Pierre Ladevèze  Jean-Pierre Pelle

Mastering Calculations in Linear and Nonlinear Mechanics

Translated by Theofanis Strouboulis

With 143 Figures
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Mastering calculations in linear and nonlinear mechanics

* A posteriori errors
  Adaptive control of parameters

Pierre Ladevèze & Jean-Pierre Pelle

Translated by
Theofanis Strouboulis
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Introduction

Today more than ever, modeling and simulation are central to a mechanical engineer’s activity. Increasingly complex models are being used routinely on a daily basis. This revolution, which has just begun, is the result of the extraordinary progress in computer technology in terms of both hardware and software.

In order to represent a real problem, one does not use just a single model, but a series of models. Starting from a first model, called the reference model, practical or economic considerations, along with the wish to take advantage of certain particular situations, often lead to the introduction of additional simplifying hypotheses, called condensation hypotheses, which result in a new, more manageable model. This, for example, is the case of hypotheses which, starting from a continuous model of a medium subjected to a given environment, lead to a “finite element” model involving parameters such as the size and type of the elements, the number of iterations, the duration of the time increments....
Of course, it is imperative not to alter the reference model completely. Therefore, controlling the additional simplifying hypotheses is an obvious and major issue. This has been a constant preoccupation on the industrial level as well as in research. The new situation is that over the last twenty years truly quantitative tools for assessing the quality of a model compared to another reference model have appeared.

This work deals with the control of the hypotheses leading from a mechanical model, usually coming from continuum mechanics, to a numerical model, i.e. the mastery of the mechanical computation process itself. Particular attention is given to structural analysis which, in this context, is the most advanced domain. The term “structure” designates the material envelope, which can consist of metallic materials, composite materials, biomaterials … in solid, fluid or gaseous environments. The models being studied are not necessarily linear and high degrees of nonlinearity may be present (plasticity, viscoplasticity, unilateral contact…). The objective of structural analysis is to simulate the behavior of a structure subject to various solicitations (prescribed displacements and forces) numerically; in particular, the aim is to evaluate the state of damage of the structure and compare it with one or several limit states. The final stage consists in optimizing the structural parameters. The practical problems concern the dimensioning, optimization, reliability and even the manufacturing process of the object being designed or built.

The basic problem consists in defining and evaluating a measure of the error due to the discretization performed, in this case, by the finite element method.

Two situations must be dealt with, depending on whether the error is evaluated before or after the finite element calculation has been performed.

Today, for the first situation corresponding to what one calls “a priori” errors, only coarse evaluations are available. The second situation is more favorable: the finite element solution constitutes an additional piece of information. It is in the corresponding field of “a posteriori” error evaluation that the first research works on linear problems were published about twenty years ago.
The numerous techniques proposed can be categorized into three approaches:

- the first approach relies on the concept of error in constitutive relation and on related field construction techniques [LADÉVEZE, 1975];
- the second approach relies on the concept of error indicator associated with the satisfaction of the equilibrium equations [BABUSKA - RHEINBOLDT, 1978];
- the third approach is based on the unevenness of the finite element solution [ZIENKIEWICZ - ZHU, 1987].

In the present work, after having described the various approaches, we focus on the first family of estimators because, on the one hand, it has the strongest mechanical meaning and, on the other hand, contrary to the other two families, it can be extended without much difficulty to nonlinear evolution problems. This approach is based on a partition of the equations of the reference problem into:

- admissibility conditions (kinematic constraint equations, equilibrium equations, initial conditions);
- the constitutive relation.

Indeed, the constitutive relation has a special status: in practice, this is often the least reliable equation. Therefore, it is natural to set this equation apart and seek an approximate displacement-stress solution over the time interval being considered which verifies the most reliable group of equations (i.e. the admissibility conditions) exactly. This solution verifies a constitutive relation which, in general, differs from the constitutive relation of the material; thus, the quality of the approximation can be assessed by comparing this constitutive relation to that of the material. Energy norms or other norms which have a deep physical meaning are used to quantify this error.

An a priori obstacle is that it is difficult to construct admissible approximate solutions, i.e. solutions which verify the admissibility conditions exactly. Indeed, the usual approximations – particularly the approximations
resulting from the application of the finite element method – fail to verify these conditions exactly because the calculated stresses are not in equilibrium with the applied forces. One circumvents this difficulty by a very general technique which enables one to construct an admissible approximate solution explicitly, therefore very inexpensively, starting from the approximate solution obtained by the finite element method. It should be noted that this construction technique takes advantage of the properties of the finite element solution.

Of course, we also present the other error estimators proposed in the literature and compare them to the constitutive relation error estimators.

Most error estimators do not provide information on local errors such as errors in the stresses. The construction of local error estimators is one of today’s key issues on the research level. Very few works have been dedicated to this problem. Here, we present a recent theory which is an extension of the approach which led to our constitutive relation error estimators.

A significant part of this work is dedicated to the application of error estimators – whatever these estimators may be – to the control of the various parameters involved in a calculation, beginning with the parameters related to the mesh. Some examples illustrate the current state of the art.

Many developments presented here for the first time stem from recent research by the authors. Let us mention, for example, the extension of the concept of error in constitutive relation to nonlinear evolution problems and to dynamic problems, the adaptive improvements to nonlinear calculations in mechanics, the evaluation of local errors….

This work is addressed to all – students, researchers, engineers – who are interested in mechanics, from the construction of models to their simulation for industrial purposes.

The first chapter describes the reference problems, the approximate models obtained by the finite element method and the main sources of discretization errors.

Chapter 2 presents the bases of the constitutive relation error method for linear problems and outlines the techniques of construction of admissible
fields. The concept of error in constitutive relation is also used to establish the theorems known as “energy theorems” which, in fact, result directly from the global formulation of the constitutive relation using overpotentials.

The other two major error estimation methods for linear problems proposed in the literature (error estimators based on the equilibrium deficiency and error indicators based on the smoothing of the finite element stresses) are presented in detail in Chapter 3.

For linear problems, simple examples of the use of the constitutive relation error measure and some elements of comparison of the global effectiveness of the various estimators are given in Chapter 4.

Chapter 5 is dedicated to the various techniques of finite element mesh adaptation. Particular emphasis is put on the “$h$” method, which is the most widely used today. Using the error estimates obtained in a preliminary calculation, it is possible to predict the element sizes necessary to achieve a predetermined level of quality. Examples of mesh adaptation in 2D and in 3D are given.

Chapter 6 (for nonlinear problems) and Chapter 7 (for vibration and transient dynamics problems) show how the constitutive relation error method enables one to derive consistent error estimates in these difficult situations.

Chapter 8 details the techniques used in the construction of admissible fields, whose central aspect is the construction of force densities on the interfaces between elements. The method is first introduced for the simpler case of 2D thermal problems, then detailed for 2D or 3D elasticity, incompressible elasticity and elastic plate problems. In this chapter, we also describe in detail an improved method of constructing the densities which increases the effectiveness of the error estimators in difficult situations (e.g. for elements with very high aspect ratios).

Chapter 9 presents recent works on the evaluation of local quantities (stresses, displacements...). Access to such estimates is crucial for industrial applications.
This book, as well as most of the corresponding work, was produced at LMT-Cachan (Ecole Normale Supérieure de Cachan/CNRS/Université Paris 6).

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April 13, 2001 Pierre Ladevèze and Jean-Pierre Pelle

Finally, we thank our colleague and friend T. Strouboulis warmly for the English translation of this book.

November 20, 2003 Pierre Ladevèze and Jean-Pierre Pelle
Chapter 1

The notion of quality of a finite element solution

1.1 Introduction

The process which leads one, starting from a physical problem, to carry out a finite element calculation is generally complex. It depends on the situations being considered and is the result of various hypotheses and simplifications made in order to represent the physical problem more or less accurately. Between the real problem and the finite element computational model, many modeling stages take place (Figure 1.1).

For example, in order to design a mechanical structure, one uses:

- a model of the geometry;
- a model of the loading case (or of the different loading cases) involved in the dimensioning;
- a model of the connections with the outside world;
- a model of the behavior of the material (or materials) the structure is made of.
Depending on the choices made in these different modeling stages, one ends up with various mechanical models. In most cases, the mechanical models constructed in this manner pertain to the mechanics of continuous media and lead to equations (differential equations, partial differential equations...) relating the unknowns (which in the most common cases are the displacement field $U$ and the stress field $\sigma$) and the problem data (initial conditions, kinematic constraints, loads, coefficients characterizing the material behavior...).

Figure 1.1. From the physical problem to the numerical model.

Since it is impossible, except in very simple situations, to determine the exact solution $(U_{ex}, \sigma_{ex})$ of such a continuous mechanical model analytically, the user must be satisfied with approximations of $(U_{ex}, \sigma_{ex})$. In practice, this is equivalent to replacing the continuous mechanical model by a simpler approximate model whose solution, considered to be an approximation of $(U_{ex}, \sigma_{ex})$, can be determined. For example, the finite element method, which is currently the most widespread method for obtaining approximations, consists in replacing the continuous mechanical model by a discrete approximate model whose solution $(U_h, \sigma_h)$ constitutes an approximation of $(U_{ex}, \sigma_{ex})$. 
Controlling the quality of the finite element calculation consists in considering the continuous mechanical model as a *reference model* and evaluating the quality of the calculated solution \((U_h, \sigma_h)\) as an approximation of the exact solution \((U_{ex}, \sigma_{ex})\) of that reference model.

### 1.2 The reference model

The medium being studied, which in our case can usually be described as a solid medium, occupies at the initial time \(t = 0\) a domain \(\Omega\) bounded by \(\partial \Omega\).

In most of this work, we will be assuming small strains. Therefore, the various configurations occupied by the medium can be assumed to coincide with the initial configuration \(\Omega\). The evolution of the medium is studied over the time interval \([0, T]\).

We assume that the medium is subjected to a given environment which, at each instant \(t\), can be typically represented (see Figure 1.2) by:

- a surface displacement field \(U_d\) on a part \(\partial_1 \Omega\) of the boundary \(\partial \Omega\);
- a surface force density \(F_d\) on the part \(\partial_2 \Omega = \partial \Omega - \partial_1 \Omega\);
- a volume force density \(f_d\) inside the domain \(\Omega\).

Figure 1.2. The reference problem.
At the initial time $t = 0$, we assume that the initial position $U_0$ and the initial velocity $V_0$ at any point $M$ of $\Omega$ are also given

$$\forall M \in \Omega, \quad U|_{t=0} = U_0, \quad \text{and} \quad \left( \frac{d}{dt} U \right)|_{t=0} = \dot{U}|_{t=0} = V_0$$

and that the reference system is Galilean.

The problem which describes the evolution of the medium in $[0, T]$ can be formulated as follows:

*Find a displacement field $U(M, t)$ and a stress field $\sigma(M, t)$ defined on $\Omega \times [0, T]$ such that:*

- **Kinematic constraint equations and initial conditions**
  $$U \in U^{[0, T]}$$
  $$\forall t \in [0, T], \quad U|_{\partial \Omega} = U_d$$
  $$\forall t \in [0, T], \quad U|_{t=0} = U_0, \quad \dot{U}|_{t=0} = V_0$$ (1.1)

- **Equilibrium equations**
  $$\sigma \in S^{[0, T]}$$
  $$\forall t \in [0, T], \quad \forall \sigma^r \in U_0$$
  $$- \int_{\Omega} \text{Tr}[\sigma^r(U^r)] \, d\Omega + \int_{\Omega} f_d \cdot U^r \, d\Omega + \int_{\partial \Omega} F_d \cdot U^r \, dS = \int_{\Omega} \rho \ddot{U} \cdot U^r \, d\Omega$$ (1.3)

- **Constitutive relation**
  $$\forall t \in [0, T], \quad \forall M \in \Omega$$
  $$\sigma|_{t} = A \left( \varepsilon(U) \right) |_{t; \tau \in [0, t]}$$ (1.4)

In this formulation:

- the strain $\varepsilon(U)$ associated with $U$ under the assumption of small
The notion of quality of a finite element solution

perturbations is defined by\(^1\)

\[
\mathcal{S}(\mathcal{U}) = \frac{1}{2} \left[ \frac{\partial \mathcal{U}}{\partial \mathcal{M}} + \left( \frac{\partial \mathcal{U}}{\partial \mathcal{M}} \right)^T \right]
\]

or, in indicial notation\(^2\),

\[
[\mathcal{S}(\mathcal{U})]_{ij} = \frac{1}{2} \left( U_{i,j} + U_{j,i} \right)
\]

- \(\rho\) designates the mass density, which here is constant with respect to \(t\); 
- \(\mathcal{U}^{[0,T]}\) designates the space in which the displacement field is sought; 
- \(\mathcal{S}^{[0,T]}\) designates the space in which the stress field is sought; 
- \(\mathcal{U}_0\) designates the space of the virtual fields chosen, which is of the form

\[
\mathcal{U}_0 = \{ U^* \text{ "regular" and such that } U^* |_{\partial \Omega} = 0 \};
\]

- \(\mathbf{A}\) is an operator which depends on the material and characterizes its behavior; the value of the stress at time \(t\) is a function of the history of the strain rate until time \(t\).

REMARKS

1. Relation (1.4) is a functional formulation of the constitutive behavior. For virtually all the constitutive relations commonly used, this formulation is equivalent to a formulation with internal variables using a free energy and a dissipation pseudopotential.

2. The regularity required depends on the problems being studied and is expressed in the choice of the spaces \(\mathcal{U}^{[0,T]}, \mathcal{S}^{[0,T]}\) and \(\mathcal{U}_0\).

Usually, this choice consists in imposing that the free energy and the kinetic energy of the fields being considered have finite values throughout the domain \(\Omega\) at each time \(t\) in \([0, T]\). For more details, one can refer, for example, to

---

\(^1\) The notation \(\partial \mathcal{U}/\partial \mathcal{M}\) designates the gradient of field \(\mathcal{U}\). \(\cdot^T\) designates the Euclidean transpose.

\(^2\) The notation \(\cdot, j\) designates the partial derivative with respect to the \(i\)th coordinate.
[DUVAUT - LIONS, 1972], [BREZIS, 1973], [EKELAND - TEMAM, 1974], [NECAS - HLAVACEK, 1981], [DAUTRAY - LIONS, 1984], [POGU - TOURNEMINE, 1992]. Further on, we will assume that the mathematical framework chosen ensures the existence and uniqueness of the solution \((U, \sigma)\) of the reference problem. Let us note that a very general uniqueness condition can be found in [LADEVEZE, 1996].

3. The equilibrium equations (1.3) are written in global form; under the condition of regularity, they are equivalent to the local equations

\[
\begin{align*}
\forall t \in [0, T] & \\
\text{div } \sigma + f_d &= \rho \ddot{U} \quad \text{in } \Omega \\
\sigma n &= F_d \quad \text{on } \partial \Omega 
\end{align*}
\]  

(1.5)

where \(n\) denotes the unit outward normal to the boundary \(\partial \Omega\).

4. We are using the language and notations of structural mechanics. However, the concepts and methods we will present are very general and can be applied, with some relatively simple adjustments, to numerous other physical situations.

Two simplified versions of the reference model [Equations (1.1) - (1.4)] will be used especially throughout this work.

The first, very simple version is that of linear statics in which the data \(U_d, f_d, F_d\) and the unknown fields \(U, \sigma\) are independent of time, and the constitutive relation is a linear relation between the stress \(\sigma\) and the strain \(\varepsilon\) (linear elasticity). Of course, in this case, the initial conditions (1.2) are irrelevant and one is interested in the final configuration at time \(T\). Thus, the reference problem becomes:

Find a displacement field \(U(M)\) and a stress field \(\sigma(M)\) defined on \(\Omega\) such that:

- Kinematic constraint equations

\[
\begin{align*}
U \in \mathcal{U} & \quad \text{and} \quad U |_{\partial \Omega} = U_d 
\end{align*}
\]  

(1.6)
The notion of quality of a finite element solution

- \textbf{Equilibrium equations}

\[ \sigma \in \mathcal{S}, \quad \forall \mathcal{U}^* \in \mathcal{U}_0 \]

\[ \int_\Omega \text{Tr}[\sigma_{\mathcal{U}}(\mathcal{U}^*)] \, d\Omega = \int_\Omega f_{\mathcal{U}} \cdot \mathcal{U}^* \, d\Omega + \int_{\partial \Omega} F_{\mathcal{U}} \cdot \mathcal{U}^* \, dS \quad (1.7) \]

- \textbf{ Constitutive relation}

\[ \sigma = \mathbf{K}_{\mathcal{U}}(\mathcal{U}) \quad (1.8) \]

In this formulation, \( \mathbf{K} \) designates the Hooke operator of the material.

For example, for an isotropic elastic material, one has

\[ \mathbf{K}_{\mathcal{U}} = \lambda \text{Tr}(\mathcal{U}) \mathbf{I} + 2\mu \mathcal{U} \]

where \( \mathbf{I} \) is the identity operator and \( \lambda \) and \( \mu \) are two Lamé coefficients classically connected to the Young modulus \( E \) and the Poisson ratio \( \nu \) by the relations

\[ \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1 + \nu)} \]

The spaces \( \mathcal{U} \) and \( \mathcal{S} \) are homologous to the spaces \( \mathcal{U}^{[0,T]} \) and \( \mathcal{S}^{[0,T]} \) for time-independent fields. For example, for linear static problems in 3D, one has

\[ \mathcal{U} = \left[ H^1(\Omega) \right]^3; \quad \mathcal{S} = \left[ L^2(\Omega) \right]^6 \text{ and } \mathcal{U}_0 = \left\{ \mathcal{U}^* \in \left[ H^1(\Omega) \right]^3 \mid \mathcal{U}^* \big|_{\partial \Omega} = 0 \right\} \]

where \( L^2(\Omega) \) designates the space of the square-integrable functions in \( \Omega \), and \( H^1(\Omega) \) the Sobolev space of the functions of \( L^2(\Omega) \) with derivatives in the space \( L^2(\Omega) \).

The second simplified version corresponds to the case where the acceleration terms can be considered negligible.

Under these conditions, one obtains the problem:

\textit{Find a displacement field} \( \mathcal{U}(M, t) \text{ and a stress field } \sigma(M, t) \text{ defined on } \Omega \times [0, T] \text{ such that}

- \textbf{Kinematic constraint equations and initial conditions}

\[ \mathcal{U} \in \mathcal{U}^{[0,T]} \]
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\[ \forall t \in [0, T], \quad U_{|\partial\Omega} = U_d \quad \text{(1.9)} \]
\[ \forall M \in \Omega, \quad U|_{t=0} = U_0, \quad \dot{U}|_{t=0} = V_0 \quad \text{(1.10)} \]

- **Equilibrium equations**
\[ \sigma \in S^{[0, 1]} \]
\[ \forall t \in [0, T], \forall U^* \in U_0 \]
\[ \int_{\Omega} \text{Tr} \{ \sigma \mathbb{S}(U^*) \} \, d\Omega = \int_{\Omega} f_d \cdot U^* \, d\Omega + \int_{\partial\Omega} F_d \cdot \dot{U}^* \, dS \quad \text{(1.11)} \]

- ** Constitutive relation**
\[ \forall t \in [0, T], \forall M \in \Omega \]
\[ \sigma|_{t} = A\big( \mathbb{S}(\dot{U})|_{\tau}; \tau \in [0, t] \big) \quad \text{(1.12)} \]

This is called a “quasi-static” problem. Plasticity and viscoplasticity problems are often treated in this framework.

### 1.3 The approximate problem and discretization errors

Except in very exceptional cases, the solution \((U_{ex}, \sigma_{ex})\) to the reference problem [Equations (1.1) - (1.4)], even in its simplest form [Equations (1.6) - (1.8)], cannot be obtained explicitly. In practice, by introducing simplifying hypotheses called *condensation hypotheses*, the reference problem is replaced by an approximate, simpler problem whose solution, which we will designate by \((U_h, \sigma_h)\), can be obtained numerically. Of course, there are numerous methods for constructing approximate problems; these depend not only on the nature of the problem being considered, but also on the types of results being sought.

We are going to examine the cases of linear problems and of quasi-static nonlinear evolution problems, limiting ourselves to the most commonly used approximation methods. Here, for two characteristic cases, our objective is to emphasize the impact of a numerical approximation on the quality of the calculated approximate solution.
1.3.1 Linear problems

Today, the method used most often to obtain approximations of Equations [(1.6) - (1.8)] is the finite element displacement method.

The finite element displacement method

Let us formulate Problem [(1.6) - (1.8)] using the potential energy\(^3\)

\[
E_p(U) = \frac{1}{2} \int_{\Omega} \text{Tr} \left[ \varepsilon(U) K \varepsilon(U) \right] d\Omega - \int_{\Omega} f_d \cdot U d\Omega + \int_{\partial \Omega_d} E_d \cdot U dS
\]

Indeed, the displacement \(U_{ex}\) is the solution of the minimization problem

\[
E_p(U_{ex}) = \min_{U \in K\alpha} E_p(U)
\]

(1.13)

where \(U K\alpha\) designates the Kinematically Admissible displacement fields, i.e. the fields which verify the kinematic constraint equations (1.6). Then, the solution stress field \(\sigma_{ex}\) is obtained through the constitutive relation

\[
\sigma_{ex} = K \varepsilon(U_{ex})
\]

The finite element displacement method consists in seeking the minimum of the potential energy only on a finite-dimension subspace of \(K\alpha\) displacement fields rather than on the set of all the \(K\alpha\) fields. Thus, the approximate field \(U_h\) is the solution of the problem

\[
E_p(U_h) = \min_{U \in K\alpha \text{ and } \text{of } PE \text{ type}} E_p(U)
\]

(1.14)

or, in terms of extremum conditions:

\[
\text{Find a displacement field } U_h \text{ defined on } \Omega \text{ such that:}
\]

\[
U_h \in U_h \quad \text{and} \quad \forall U_h' \in U_{h0}
\]

\[
\int_{\Omega} \text{Tr} \left[ \varepsilon(U_h) K \varepsilon(U_h) \right] d\Omega = \int_{\Omega} f_d \cdot U_h d\Omega + \int_{\partial \Omega_d} E_d \cdot U_h dS
\]

(1.15)

where \(U_h\) is the affine finite-dimension subspace of \(U\) chosen and \(U_{h0}\) is the subspace of \(U_0\).

\(^3\) An introduction to the potential energy is proposed in Chapter 2.
The stress field $\sigma_h$ is then obtained, element by element, through the constitutive relation

$$\sigma_h = K \mathbf{z}(U_h)$$

(1.16)

In practice, Problem (1.15) is expressed by the linear system

$$Kq = F$$

(1.17)

where:

- $q$ is the vector of nodal displacements (degrees of freedom);
- $K$ is the stiffness matrix;
- $F$ is the vector of generalized loads.

For example, for $U_d = 0$, one has

$$K = \int_{\Omega} B^T K B \, d\Omega; \quad F = \int_{\Omega} N^T f_d \, d\Omega + \int_{\partial\Omega} N^T F_d \, d\Gamma$$

(1.18)

with

$$U_h(M) = N(M)q \quad \text{and} \quad \mathbf{z}(U_h(M)) = B(M)q$$

(1.19)

where $N$ is the matrix of shape functions.

For more details on the finite element displacement method, the reader may refer, for example, to [BATHE, 1982], [IMBERT, 1984], [HUGHES, 1987], [BATOZ - DHATT, 1990], [ZIENKIEWICZ - TAYLOR, 1988], and for the stochastic aspects to [KEIBER - HEIN, 1992].

**Nonsatisfaction of the equilibrium equations**

If one compares the reference problem and the approximate problem, one can observe that the approximate solution $(U_h, \sigma_h)$ verifies, as does the exact solution $(U_e, \sigma_e)$, the kinematic constraint equations and the constitutive relation. However, the field $\sigma_h$ does not verify the equilibrium equations:

In the displacement-type finite element method, the main approximation concerns the equilibrium equations.
More specifically, the stress $\sigma_h$ presents three types of equilibrium deficiencies:

- the interior equilibrium equation is not verified
  \[ \text{div}\sigma_h + f_d \neq 0 \quad \text{in} \ \Omega \]  \hspace{1cm} (1.20)
- the stress vector is not in equilibrium with the applied loads
  \[ \sigma_h n = E_d \quad \text{on} \ \partial \Omega \]  \hspace{1cm} (1.21)
- the stress vector is discontinuous at the interface between two elements
  \[ \left[ \sigma_h n \right]_{|E_1} + \left[ \sigma_h n \right]_{|E_2} \neq 0 \quad \text{on} \ \Gamma_{E_1E_2} \]  \hspace{1cm} (1.22)

with the notations defined in Figure 1.3.

![Figure 1.3. Notations at the interface between two elements.]

**Other sources of discretization errors**

In fact, the above equilibrium deficiencies are not the only possible sources of errors. Depending on the continuum mechanics model being used, other sources of error exist or can exist:

- *Errors due to nonrespect of the geometry:* For example, the border of a circular plate is replaced by a polygon if one uses elements with straight edges, or by parabolic arcs if one uses isoparametric elements of degree 2.
- *Errors due to the approximation of the displacement boundary conditions:* If the prescribed displacements are nonzero, it is necessary, in order for the fields of
the finite element type to be $KA$, that the prescribed $U_d$ be compatible with the
type of element chosen; for example, on the boundary of an element in $\partial_1 \Omega$,
the given displacement $U_d$ should be linear for first-order elements, or
parabolic for second-order elements.

- **Errors due to the approximation of the applied loads** In practice, the loads
actually taken into account in finite element programs are approximations of
the real loads; for example, for first-order elements, the loads $f_d$ are assumed
to be constant within each element.

- **Errors due to the numerical treatment of the approximate problem** Numerical
integration errors during the calculation of $\mathbf{K}$ and $\mathbf{F}$, errors during the
resolution of the linear system (errors due to the ending of the iterations if an
iterative method is being used, and, in all cases, roundoff errors).

In a preliminary stage, the first three types of errors can be disregarded,
which is equivalent to assuming that the corresponding approximations have
been made on the level of the continuum mechanics reference model. For
more details on these types of errors, the reader could refer, for example, to
[STRANG - FIX, 1976], [CIARLET, 1978]. The last type is of a different nature and
cannot be avoided. However, considering the precision of modern computers,
in numerous everyday situations and for most of the problems that we are
considering here, these errors are completely negligible compared to the
discretization errors due to the nonrespect of the equilibrium equations of the
reference model. Nevertheless, methods of evaluation – or, at least, detection –
of this type of error exist, for example, the method of [LA PORTE - VIGNE,
1974], which is stochastic. In practice, it is therefore necessary to perform
several calculations in order to propagate the roundoff errors differently
[DAUMAS - MULLER, 1997].

**Remarks and comments**

The finite element displacement method as described in the previous
paragraphs is used in mechanics for the resolution of 2D and 3D linear elasticity
problems. One should note that 2D and 3D thermal equilibrium problems also
fall within the domain of application of these types of methods.
Nevertheless, there are also many other finite element methods: “equilibrium” methods based on the minimization of the complementary energy, originally developed by FRAEIJS DE VEUBEKE [FRAEIJS DE VEUBEKE, 1965], [FRAEIJS DE VEUBEKE - SANDER, 1968], [FRAEIJS DE VEUBEKE - HOGGE, 1970], as well as numerous variations of mixed methods based on more or less sophisticated mixed principles [WASHIZU, 1975], [ZIENKIEWICZ - TAYLOR, 1988], [VALID, 1995]. Let us make a special mention of the family of elements developed by JIROUSEK [JIROUSEK, 1985], [JIROUSEK - WROBLESKI, 1996].

For structures of the beam, plate and shell types, the numerous finite elements proposed in the literature can be interpreted in the framework of mixed formulations [VALID, 1995], [BATOZ - DHATT, 1990], [CRISFIELD, 1991].

Another large family of finite elements is that of boundary elements associated with the integral equation method, which is extremely effective for 3D homogeneous and isotropic media [BREBBIA - TELLES - WROBEL, 1984], but loses much of its interest outside of these assumptions, despite some exceptions, such as [BONNET, 1999].

1.3.2 Nonlinear problems

Here, we will consider the example of a nonlinear quasi-static problem [Equations (1.9) - (1.12)]. The classical treatment of this type of problem by the incremental method requires both a space discretization and a time discretization.

**Discretization of the problem**

In this section, to simplify the presentation, we will assume that the prescribed displacements are zero, \( U_d = 0 \).

**Discretization in space**

The displacement field is sought in the form

\[
U_h(M, t) = N(M)q(t)
\] (1.23)