solutions 165

5. INVARIANT AND PARTIALLY INVARIANT SOLUTIONS 169

1 The main definitions 170
   1.1 Local Lie group of transformations 170
   1.2 Invariant manifolds 177
   1.3 Admitted Lie group 179
   1.4 Algorithm of finding an admitted Lie group 181
   1.5 Example of finding an admitted Lie group 182
   1.6 Lie algebra of generators 184
   1.7 Classification of subalgebras 187
   1.8 Classification of subalgebras of algebra (5.19) 191
   1.9 On classification of high dimensional Lie algebras 192

2 Group classification 193
   2.1 Equivalence transformations 193
      2.1.1 Examples and remarks about an equivalence group 196
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1.2</td>
<td>Group classification of equation (5.16)</td>
<td>198</td>
</tr>
<tr>
<td>3</td>
<td>Multi-parameter Lie group of transformations</td>
<td>200</td>
</tr>
<tr>
<td>4</td>
<td>Invariant solutions</td>
<td>202</td>
</tr>
<tr>
<td>4.1</td>
<td>The main definitions</td>
<td>202</td>
</tr>
<tr>
<td>4.2</td>
<td>Invariant solutions of equation (5.16)</td>
<td>204</td>
</tr>
<tr>
<td>5</td>
<td>Group classification of two–dimensional steady gas dynamics equations</td>
<td>205</td>
</tr>
<tr>
<td>5.1</td>
<td>Equivalence transformations</td>
<td>205</td>
</tr>
<tr>
<td>5.2</td>
<td>Admitted group</td>
<td>206</td>
</tr>
<tr>
<td>5.3</td>
<td>Optimal system of subalgebras</td>
<td>209</td>
</tr>
<tr>
<td>5.4</td>
<td>Invariant solutions</td>
<td>211</td>
</tr>
<tr>
<td>6</td>
<td>Partially invariant solutions</td>
<td>212</td>
</tr>
<tr>
<td>7</td>
<td>Partially invariant solutions of a non admitted Lie group</td>
<td>214</td>
</tr>
<tr>
<td>8</td>
<td>Some classes of partially invariant solutions</td>
<td>216</td>
</tr>
<tr>
<td>8.1</td>
<td>The Navier–Stokes equations</td>
<td>216</td>
</tr>
<tr>
<td>8.1.1</td>
<td>One class of solutions</td>
<td>217</td>
</tr>
<tr>
<td>8.1.2</td>
<td>Compatibility conditions</td>
<td>218</td>
</tr>
<tr>
<td>8.2</td>
<td>One class of irregular partially invariant solutions</td>
<td>220</td>
</tr>
<tr>
<td>9</td>
<td>The Pukhnachov method</td>
<td>222</td>
</tr>
<tr>
<td>9.1</td>
<td>Rotationally symmetric motion of an ideal incompressible fluid</td>
<td>223</td>
</tr>
<tr>
<td>9.2</td>
<td>Application to a one dimensional gas flow</td>
<td>225</td>
</tr>
<tr>
<td>10</td>
<td>Nonclassical, weak and conditional symmetries</td>
<td>226</td>
</tr>
<tr>
<td>10.1</td>
<td>Nonclassical symmetries</td>
<td>227</td>
</tr>
<tr>
<td>10.1.1</td>
<td>Remark about involutive conditions</td>
<td>228</td>
</tr>
<tr>
<td>10.2</td>
<td>Illustrative example of nonclassical symmetries</td>
<td>228</td>
</tr>
<tr>
<td>10.3</td>
<td>Weak and conditional symmetries</td>
<td>230</td>
</tr>
<tr>
<td>10.3.1</td>
<td>Weak symmetries</td>
<td>230</td>
</tr>
<tr>
<td>10.3.2</td>
<td>Conditional symmetries</td>
<td>231</td>
</tr>
<tr>
<td>10.4</td>
<td>B–symmetries</td>
<td>231</td>
</tr>
<tr>
<td>11</td>
<td>Group of tangent transformations</td>
<td>232</td>
</tr>
<tr>
<td>11.1</td>
<td>Lie groups of finite order tangency</td>
<td>232</td>
</tr>
<tr>
<td>11.2</td>
<td>An admitted Lie group of tangent transformations</td>
<td>237</td>
</tr>
<tr>
<td>11.3</td>
<td>Contact transformations of the Monge–Ampere equation</td>
<td>239</td>
</tr>
<tr>
<td>11.4</td>
<td>Lie–Bäcklund operators</td>
<td>242</td>
</tr>
<tr>
<td>11.4.1</td>
<td>Boussinesq equation</td>
<td>244</td>
</tr>
<tr>
<td>11.4.2</td>
<td>Nontrivial Lie–Bäcklund operators</td>
<td>244</td>
</tr>
</tbody>
</table>
6. SYMMETRIES OF EQUATIONS WITH NONLOCAL OPERATORS

1 Definitions of an admitted Lie group
   1.1 The geometrical approach
   1.2 The approach based on a solution
2 Symmetry groups for integro–differential equations
   2.1 Short review of the methods
   2.2 Admitted Lie group
   2.3 The kinetic Vlasov equation
3 Homogeneous isotropic Boltzmann equation
   3.1 Admitted Lie group
   3.2 Invariant solutions
4 One-dimensional motion of a viscoelastic medium
   4.1 The case $z = 0$
   4.2 The case $z = -\infty$
5 Delay differential equations
   5.1 Example
   5.2 Admitted Lie group
   5.3 Continuation of the study of equation (6.75)
6 Group classification of the delay differential equation
   6.1 Two dimensional case
   6.2 An equivalence group
7 Stochastic differential equations

7. SYMBOLIC COMPUTER CALCULATIONS

1 Introduction to Reduce
   1.1 Reduce commands
   1.2 Some remarks
   1.3 Example of a code
2 Linearization of a third order ODE
   2.1 Introduction to the problem
   2.1.1 Second order equation: the Lie linearization test
   2.1.2 Invariants of the equivalence group
   2.2 Third order equation: linearizing point transformations
   2.2.1 The linearization test for equation (7.15)
   2.2.2 The linearization test for equation (7.20)
   2.2.3 Applications of the linearization theorems
   2.3 Third order equation: linearizing contact transformations
2.3.1 Second order invariants of the equivalence group 314
2.3.2 Conditions for linearization 316
  The linearization test with $\alpha = 0$ 317
  The linearization test with $\alpha \neq 0$ 318
  Proof of the linearization theorems 320
2.3.3 Applications of contact transformations to linearization 323

8. APPENDIX 331
  1 Reduce code for solving systems of linear homogeneous equations 331
     1.1 Procedures for solving linear homogeneous equations 331
     1.2 Reconstitution of the original independent variables 338

References 339
Index 351
Differential equations, especially nonlinear, present the most effective way for describing complex physical processes. Each solution of a system of differential equations corresponds to a particular process. Therefore, methods for constructing exact solutions of differential equations play an important role in applied mathematics and mechanics. This book aims to provide scientists, engineers, and students with an easy to follow, but comprehensive, description of the methods for constructing exact solutions of differential equations. The emphasis is on the methods of differential constraints, degenerate hodograph, and group analysis. These methods have become a necessary part of applied mathematics and mathematical physics. The book is primarily designed to present both fundamental theoretical and algorithmic aspects of these methods. The description of algorithms contains illustrative examples which are typically taken from continuum mechanics. Some sections of the book introduce new applications and extensions of these methods. For example, the sixth chapter presents integro-differential and functional differential equations, a new area of group analysis.

Nonlinear partial differential equations is a vast area. There is a great number of classical and recent results on obtaining exact solutions for this type of equations. Being both selective and comprehensive is a challenge. While I drew upon multitude of sources for this book, still many results are omitted due to space constraints. It should also be noted that the method of differential constraints is not well-known outside Russia; there are only a few books in English where the idea behind this method (without analysis) is briefly mentioned. This book is a result of an effort to introduce, at a fairly elementary level, many methods for constructing exact solutions, collected in one book. It is based on my research and various courses and lectures given to undergraduate and graduate students as well as professional audiences over the past twenty five years. The book is assembled, in a coherent and comprehensive way, from results that
are scattered across many different articles and books published over the last thirty years.

The approach is analytical. The material is presented in a way that will enable the readers to succeed in their study of the subject. Introductions to theories are followed by examples. The target reader of the book are students, engineers, and scientists with diverse backgrounds and interests. For a deeper coverage of a particular method or an application the readers are referred to special-purpose books and/or scientific articles referenced in the book. The prerequisites for the study are standard courses in calculus, linear algebra, and ordinary and partial differential equations.

1. Notes to the reader

1. Analytical studies of properties of partial differential equations play an important role in applied mathematics and mathematical physics. Among them, analytical study based on the knowledge of particular classes of solutions has received a widespread attention. Each exact solution has several meanings, including an exact description of a real process in the framework of a given model, a model to compare various numerical methods, and a theory to improve the models used. This book focuses on the methods for constructing an exact solution of differential equations provided that the solution satisfies additional differential or finite constraints.

2. Most manifolds, differential equations, and other objects in the book are considered locally. All functions are assumed to be continuously differentiable a sufficient number of times. The requirement of a local study is mainly related to the inverse function theorem and the existence theorem of a local solution of an initial value problem. The local approach makes the apparatus of the study both flexible and generalizable.

3. The notion of an exact solution is not strictly defined. The concept of an exact solution is changing along with the development of mathematics. Different authors include different meaning in this notion. The exact solutions can be:

   a) explicit formulae in terms of elementary functions, their quadratures, or special functions;
   b) convergent series with effectively computed coefficients;
   c) solutions for which the process of their finding is reduced to integration of ordinary differential equations.

   The author assumes that an exact solution is a solution which has a reduced number of dependent or independent variables.

4. Particular solutions are being sought with the greatest possible functional or constant arbitrariness. Notice that any particular solution is defined by the initial differential equations and some additional analytical, geometrical, kinematic, or physical properties that lead to either the reduction of the dimension
of a problem, or the simplification of the initial equations. After finding the representation of a solution one can try to satisfy specific initial and boundary conditions by a special selection of arbitrary elements of the solution. Sometimes these methods are called half-inverse methods.

5. Compatibility analysis is one of the main techniques for constructing exact solutions. The general theory of compatibility is a special subject of algebraic analysis. Only an introduction into this theory is given in this book.

6. One of the features of a compatibility analysis is a large volume of analytical calculations. The analytical calculations include sequential executions of several algebraic operations. Since these operations are very labor intensive one has to use a computer for symbolic manipulations. Using a computer allows a considerable reduction of expense in an analytical study of systems of partial differential equations. Nowadays, obtaining new results is impossible without using a computer for analytical calculations.

2. Organization of the book

The book is divided into several chapters covering the main topics of the methods for constructing exact solutions of partial differential equations. These are united by the idea that a solution satisfies additional differential or finite constraints. For various methods the constraints are built in different ways.

The first chapter introduces methods for constructing exact solutions of partial differential equations with a single dependent function and applies these methods to studying systems of partial differential equations. For example, the Cauchy method (method of characteristics) is the main tool for finding solutions of nonlinear partial differential equations. For finding an invariant solution one has to be able to solve an overdetermined system of linear partial differential equations. Such systems can be solved by using Poison brackets. Many methods for solving differential equations along with point transformations use tangent transformations. The classical tangent transformations are the Legendre, the Hopf-Cole and the Laplace transformations. The second part of the chapter presents methods for constructing particular solutions. These methods are based on some assumptions about solutions. The assumptions are related to different representations of a solution (e.g., separation of variables, self-similar solutions, travelling waves, or partial representation) or to different requirements for a solution to satisfy such as additional functional or differential properties. The second chapter is devoted to systems of partial differential equations. If a system is written in Riemann invariants, then for homogeneous systems one obtains Riemann waves. The well-known problem of a decay of arbitrary discontinuity of a gas is solved in terms of Riemann waves. Another method that plays a very important role in gas dynamics is the hodograph method, when the hodograph is not degenerate. Presentation of self-similar solutions is given from a group analysis point of view. This way of
The third chapter considers the method of degenerate hodograph. This method deals with solutions that are distinguished by finite relations between the dependent variables. They form a class of solutions called multiple waves. The Riemann waves and the Prandtl–Meyer flows belong to this class of solutions. The first application of simple waves for multi-dimensional flows was made for isentropic flows of an ideal gas: simple and double waves. For double waves the Ovsiannikov theorem plays a very important role. The practical meaning of this theorem is demonstrated in the chapter by several examples. Applications of double waves in gas dynamics are followed by applications of double waves in a rigid plastic body. The chapter is completed by the study of triple waves of isentropic potential gas flows.

The fourth chapter is devoted to the method of differential constraints. Since the theory of involutive systems is the basis of the method, the first section introduces this theory. The theory of compatibility is followed by the basic definitions of the method of differential constraints. The first problem to arise in applications of the method of differential constraints is the involutiveness problem of an original system of partial differential equations with differential constraints. Since the Cartan-Kähler theorem only provides the existence of a solution for analytic systems, the existence problem of a solution for nonanalytic involutive systems appears. This problem is solved by using the notion of characteristics for an overdetermined system of partial differential equations. Characteristic curves also play the main role in defining a class of solutions that generalizes simple waves. The generalized simple waves have properties similar to simple waves. For example, the solution of the Goursat problem can be given in terms of generalized simple waves. The general study of generalized simple waves is followed by a section devoted to deriving this class of solutions for gas dynamic equations. The second part of the chapter considers applications of the method of differential constraints to systems of quasilinear equations with more than two independent variables. After the general study one finds examples of differential constraints for the system of multi-dimensional gas dynamic equations. As mentioned above, invariant solutions also can be described by differential constraints. Relations between the method of differential constraints and Lie-Bäcklund groups of transformations are studied in this chapter.

The fifth chapter presents a concise form of the basic algorithms that form the core of group analysis. The problem of finding an admitted Lie group is the first step in applications of group analysis for constructing exact solutions. The algebraic structure of the admitted Lie group introduces an algebraic structure into the set of all solutions. This algebraic structure is used to find invariant
and partially invariant solutions. The main feature of these classes of solutions is that they reduce the number of independent and dependent variables. In this sense the problem of finding these solutions is simpler than the ones for the general solution. A new way of using partially invariant solutions as a means of finding exact solutions is also discussed. Finally, involving derivatives in the transformation generalizes the notion of a Lie group of point transformations and leads to the notions of Bäcklund and a group of Lie-Bäcklund transformations.

The algorithmic approach of group analysis was developed specifically for differential equations. The sixth chapter discusses an extension of group analysis for equations having nonlocal terms. As for partial differential equations, the first step involves constructing an admitted Lie group. The first section of the chapter discusses different approaches to the definition of an admitted Lie group. This discussion assists in establishing a definition of an admitted Lie group for integro-differential and functional differential equations. As for partial differential equations the main difficulty in finding an admitted Lie group consists of solving the determining equations. In contrast to partial differential equations a method for solving the determining equations depends on the nonlocal equations under study. Three different examples of solving determining equations are considered in the chapter. The last part of the chapter focuses on functional differential equations and, particularly, on delay differential equations. By example, it is shown that the method for solving determining equations for delay differential equations is similar to the one for partial differential equations.

One of the features of compatibility analysis of differential equations is the extensive analytical manipulations involved in the calculations. Computer algebra systems have become an important computational tool in analytical calculations. The goal of the seventh chapter is to demonstrate computer symbolic calculations in the study of compatibility analysis. This is demonstrated by solving the problem of linearization of a third order ordinary differential equation.

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Nakhon Ratchasima
December 2004

Sergey V. Meleshko
Chapter 1

EQUATIONS WITH ONE DEPENDENT FUNCTION

This chapter introduces methods for constructing exact solutions of partial differential equations with one dependent function. Application of these methods is one of the steps for studying systems of partial differential equations. The methods are introduced by considering simple examples. The theory of the methods is discussed in the following chapters.

Linearity, quasilinearity, order of equations and other preliminary notions are considered in the first section. Such properties of solutions as replacement of variables and functional dependence, often used for obtaining exact solutions, are also introduced here. The next section is devoted to the Cauchy method (method of characteristics). This method is one of the main methods applied for constructing exact solutions of first order partial differential equations. The Cauchy method reduces a Cauchy problem for a partial differential equation to the Cauchy problem for a system of ordinary differential equations. This method is illustrated by the Hopf equation. The Cauchy method allows finding exact solutions with arbitrary functions. However, even knowledge of solutions with arbitrary constants can assist in constructing the general solution. This leads the reader to the solutions called complete and singular integrals. The section devoted to these solutions also contains the Lagrange-Charpit method for obtaining the complete integral. Practically, for finding any invariant solution, one has to be able to solve an overdetermined system of linear partial differential equations. For a system of quasilinear equations with a single dependent variable the problem of compatibility is solved through the concepts of Poisson brackets and complete systems.

Many methods of solving differential equations use a change of the dependent and independent variables that transforms a given differential equation into another equation with known properties. The change of variables, which also involves derivatives in the transformation, is called a tangent
transformation. The classical tangent transformations such as the Legendre transformation, the Hopf-Cole transformation, and the Laplace transformations are studied in the first part of chapter 1.

The second part of the chapter is devoted to methods for constructing particular solutions. These methods are based on certain assumptions about solutions. The assumptions can be about the representation of a solution (separation of variables, self-similar solutions, travelling waves or partial representation) or they can be based on the requirements for a solution to satisfy additional functional or differential properties. The first chapter discusses functionally invariant solutions or solutions having intermediate integral.

1. Basic definitions and examples

The purpose of the section is to give introductory remarks on exact solutions of partial differential equations

\[ F_k(x, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \ldots) = 0, \quad (k = 1, 2, \ldots, m). \]  

with \( n \) independent variables \( x = (x_1, x_2, \ldots, x_n) \) and one dependent function \( u(x) \).

**Definition 1.1.** A solution of equations (1.1) is a function \( u(x_1, x_2, \ldots, x_n) \), which being substituted into (1.1) reduces them to identities with respect to the independent variables \( x_1, x_2, \ldots, x_n \).

There is also a geometrical definition of a solution, considered as a manifold. A function \( u(x_1, x_2, \ldots, x_n) \) satisfying Definition 1.1 that is assumed to be sufficiently many times continuously differentiable in some domain \( D \) in \( \mathbb{R}^n \) is called a classical solution or a genuine solution. Graphically, any solution \( u = u(x_1, x_2, \ldots, x_n) \) of (1.1) can be represented as a smooth surface in \( \mathbb{R}^{(n+1)} \) lying over the domain \( D \) in the \( (x_1, x_2, \ldots, x_n) \)-hyperplane.

The maximal order of the derivatives, included in the differential equation, is called the order of this equation. If the function \( F_k \) is linear with respect to the unknown function \( u \) and its derivatives, then this equation is called a linear equation, otherwise it is called nonlinear. A nonlinear equation \( F_k \), which is only linear with respect to the maximal order derivatives, is called a quasilinear equation.

Among the methods for constructing exact solutions of nonlinear partial differential equations that should be noted are the classical methods of finding the general solution of first order equations: the Cauchy method, complete and singular integrals, the Lagrange–Charpit method and Poisson brackets. Before giving a short introduction to these methods\(^1\) let us consider some examples.

\(^1\)The detail theory of these methods one can find, for example, in [32] and [163].
1.1 Replacement of the independent variables

Assume that one needs to solve the partial differential equation
\[ \alpha u_x + \beta u_y = 0, \]
where \( \alpha \) and \( \beta \) are constant, and \( \alpha^2 + \beta^2 \neq 0 \). Using the change of the independent variables
\[ \xi = \beta x - \alpha y, \quad \eta = \alpha x + \beta y \]
one obtains the equation
\[ (\alpha^2 + \beta^2) \omega_\eta = 0, \]
with \( \omega(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)) \). The general solution of the last equation is \( \omega = \omega(\xi) \). The function \( \omega = \omega(\xi) \) is arbitrary. Hence, the general solution of the original equation is \( u = \omega(\beta x - \alpha y) \).

Remark 1.1. Formulae for the transformed derivatives are easily obtained by using the invariance of the differential. In fact, let us consider an arbitrary function \( f(x_1, x_2, \ldots, x_n) \) and the new independent variables \( \xi_i = \xi_i(x_1, x_2, \ldots, x_n), \quad (i = 1, 2, \ldots, n) \). The invariance of the differential with respect to the replacement of the independent variables means
\[ df = \sum_{i=1}^{n} f_{x_i} \, dx_i = \sum_{j=1}^{n} f_{\xi_j} \, d\xi_j. \]  
(1.2)

Substituting the differentials
\[ d\xi_j = \sum_{i=1}^{n} \frac{\partial \xi_j}{\partial x_i} \, dx_i \]
into (1.2), one obtains
\[ df = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} f_{\xi_j} \frac{\partial \xi_j}{\partial x_i} \right) dx_i = \sum_{i=1}^{n} f_{x_i} \, dx_i. \]

By virtue of the independence of the differentials \( dx_i \), one finds
\[ f_{x_i} = \sum_{j=1}^{n} \frac{\partial \xi_j}{\partial x_i} f_{\xi_j}, \quad (i = 1, 2, \ldots, n). \]

Another very well-known example where the equation is transformed to a simple form is the wave equation
\[ u_{tt} - c^2 u_{xx} = 0, \]
where $c$ is constant. Replacing the independent variables $(x, t)$ with $(\xi, \eta)$, where
\[ \xi = x + ct, \quad \eta = x - ct, \]
one obtains the general solution of the wave equation (the d’Alembert formula)
\[ u = f_1(x + ct) + f_2(x - ct). \]
Here the functions $f_1$ and $f_2$ are arbitrary functions, which are defined by auxiliary initial or boundary conditions. Additional conditions (initial and boundary data) are usually related with the underlying physical problem.

The integration of some differential equations can be also simplified by including in the transformation not only the independent variables, and also some unknown functions. For example, applying the Kirchhoff transformation
\[ \psi = \int_{u_0}^{u} k(\rho)d\rho, \]
to the nonlinear equation
\[ div (k(u)\nabla u) = 0, \] (1.3)
the function $\psi$ satisfies the linear Laplace equation $\Delta \psi = 0$, which is well studied. Thus, all properties of solutions of equation (1.3) can be discussed on the basis of the solutions of the Laplace equation.

1.2 Functional dependence.

Functional dependence is often used for constructing the general solutions. For example, the partial differential equation with respect to the function $u(x, y)$
\[ g_y u_x - g_x u_y = 0 \] (1.4)
means, that the Jacobian $\partial(u, g)/\partial(x, y)$ vanishes. Here $g = g(x, y)$ is some given function of the independent variables $x$ and $y$. The general solution of this equation is $u = \omega(g(x, y))$ with an arbitrary function $\omega(\xi)$. The proof is obtained by the replacement of the independent variables. Without loss of generality one can assume that $g_y \neq 0$. Taking
\[ \xi = g(x, y), \quad \eta = x, \]
equation (1.4) is reduced to the equation $\omega_\eta = 0$, where $u(x, y) = \omega(g(x, y), x)$. The representation
\[ u = \omega \circ g \]
also gives the general solution of equation (1.4) in the more general case where $g = g(u, x, y)$. 

2. The Cauchy method

One of the main tools of solving partial differential equations is the method of solving the first order nonlinear partial differential equation

\[ F(x, u, p) = 0. \]  

Let the initial data be given parametrically on some hypersurface

\[ u = u(t), \quad x_i = x_i(t), \quad (i = 1, 2, \ldots, n). \]  

Here \( x = (x_1, x_2, \ldots, x_n) \) are the independent variables, \( t = (t_1, t_2, \ldots, t_{n-1}) \) are the parameters describing the initial values, \( p = (p_1, p_2, \ldots, p_n) \), and \( p_i = \partial u/\partial x_i, \quad (i = 1, 2, \ldots, n) \) are partial derivatives. The functions \( u(t) \), \( x_i(t) \) and \( F(x, u, p) \) are assumed to be sufficiently many times continuously differentiable.

**Definition 1.2.** The problem of finding a solution of equation (1.5) satisfying the initial data (1.6) is called a Cauchy problem.

The Cauchy method for constructing the solution of the Cauchy problem (1.5), (1.6) reduces this problem to finding a solution of the Cauchy problem of the system of ordinary differential equations, which is called a characteristic system,

\[ \frac{dx_i}{ds} = F_{pi}, \quad \frac{du}{ds} = p_\alpha F_{p_\alpha}, \quad \frac{dp_i}{ds} = -(F_u p_i + F_{x_i}), \quad (i = 1, 2, \ldots, n), \]  

with the initial data at the point \( s = 0 \):

\[ x = x(t), \quad u = u(t), \quad p = p(t). \]

Here \( x = x(t) \) and \( u = u(t) \) are defined by (1.6), and summation with respect to a repeat index is assumed. The initial data \( p(t) \) are found by solving equation (1.5) and the tangent conditions:

\[ F(x(t), u(t), p(t)) = 0, \quad u_{tk}(t) = p_\alpha(t) \frac{\partial x_\alpha}{\partial t_k}(t), \quad (k = 1, 2, \ldots, n-1). \]

As the result of solving the Cauchy problem for the characteristic system one obtains the functions \( u(s, t_1, \ldots, t_{n-1}) \) and \( x_i(s, t_1, \ldots, t_{n-1}) \), \( (i = 1, 2, \ldots, n) \).

**Definition 1.3.** The curve \( x(s, t) \) in the space of the independent variables with fixed \( t \), is called a characteristic.

The solution \( u = u(x) \) of the Cauchy problem (1.5), (1.6) is constructed by eliminating the parameters \( s, t_1, \ldots, t_{n-1} \) from the equations \( x = x(s, t) \) and \( u = u(s, t) \). By virtue of the inverse function theorem for the elimination it is sufficient to require the inequality

\[ \Delta(s, t_1, \ldots, t_{n-1}) \equiv \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(s, t_1, \ldots, t_{n-1})} = \text{det} \left( \begin{array}{c} \frac{F_{pi}}{\partial x_i/\partial t_k} \end{array} \right) \neq 0. \]
**Theorem 1.1.** Let the initial data (1.6), (1.8) satisfy the condition

\[
\Delta(0, t_1^0, \ldots, t_{n-1}^0) \neq 0
\]

at some point \( t_0 = (t_1^0, \ldots, t_{n-1}^0) \). The solution \( x = x(s, t), u = u(s, t), p = p(s, t) \) of the initial value problem (1.6), (1.8) of the characteristic system (1.7) gives the solution \( u(x) \) of the Cauchy problem (1.5), (1.6) in some neighborhood of the point \( x(t_0) \).

**Proof.**

By virtue of system (1.7) one finds

\[
\frac{dF}{ds} = F_{x_\alpha} \frac{dx_\alpha}{ds} + F_u \frac{du}{ds} + F_p \frac{dp_\alpha}{ds} = 0.
\]

This means that the function \( F(x(s, t), u(s, t), p(s, t)) \) is an integral of system (1.7). By virtue of the choice of the initial data, one has \( F(x(s, t), u(s, t), p(s, t)) = 0 \). For the proof of the theorem it is enough to show that the functions \( p_i \) coincide with the derivatives \( \partial u / \partial x_i, (i = 1, 2, \ldots, n) \) of the function \( u = u(x_1, x_2, \ldots, x_n) \), which is recovered from the solution of the Cauchy problem (1.6)--(1.8).

Notice that the determinant of the linear system of the algebraic equations with respect to \( y_1, y_2, \ldots, y_n \):

\[
y_\alpha x_{\alpha s} = u_s, \ y_\alpha x_{\alpha t_k} = u_{t_k}, \ (k = 1, \ldots, n - 1)
\]

is equal to \( \Delta(s, t_1, \ldots, t_{n-1}) \). Since \( \Delta(0, t_1^0, \ldots, t_{n-1}^0) \neq 0 \), the determinant of system (1.10) is not equal to zero in some neighborhood of the point \( (0, t_1^0, \ldots, t_{n-1}^0) \). Hence, the linear system (1.10) has an unique solution. Because of the chain rule, the change of the variables \( (s, t_1, \ldots, t_{n-1}) \) with \( (x_1, \ldots, x_n) \) in the function \( u(s, t) \) leads to

\[
\frac{\partial u}{\partial x_\alpha} x_{\alpha s} = u_s, \ \frac{\partial u}{\partial x_\alpha} x_{\alpha t_k} = u_{t_k}, \ (k = 1, \ldots, n - 1).
\]

Hence, the solution of (1.10) is \( y_i = \partial u / \partial x_i, (i = 1, 2, \ldots, n) \).

To complete the proof of the theorem one needs to prove that the expressions \( U_0 = u_s - p_\alpha x_{\alpha s}, \ U_k = u_{t_k} - p_\alpha x_{\alpha t_k}, \ (k = 1, \ldots, n - 1) \) also vanish.

In fact, by virtue of (1.7) one has \( U_0 \equiv 0 \) and

\[
\frac{\partial U_k}{\partial s} - \frac{\partial U_0}{\partial t_k} = (F_u p_\alpha + F_{x_\alpha}) \frac{\partial x_{\alpha s}}{\partial t_k} + F_p \frac{\partial p_\alpha}{\partial t_k}, \ (k = 1, \ldots, n - 1).
\]

Since \( F(x(s, t), u(s, t), p(s, t)) = 0 \), the differentiation it with respect to \( t_k \) gives

\[
\frac{\partial F}{\partial t_k} = F_{x_\alpha} \frac{\partial x_{\alpha}}{\partial t_k} + F_u \frac{\partial u}{\partial t_k} + F_p \frac{\partial p_\alpha}{\partial t_k} = 0, \ (k = 1, 2, \ldots, n - 1).
\]
Substituting $F_{x_{a}} \frac{\partial x_{a}}{\partial t_k} + F_{p_{a}} \frac{\partial p_{a}}{\partial t_k}$ found from these equations into (1.11), they can be rewritten as follows

$$\frac{\partial U_k}{\partial s} = -F_{u} \left( \frac{\partial u}{\partial t_k} - P_{a} \frac{\partial x_{a}}{\partial t_k} \right) = -F_{u} U_k, \quad (k = 1, 2, \ldots, n - 1). \quad (1.12)$$

Because of the choice of the initial data, $U_k(0, t) = 0$. Because of the uniqueness of the solution of the Cauchy problem, the last equations (1.12) have the unique solution $U_k(s, t) = 0$.

Comparing the expressions $U_0 = 0$, $U_k = 0$ and system (1.10), one obtains $p_i = \partial u / \partial x_i$, $(i = 1, 2, \ldots, n)$.

**Remark 1.2.** Another representation of the characteristic system (1.7) is

$$\frac{d u}{P_{a} F_{p_{a}}} = \frac{d x_i}{F_{p_{i}}} = \frac{d p_{i}}{-(F_{u} P_{i} + F_{x_i})} = d s, \quad (i = 1, 2, \ldots, n).$$

**Remark 1.3.** Let the function $F(x, u, p)$ be linear with respect to the partial derivatives (equation (1.5) is a quasilinear partial differential equation)

$$F = a_{\alpha}(x, u) u_{x_{\alpha}} - a(x, u).$$

Since $F(x, u, p) = 0$ is an integral of the characteristic system (1.7), the equation $\frac{d u}{d s} = a_{\alpha}(x, u) p_{a}$ in the characteristic system can be exchanged with the equation $\frac{d u}{d s} = a(x, u)$. Hence, the part of the equations for the functions $x = x(s, t), u = u(s, t)$ in system (1.7) forms a closed system. For these equations there is no necessity to set initial values for the variables $p_i$, $(i = 1, 2, \ldots, n)$. An application of the Cauchy method to such a type of equations becomes simpler.

**Remark 1.4.** If the equation $F(x, u, p) = 0$ is linear and homogeneous$^2$, i.e.,

$$F = a_{\alpha}(x) u_{x_{\alpha}},$$

the general solution of this equation has the form

$$u = \Phi(\varphi_1(x), \varphi_2(x), \ldots, \varphi_{n-1}(x)).$$

Here $\Phi$ is an arbitrary function with $n - 1$ arguments, the functions $\varphi_i(x)$, $(i = 1, 2, \ldots, n - 1)$ are functionally independent solutions of this equation, and

---

$^2$Linear homogeneous equations play a special role in solving a complete system and in using group analysis method.
they are called integrals of equation (1.5). In fact, for a linear homogeneous equation the characteristic system (1.7) is reduced to the system

$$\frac{du}{ds} = 0, \quad \frac{dx_i}{ds} = a_i(x), \quad (i = 1, 2, \ldots, n).$$

The system of ordinary differential equations

$$\frac{dx_i}{ds} = a_i(x), \quad (i = 1, 2, \ldots, n).$$

only has $n - 1$ independent integrals $\varphi_i(x) = c_i, \quad (i = 1, 2, \ldots, n - 1)$.

Since $\frac{du}{ds} = 0$, the function $u(x)$ is also an integral. Hence, $u(x)$ depends on $\varphi_i(x), \quad (i = 1, 2, \ldots, n - 1)$:

$$u = \Phi(\varphi_1(x), \varphi_2(x), \ldots, \varphi_{n-1}(x)).$$

Let us apply the Cauchy method to the equation

$$\rho_t + c(\rho)\rho_x = 0, \quad (1.13)$$

where $c(\rho)$ is some function of the argument $\rho$. Analysis of this equation gives the majority of the basic ideas arising in studies of nonlinear hyperbolic equations: numerous physical problems are modelled by this equation. In numerical methods this equation often serves as a model equation on which those or other numerical methods are tested.

The initial data for equation (1.13) are taken on the line $t = 0$. Continuously differentiable solutions of the Cauchy problem are considered. According to the method, one needs to construct the system of characteristics, issuing from the points of the line $t = 0$. These characteristics correspond to the integrals of the characteristic system. Choosing the variable $t$, instead of $s$, as the parameter along the characteristic curves, the characteristic system takes the form

$$\frac{dx}{dt} = c(\rho), \quad \frac{d\rho}{dt} = 0. \quad (1.14)$$

Let the initial values at $t = 0$ be

$$x = \xi, \quad \rho = f(\xi), \quad (1.15)$$

where $\xi$ is a parameter. Since the function $\rho(x, t)$ is constant on any characteristic curve, the function $c(\rho)$ is also constant on the characteristic. Thus any characteristic curve of equation (1.13) is a straight line in the $(x, t)$-plane

---

3If $c'(\rho) \neq 0$, the change $u = c(\rho)$ reduces equation (1.13) to the Hopf equation

$$u_t + uu_x = 0.$$
with the slope \( c(\rho) \). The general solution of equation (1.13) is reduced to the construction of the family of the straight lines in the \((x, t)\)-plane. Each of the straight lines has the slope \( g(\xi) = c(f(\xi)) \), which is defined by the value \( \rho = f(\xi) \) at the point \( t = 0, \ x = \xi \). The solution of the Cauchy problem (1.14), (1.15) is

\[
x = \xi + t g(\xi), \quad \rho = f(\xi).
\]

This is a parametric representation of the general solution of equation (1.13). Let the function \( \xi(x, t) \) be implicitly defined by the first equation (1.16). Differentiating equations (1.16) with respect to \( x \) and \( t \), one obtains

\[
1 = (1 + t g'(\xi)) \xi_x, \quad \rho_x = f'(\xi) \xi_x,
\]

\[
0 = (1 + t g'(\xi)) \xi_t + g(\xi), \quad \rho_t = f'(\xi) \xi_t.
\]

Hence,

\[
\rho_t = \frac{-g(\xi) f'(\xi)}{1 + t g'(\xi)}, \quad \rho_x = \frac{f'(\xi)}{1 + t g'(\xi)},
\]

and equation (1.13) is satisfied.

From the analytical representation of the solution one notes that for \( F'(\xi) < 0 \) the derivatives \( \rho_t, \rho_x \) can become infinite at the time \( t = t_k \equiv -1/F'(\xi) \). This means that the characteristic lines cross and, since \( \rho \) has different constants \( \rho = f(\xi) \) on each characteristic line, a contradictory result is obtained. Hence, a smooth solution cannot exist for all \( t > 0 \). The points, where the characteristics cross, is called a gradient catastrophe. The minimum value \( \min(t_k(\xi)) \) is called the breaking time. A smooth solution of the Cauchy problem of equation (1.13) does not exist from the moment the breaking time occurs, and thereafter the concept of a "solution" requires generalization.

3. **Complete and singular integrals**

Let us consider a differential equation of first order with two independent variables

\[
F(x, y, u, p, q) = 0.
\]

Here \( F_p^2 + F_q^2 \neq 0 \), and the usual notations \( p = u_x, q = u_y \) are used.

**Lemma 1.1.** If a family of solutions \( u = f(x, y, a) \) of differential equation (1.17), depending on a parameter \( a \), has an envelope, then this envelope is also a solution.

**Proof.**

The envelope of the family \( u = f(x, y, a) \) is defined by the formula

\[
\psi(x, y) = f(x, y, a(x, y)),
\]

where the function \( a(x, y) \) is found from the equation \( f_a(x, y, a) = 0 \). Taking into account

\[
f_a(x, y, a(x, y)) \equiv 0,
\]

the equation (1.17) holds.
the derivatives of the function $\psi(x, y)$ are

$$
\psi_x = f_x + f_a a_x = f_x, \quad \psi_y = f_y + f_a a_y = f_y.
$$

Since for any $a$

$$
F(x, y, f(x, y, a), f_x(x, y, a), f_y(x, y, a)) = 0,
$$

the function $u = \psi(x, y)$ is also a solution of equation (1.17). Let a family of solutions of equation (1.17) have two parameters.

**Definition 1.4.** A two–parameter family $u = \phi(x, y, a, b)$ of solutions is called a complete integral of equation (1.17), if in the considered domain the rank $r$ of the matrix

$$
\begin{pmatrix}
\phi_a & \phi_{ax} & \phi_{ay} \\
\phi_b & \phi_{bx} & \phi_{by}
\end{pmatrix}
$$

is equal to two.

Having a complete integral $u = \phi(x, y, a, b)$, one can obtain a set of solutions of equation (1.17) with one arbitrary function. In fact, assume that $a = a(x, y)$ and $b = b(x, y)$. For the function $\psi(x, y) = f(x, y, a(x, y), b(x, y))$ one finds

$$
\psi_x = \phi_x + \phi_a a_x + \phi_b b_x, \quad \psi_y = \phi_y + \phi_a a_y + \phi_b b_y.
$$

To use the property that the function $u = \phi(x, y, a, b)$ is a solution of equation (1.17), it is natural to require

$$
\phi_a a_x + \phi_b b_x = 0, \quad \phi_a a_y + \phi_b b_y = 0. \tag{1.18}
$$

If $\phi_a^2 + \phi_b^2 \neq 0$, then the determinant of the homogenous linear system (1.18) with respect to $\phi_a, \phi_b$ has to be equal to zero. Hence, the Jacobian $\partial(a, b)/\partial(x, y) = 0$. Thus, for example, $b = \omega(a)$, and equations (1.18) are reduced to the equation

$$
\phi_a + \phi_b \omega'(a) = 0.
$$

This equation defines the envelope of the family $u = \phi(x, y, a, \omega(a))$. Finding the function $a(x, y)$ from this equation, one obtains the solution $u = \phi(x, y, a(x, y), \omega(a(x, y)))$ with one arbitrary function $\omega(a)$.

The equations

$$
\phi_a = 0, \quad \phi_b = 0
$$

lead to the concept of a singular integral.

---

4The condition $r = 2$ guarantees, that the function $\phi$ essentially depends on the two independent parameters.
Definition 1.5. The envelope of a two-parameter family of solutions \( u = \phi(x, y, a, b) \), obtained by eliminating the parameters \( a \) and \( b \) from the equations
\[
\phi_a(x, y, a, b) = 0, \quad \phi_b(x, y, a, b) = 0,
\]
is called a singular integral of equation (1.17).

For some equations a singular integral can be found without knowing a two-parameter family of solutions. In fact, since a two-parameter family \( u = \phi(x, y, a, b) \) of solutions satisfies equation (1.17):
\[
F(x, y, \phi(x, y, a, b), \phi_x(x, y, a, b), \phi_y(x, y, a, b)) \equiv 0,
\]
differentiating it with respect to the parameters \( a \) and \( b \), and using the properties (1.19), one obtains
\[
F_p\phi_x + F_q\phi_y = 0, \quad F_p\phi_x + F_q\phi_y = 0.
\]
If the determinant of this linear system of algebraic equations with respect to \( F_p, F_q \)
\[
det \begin{pmatrix} \phi_{ax} & \phi_{ay} \\ \phi_{bx} & \phi_{by} \end{pmatrix} \neq 0,
\]
then \( F_p = 0 \) and \( F_q = 0 \). Therefore, in this case the singular integral \( u = \phi(x, y, a, b) \) satisfies the equations
\[
F(x, y, u, p, q) = 0, \quad F_p(x, y, u, p, q) = 0, \quad F_q(x, y, u, p, q) = 0,
\]
where \( p = \phi_x, \ q = \phi_y \).

To find a two-parameter family of solutions one can apply a method which uses the notion of a completely integrable system\(^5\). Let us consider the overdetermined system of first order equations:
\[
p = f(x, y, u), \quad q = g(x, y, u).
\]

Definition 1.6. An overdetermined system (1.20) is called completely integrable if the equation
\[
f_y + g_f u = g_x + f g_u
\]
is identically satisfied with respect to \( x, y, u \).

Theorem 1.2. If system (1.20) is completely integrable, then its solution is defined and contains one arbitrary constant.

\(^5\)The more general case of completely integrable systems is studied in the next chapter.
Proof.
Integrating the first equation of system (1.20) with respect to \( x \), one finds a solution \( u = \phi(x, y, C(y)) \). Here the function \( C(y) \) is a constant of integration with respect to the independent variable \( x \). Since \( \phi_c \neq 0 \), substituting it in the second equation, one obtains

\[
\frac{dC}{dy} = \frac{g - \phi_y}{\phi_c}.
\]  

(1.22)

Let us show that the right hand-side in (1.22) does not depend on \( x \). Since

\[
\phi_x = f_u \phi_c, \quad \phi_{xy} = f_y + f_u \phi_y,
\]

one has

\[
\frac{\partial}{\partial x} \left( \phi_c^{-1} (g - \phi_y) \right) = \phi_c^{-2} \left( (g_x + g_u \phi_x - \phi_{xy}) \phi_c - (g - \phi_y) \phi_x \right)
\]

\[
= -\phi_c^{-1} \left( (f_y + g_f u) - (g_x + q g_u) \right) = 0.
\]

Therefore equation (1.22) is an ordinary differential equation with respect to \( y \), and its solution depends on one arbitrary constant, for example, \( b \).

Assume there is the equation

\[
\Phi(x, y, u, p, q) = a
\]

(1.23)

with a parameter \( a \) such that from this equation, and the given equation

\[
F(x, y, u, p, q) = 0,
\]

(1.24)

one can find the derivatives \( p \) and \( q \):

\[
p = f(x, y, u, a), \quad q = g(x, y, u, a)
\]

such that the last overdetermined system is completely integrable. Solving this completely integrable system, one obtains a two-parameter family of solutions of the original equation (1.24): one is the parameter \( a \) and another is the constant arising when solving the totally integrable system (the parameter \( b \)).

The function \( \Phi(x, y, u, p, q) \) in (1.23) can be found by the Lagrange–Charpit method. This method is based on the following idea. Since the functions (1.20) satisfy the equations

\[
F(x, y, u, f(x, y, u), g(x, y, u)) \equiv 0, \quad \Phi(x, y, u, f(x, y, u), g(x, y, u)) \equiv a,
\]

one can find the derivatives \( f_y, g_x, f_u, g_u \) by differentiating these equations with respect to \( x, y \) and \( u \):

\[
f_y = \Delta^{-1} \left( -F_y \Phi_q + F_q \Phi_y \right), \quad g_x = \Delta^{-1} \left( -F_p \Phi_x + F_x \Phi_p \right),
\]

\[
f_u = \Delta^{-1} \left( -F_u \Phi_q + F_q \Phi_u \right), \quad g_u = \Delta^{-1} \left( -F_p \Phi_u + F_u \Phi_p \right),
\]
where $\Delta = F_p \Phi_q - F_q \Phi_p$. Substituting these derivatives into (1.21), one has
\[ \Phi_x F_p + \Phi_y F_q + \Phi_u (qF_q + pF_p) - \Phi_p (F_x + pF_u) - \Phi_q (F_y + qF_u) = 0. \]
This equation has to be satisfied when the expressions (1.20) are substituted into it. Requiring the satisfaction of this equation identically with respect to the variables $x, y, u, p, q$, one obtains a homogeneous quasilinear differential equation of first order for the function $\Phi(x, y, u, p, q)$. Notice that one of the integrals of this equation is $F(x, y, u, p, q)$.

The concept of complete integral can be generalized for first order equations with many independent variables $x = (x_1, x_2, \ldots, x_n)$:
\[ F(x, u, p) = 0. \]  
(1.25)
Here $p = (p_1, p_2, \ldots, p_n)$. $p_i = \partial u/\partial x_i$, ($i = 1, 2, \ldots, n$). Let the function $u = \phi(x_1, \ldots, x_n, a_1, \ldots, a_n)$ be a solution of equation (1.25), where the parameters $a = (a_1, \ldots, a_n)$ are fixed.

**Definition 1.7.** An $n$-parameter family of solutions $u = \phi(x_1, \ldots, x_n, a_1, \ldots, a_n)$ of equation (1.25) is called a complete integral, if the equations
\[ u - \phi(x, a) = 0, \quad p_i - \phi_x(x, a) = 0, \quad (i = 1, 2, \ldots, n) \]  
(1.26)
can be solved with respect to the parameters $a_1, \ldots, a_n$. These representations of the parameters $a_1, \ldots, a_n$ have to be such that substituting them into (1.25), one obtains an identity with respect to $2n$ variables, where first of these $n$ variables are $x_1, \ldots, x_n$.

Assume that $a = a(x)$, then $p_i = \partial \phi/\partial x_i + (\partial \phi/\partial a)(\partial a/\partial x_i)$, ($i = 1, 2, \ldots, n$). In order to use the property of the $n$-parameter family $u = \phi(x_1, \ldots, x_n, a_1, \ldots, a_n)$ to be a solution, similar to equations (1.19), one can require that the functions $a_i = a_i(x)$, ($i = 1, 2, \ldots, n$) satisfy the equations
\[ \frac{\partial \phi}{\partial a_{\alpha}} \frac{\partial a_{\alpha}}{\partial x_i} = 0, \quad (i = 1, 2, \ldots, n). \]  
(1.27)

Let us use a complete integral for constructing the solution of the Cauchy problem $^7$
\[ u = u(t_1, t_2, \ldots, t_{n-1}), \quad x_i = x_i(t_1, t_2, \ldots, t_{n-1}), \quad (i = 1, 2, \ldots, n) \]  
(1.28)
Assume that for some
\[ a_i = a_i(t_1, t_2, \ldots, t_{n-1}), \quad (i = 1, 2, \ldots, n) \]  
(1.29)

\(^{6}\)It is assumed summation with respect to a repeated index. Here and further, if it is not specified, the summation with respect to all values of the repeated index, which it can accept, is applied. For example, $a_{\alpha}^x = \sum_{\alpha=1}^n a_{\alpha} x_{\alpha}$.

\(^{7}\)This method is different from the Cauchy method for solving the Cauchy problem.
one has
\[ \frac{\partial \phi}{\partial a_\alpha}(x(t), a(t)) \frac{\partial a_\alpha}{\partial t_j}(t) = 0, \quad (j = 1, 2, \ldots, n - 1), \quad (1.30) \]
where \( t = (t_1, \ldots, t_{n-1}) \). Hence, for any assignments \( t = t(x) \) the derivatives of the function \( u(x) = \phi(x, a(t(x))) \) are
\[ u_{x_j} = \frac{\partial \phi}{\partial x_j} + \frac{\partial \phi}{\partial a_\alpha} \frac{\partial a_\alpha}{\partial t_\beta} \frac{\partial t_\beta}{\partial x_j} = \frac{\partial \phi}{\partial x_j}, \quad (j = 1, 2, \ldots, n). \]

Thus, the solution of the Cauchy problem can be constructed in the following way. First, one finds the functions \( a_i(t_1, \ldots, t_{n-1}) \) from the system of equations, which consists of the \((n - 1)\) equations (1.30), and the representation of the complete integral with the initial value data (1.28) substituted into it:
\[ u(t) = \phi(x(t), a(t)), \quad \frac{\partial \phi}{\partial a_\alpha}(x(t), a(t)) \frac{\partial a_\alpha}{\partial t_\beta}(t) = 0, \quad (i = 1, 2, \ldots, n - 1), \]
Then from the second part of the Cauchy data (1.28):
\[ x = x(t) \]
one finds \( t = t(x) \). The function \( u = \phi(x, a(t(x))) \) is the required solution of the Cauchy problem.

4. Systems of linear equations

This section is devoted to solving a linear system of homogeneous first order differential equations with one unknown function \( u(x), \quad (x \in \mathbb{R}^n) \):
\[ X_i(u) \equiv a_{i\alpha} p_\alpha = 0, \quad (i = 1, 2, \ldots, m). \quad (1.31) \]

Here \( x = (x_1, x_2, \ldots, x_n), \quad p_j = \partial u/\partial x_j \), the function \( u(x) \) and the coefficients \( a_{ij} = a_{ij}(x), \quad (i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n) \) are assumed to be sufficiently many times continuously differentiable. For the sake of simplicity it is assumed that the rank of the matrix \( A = (a_{ij}) \) composed of the coefficients \( a_{ij} = a_{ij}(x), \quad (i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n) \) is equal to the number of the equations. This, in particular, means that \( m \leq n \). Notice that with this agreement for \( m = n \) there is only the trivial solution \( u = const. \)

**Remark 1.5.** Any system of quasilinear equations with one unknown function
\[ b_{i\alpha}(x, u) p_\alpha = b_i(x, u), \quad (i = 1, 2, \ldots, m) \quad (1.32) \]
can be reduced to a linear homogeneous system of the form (1.31). For this purpose one can use an implicit representation of a solution of system (1.31), i.e., the function \( Q(x, u) \) with \( Q_u \neq 0 \), such that \( Q(x, u(x)) \) is constant for