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Optimization of Elliptic Systems

Theory and Applications

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Preface

The present monograph is intended to provide a comprehensive and accessible introduction to the optimization of elliptic systems. This area of mathematical research, which has many important applications in science and technology, has experienced an impressive development during the past two decades. There are already many good textbooks dealing with various aspects of optimal design problems. In this regard, we refer to the works of Pironneau [1984], Haslinger and Neittaanmäki [1988], [1996], Sokołowski and Zolésio [1992], Litvinov [2000], Allaire [2001], Mohammadi and Pironneau [2001], Delfour and Zolésio [2001], and Mäkinen and Haslinger [2003]. Already Lions [1968] devoted a major part of his classical monograph on the optimal control of partial differential equations to the optimization of elliptic systems. Let us also mention that even the very first known problem of the calculus of variations, the *brachistochrone* studied by Bernoulli back in 1696, is in fact a shape optimization problem.

The natural richness of this mathematical research subject, as well as the extremely large field of possible applications, has created the unusual situation that although many important results and methods have already been established, there are still pressing unsolved questions. In this monograph, we aim to address some of these open problems; as a consequence, there is only a minor overlap with the textbooks already existing in the field.

The exposition concentrates along two main directions:

- the optimal control of linear and nonlinear elliptic equations, including *variational inequalities* and *control into coefficients problems*,

- problems involving unknown and/or variable domains, like general *shape optimization problems* defined on various classes of bounded domains in Euclidean space, or *free boundary problems* arising in various physical processes.

It should be noted that many shape optimization problems occur naturally as control into coefficients problems. A large and interesting class of examples of this type, to which the whole of Chapter 6 is devoted, concerns the optimization of basic mechanical structures like beams, plates, arches, curved rods, and shells.

There are strong connections between all these seemingly different types of problems. This fact has for the first time been illustrated in the so-called *map-*

ping method introduced by Murat and Simon [1976], which makes it possible to transform domain optimization problems into control into coefficients problems. Throughout this monograph, we will try to elucidate such connections. Another classical contribution to the solution of shape optimization problems is the *speed method*, which was introduced by Zolésio [1979] and thoroughly discussed in the above-mentioned publications.

One basic feature of this textbook is the endeavor to relax the needed regularity assumptions as much as possible in order to include large classes of possible applications. We have succeeded in this aim for several fundamental questions:

- The existence theory for general domain optimization problems presented in Chapter 2 requires just the uniform continuity of the domain boundaries.

- The existence theory and the sensitivity analysis for plates and for curved mechanical structures, mainly performed in Chapter 6, is established under regularity hypotheses that are one or two degrees (depending on the case) lower than those usually postulated in the scientific literature.

Another characteristic of this book is that we have tried to stress the application of optimal control methods even in the case of problems involving variable/unknown domains. In this respect, it should be mentioned that our techniques are close to the works of Lions [1968], [1983], Cesari [1983], Barbu [1984], [1993], and Barbu and Precupanu [1986]. We are thoroughly convinced that optimal control theory may provide a rather complete and reliable approach to the challenging problems involving the optimization of systems defined on variable domains. Many of the presented results in this direction, mostly in Chapter 5, are original contributions of the authors.

In order to give the reader a comprehensive overview of the subject, we also report on other important results from the existing literature. Whenever certain theoretical developments are already available in textbook form, our discussion will be limited to the shortest possible presentation.

The book is organized in six chapters that give a gradual and accessible presentation of the material, where we have made a special effort to present numerous examples, both at the theoretical and at the numerical level. The material covers

- motivating examples of "purely" mathematical nature or originating from various applications (in Chapter 1),

- general existence results for control and shape optimization problems (in Chapter 2),

- a sensitivity analysis of linear and nonlinear control problems in the absence of differentiability assumptions, based on various penalization methods (in Chapter 3),

- the presentation of the a priori estimates technique for the numerical approximation of control problems governed by linear or nonlinear elliptic equations (in Chapter 4),
- optimal control and other approaches in unknown domain problems including free boundaries and optimal design (in Chapter 5),
- a fairly complete optimization theory of curved mechanical structures like arches, curved rods, and shells (in Chapter 6).

The three appendices collect important notions and results from the theory of function spaces and elliptic equations, from convex and nonlinear analysis, and from functional analysis, which are frequently used throughout this monograph.

In Chapters 5 and 6, several rather complex geometric optimization problems are studied in detail and are completely solved, including numerical results. We do not discuss the questions that arise from the practical implementation of the presented methods on a computer or from the solving of the associated finite-dimensional problems, as they do not enter into the objective of this book.

Let us also mention at this place that in order to keep the exposition at a reasonable length and due to other reasons, several directions of active research, such as second-order optimality conditions, a posteriori error estimates, homogenization methods, and applications of shape optimization in fluid mechanics, could not be covered in this textbook. However, we have tried to provide the reader with the corresponding relevant references in some of these subjects.

Now we comment briefly on some examples and applications, and we make a more detailed presentation of the text. The aim is to give the reader, from the very beginning, a clear image about the problems and the questions that are studied in this book, and about their motivation and difficulties.

We consider first the simplest case of an elastic shell of constant thickness that admits a general cylindrical surface as its midsurface. We assume that the shell is clamped along two of its generators and the forces acting on it are constant along the generators and perpendicular to them. Consequently, it is clear that the resulting deformation of the shell is also constant along the generators.

It is enough to investigate a two-dimensional section perpendicular to the generators. The obtained structure in $\mathbf{R}^2$ is called an arch, and its deformation is described by the so-called Kirchhoff–Love model. We mention bridges, roads, industrial tubes, windows, roofs, among others, as real-life examples entering this description. The design of such structures puts several important questions to the engineer or the architect: maximize the mechanical resistence of the structure, minimize the total cost, fulfill all the (technological) constraints that are imposed, etc. In general, a "compromise" among the sometimes conflicting aims has to be found.

We indicate now the mathematical formulation of the Kirchhoff–Love model. If $\varphi = (\varphi_1, \varphi_2) : [0, 1] \to \mathbf{R}^2$ is the parametrization of the arch with respect to its arc length and $c : [0,1] \to \mathbf{R}$ denotes its curvature, then the deformation vector $\bar{v} = (v_1, v_2) \in H_0^1(0,1) \times H_0^2(0,1)$ is the solution of

$$\int_0^1 \left[\frac{1}{\varepsilon} (v_1' - c\,v_2)(s)(u_1' - c\,u_2)(s) \;+\; (v_2' + c\,v_1)'(s)(u_2' + c_1\,u_1)'(s) \right] ds$$
$$= \int_0^1 (f_1\,u_1 + f_2\,u_2)(s)\,ds\,, \qquad \forall\; u_1 \in H_0^1(0,1)\,,\; \forall\; u_2 \in H_0^2(0,1)\,.$$

Here, $\sqrt{\varepsilon}$ represents the constant thickness of the arch and $[f_1, f_2] \in L^2(0,1)^2$ are, respectively, the tangential and normal components of the forces loading the clamped arch (assumed to act in its plane), while the tangential component v_1 and the normal component v_2 perform a similar representation for the deformation. The arbitrary functions $u_1 \in H_0^1(0,1)$ and $u_2 \in H_0^2(0,1)$ are test functions specific to the weak (variational) formulation of differential equations. Let us also mention that a complete study of this problem may be found in Ciarlet [1978, p. 432].

As the shape of the arch is completely characterized by its curvature c, the corresponding geometric optimization problems may be formulated as the minimization of some functional subject to the Kirchhoff–Love model as a side constraint and with the function c as the minimization parameter (control). For instance, one integral cost functional of interest is

$$\int_0^1 [v_2(s)]^2\,ds\,.$$

This means to find the form of the arch that has a minimal normal displacement in the sense of the above norm under the action of some known load (f_1, f_2). This is a natural safety requirement in many applications. Further (technological) constraints may be imposed directly on the admissible controls c or on the corresponding state (v_1, v_2).

We notice that the mere formulation of these problems requires the curvature c and its derivative (in the second term on the left side of the above equation). To ensure the integrability of such expressions one needs $\varphi \in W^{3,\infty}(0,1)^2$ or $\varphi \in C^3[0,1]^2$ for the corresponding parametrization. It is obvious that such requirements are inappropriate to the potential applications (see Figure 1.1 in Chapter 6, the Gothic arch). Moreover, some of the simplest and most popular discretization approaches (see Chapter 4) introduce nonsmooth approximations of φ in a natural way, and again the Kirchhoff–Love model cannot be applied. Such examples show that new mathematical methods have to be developed in order to relax the regularity hypotheses and to ensure a broad class of applications. In this book, a more sophisticated variational technique called the control variational method, based on control theory, is discussed. It is due to the authors and represents an alternative to the classical Dirichlet principle in the theory of elliptic equations. It is used for the analysis and optimization of Lipschitzian arches in Section 6.1 and of a simplified model of plates with discontinuous thickness in §3.4.2. More geometric optimization problems with mechanical background, such as optimal design of three-dimensional elastic curved rods and of general elastic shells, are studied by other methods in Sections 6.2 and 6.3. Thickness optimization problems for plates are investigated in §2.2.2 and Section 3.4. They are highly nonconvex optimization problems, but they still enjoy the property that they are defined in some known domain in the Euclidean space $\mathbf{R}^d$, $d \in \mathbf{N}$. In the above example, $d = 1$ and the domain is $]0,1[$.

We now present another example that involves unknown/variable domains. The application is related to the confinement of plasma in a tokamak machine. We denote by $\Omega \subset \mathbf{R}^2$ the smooth and bounded domain representing the cross section of the void chamber and by $D \subset \Omega$ its (unknown) subdomain occupied by the confined plasma (see Figure 2.1 in Chapter 1). Within the void region $\Omega \setminus \bar{D}$, the poloidal flux ψ satisfies (cf. Blum [1989, Ch. V]) the elliptic equation

$$-\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial \psi}{\partial y}\right) = 0 \quad \text{in } \Omega \setminus \bar{D},$$

which is nonsingular $(x > c > 0)$ due to the natural choice of coordinates, based on the symmetry of the tokamak in $\mathbf{R}^3$. The boundary ∂D of the plasma is one of the unknowns of the problem, and this is an example of a free boundary problem. In order to identify it, one uses supplementary measurements on the outer boundary $\partial\Omega$:

$$\psi = f, \quad \frac{1}{x}\frac{\partial \psi}{\partial n} = g \quad \text{on } \partial\Omega.$$

One can introduce a shape optimization problem with minimization parameter given by the unknown domain $D \subset \Omega$, with performance index

$$\int_{\partial\Omega} \left| \frac{1}{x}\frac{\partial \psi}{\partial n} - g \right|^2 d\tau$$

obtained by the penalization of the second boundary condition and with side conditions given by the first boundary condition and the elliptic equation for ψ in $\Omega \setminus \bar{D}$. This formulation can be further refined by introducing a fictitious control variable and a Tikhonov regularization as in Example 1.2.6 in Chapter 1. Other simple examples of variable domain optimization problems may be found in §2.3.1. In Section 5.1, the relationship between free boundary problems and shape optimization problems is further explored, while §5.3.1 presents the connection between variable domain problems and control into coefficients problems via the classical mapping and speed methods. Since such a procedure demands high regularity properties for the unknown domains, we introduce in Section 5.2 several alternative approaches, based on control theory, which may be applied in more general situations. Moreover, in Section 2.3 a rather complete existence theory for variable domains optimization problems is developed under the mere (uniform) continuity assumption for the unknown boundaries. In Sections 2.1, 2.2 (existence), and Chapter 3 (optimality conditions), a rather complete presentation of control problems for linear and nonlinear elliptic equations, including variational inequalities, is given.

Although all of us have been actively involved in the study of optimization problems in infinite-dimensional spaces for many years, the origin of this book can be traced back to the lectures delivered by one of us in 1995 during the summer school that is organized annually by the University of Jyväskylä. These lectures have been published in the form of the report Tiba [1995b]. The following ten years were marked by an intensive cooperation between us that is witnessed by the publication of numerous papers in all of the research directions forming the subject of this monograph.

Much of the material covered in this volume is original and resulted from our studies when we were affiliated with the University of Jyväskylä, the Humboldt University Berlin, the Institute for Mathematics of the Romanian Academy of Sciences in Bucharest, and the Weierstrass Institute in Berlin. The financial support of these institutions, of the Academy of Finland, of the Alexander-von-Humboldt Foundation, and of the DFG Research Center MATHEON in Berlin, is gratefully acknowledged.

This monograph is addressed to a large readership, primarily to master's or doctoral students and researchers working in this field of mathematics. Much of this material will prove useful also to scientists from other fields where the optimization of elliptic systems occurs, such as physics, mechanics, and engineering.

During the preparation of this monograph, we obtained much encouragement and many helpful hints from a number of colleagues who cannot be named here. We are also indebted to Springer-Verlag, especially to Achi Dosanjh (New York), for their continuing encouragement.

Finally, we would like to thank Marja-Leena Rantalainen (Jyväskylä) and Jutta Lohse (WIAS Berlin) for their efforts in the excellent LaTeX setting of this text. We are also indebted to Dipl.-Math. Gerd Reinhardt (WIAS Berlin) for his help in solving the problems arising from the inclusion of the figures in the text. Of course, the authors carry the full responsibility for each occasional misprint or other possible mistake in this monograph.

Jyväskylä, Berlin, and Bucharest, March 2005

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A Brief Reader's Guide

The authors are fully aware of the fact that the reader of this volume will usually be interested in only a certain part of it. Therefore, we give some hints in order to facilitate the reader's orientation within the text.

The book is divided into six chapters, referred to as Chapter 1 to Chapter 6, and three appendices, referred to as Appendix 1 to Appendix 3. Each of the chapters consists of several "sections," called Section 1.1, Section 6.1, and so on. The sections themselves may be divided into several subsections, called "paragraphs" and referred to, for example, as §3.1.3. Also, these paragraphs may have subparagraphs denoted, for instance, by §3.1.3.1. Clearly, the latter refers to the first subparagraph of the third paragraph in the first section of Chapter 3.

Let us also comment on the numbering used in this textbook. Equations are numbered by three integers that refer to the corresponding chapter, section, and equation, in that order. If, for example, we refer to equation (4.2.6), then we mean the sixth equation in the second section of Chapter 4. Definitions, Theorems, Lemmas, Propositions, Corollaries, and Examples, are also numbered sectionwise within each chapter; typical examples are Theorem 5.2.1, Lemma 6.2.4, Definition 2.2.1, and so on. An exception to this rule is the numbering within the three appendices, where references are made in the form Proposition A1.1, Theorem A2.3, Definition A3.1, and the like, with obvious meaning. Remarks are not numbered. Finally, figures are numbered sectionwise within each chapter.

Contents

Chapter 1

Introductory Topics

This first chapter brings a brief introduction to the problems to be studied in the following chapters. We present a large variety of examples involving different types of controls (distributed, boundary, pointwise, by the coefficients, linear, nonlinear, ...). All of them are governed by elliptic differential equations that are either defined in a given (fixed) spatial domain or in an a priori unknown domain. We also consider cases in which the domain itself is the minimization parameter (so-called *shape optimization*). For some of the examples, physical origin and practical relevance will be pointed out.

To avoid any unnecessary technicalities, we introduce the mathematical terminology mainly in the examples, in an informal manner. A brief rigorous account of the basic mathematical notions and results used throughout this monograph is contained in the three appendices at the end of the book, where relevant references are also given. It is, however, assumed that the reader has a working knowledge of the fundamental elements of analysis and functional analysis as presented, for instance, in the standard monographs by Rudin [1987] and Yosida [1980].

1.1 Some General Notions

We now discuss several definitions that are related to general optimal control problems. The setting adopted in this section simplifies the presentation and the systematization of the fundamental notions and is also motivated by a large class of examples and applications that will be described below in the next sections.

To begin with, let us consider three reflexive Banach spaces U, V, Z together with their respective dual spaces U^*, V^*, Z^*. By renorming, if necessary, we may assume without loss of generality that U, V, Z and their duals are strictly convex spaces. Moreover, let a Hilbert space H be given that is identified with its dual space and satisfies $V \subset H$ with continuous embedding. The scalar product in H and the pairing between V and its dual are denoted by $(\cdot\,,\cdot)_H$

and $(\cdot\,,\cdot)_{V^*\times V}$, respectively. The corresponding norms are denoted by $|\cdot|_H$, $|\cdot|_V$, $|\cdot|_U$, and so on; by $[\cdot\,,\cdot]$ we denote ordered pairs in product spaces.

Let $B : U \to Z$ be a linear and bounded operator, and let $A : V \to Z$ denote some (possibly nonlinear) operator. In many examples, we will have $Z = V^*$. We assume that for any fixed $f \in Z$ and any $u \in U$ (called *control*), the equation

$$Ay = Bu + f \tag{1.1.1}$$

has a unique solution $y \in V$ in a sense to be made precise (which is called the *state*). Consequently, (1.1.1) is sometimes named the *state equation*. In the applications to follow, y will be a weak solution to an elliptic problem. It may be defined in various ways, as one can see in Appendix 2 and in the subsequent examples. Later, we will also consider operators A depending directly on u, $Ay = A(u)y$, nonlinear operators B, and further generalizations.

Let a proper, convex, and lower semicontinuous mapping $L : V \times U \to\,]-\infty, +\infty]$ be given. We then introduce the abstract *control problem* (P) by

$$\inf \{L(y,u)\} \tag{1.1.2}$$

over all the pairs $[y,u]$ satisfying the state equation (1.1.1).

If $E = \mathrm{dom}\,(L) \subset V \times U$ denotes the closed convex set given by the effective domain of L (cf. Appendix 1), then we see that not all of the pairs $[y,u]$ satisfying (1.1.1) are meaningful for (1.1.2); indeed, some may give $L(y,u) = +\infty$. Consequently, the minimization in (1.1.2) is in fact considered only over all pairs $[y,u] \in E$ that satisfy (1.1.1). Such pairs are called *admissible* for the *cost functional* L or for the optimal control problem (P). We call E the *constraints set*, and we say in this case that the constraints are *mixed* since they involve both the state y and the control u.

It is quite standard in control theory to formulate the constraints explicitly, since they have their own motivation in the underlying applications. In general, $u = 0$ should be allowed as admissible control, corresponding to the case that no external influence is acting on the system.

Suppose now that some nonempty, closed, and convex sets $C \subset H$, $U_{ad} \subset U$ are given. We may then consider the separate constraints

$$\textit{control constraints} \quad u \in U_{ad}, \tag{1.1.3}$$

$$\textit{state constraints} \quad y \in C. \tag{1.1.4}$$

We get an equivalent formulation of the control problem (P) by including the constraints in the cost functional with the help of the *indicator function* $I_{C\times U_{ad}}$ of the set $C \times U_{ad} \subset H \times U$. To this end, we replace L by a new cost functional, namely by

$$L(y,u) + I_{C\times U_{ad}}(y,u). \tag{1.1.5}$$

In the following, the new cost functional (1.1.5) will again be denoted by L; this will not lead to any confusion. Now recall the definition of the indicator function (cf. Appendix 1) to see that $L(y,u) < +\infty$ only if $(y,u) \in C \times U_{ad}$, which means that any solution of the control problem (1.1.2) with the cost functional (1.1.5) automatically satisfies the constraints (1.1.3), (1.1.4). Of course, using the new cost functional (1.1.5) does not exclude the possibility that within the definition of the set E further (implicit) constraints are hidden.

In many cases it is advantageous to include only a part of the constraints in the cost functional while preserving the others in explicit form. If, for instance, only the control constraints are to be included, one considers the cost functional

$$L(y,u) + I_{V\times U_{ad}}(y,u). \tag{1.1.5'}$$

Also for this cost functional the generic notation L may be preserved with no danger of confusion.

Let us summarize: a general formulation of the optimal control problem (P) consists of the following ingredients:

- a *cost functional* to be minimized ((1.1.2)),
- a *state system* ((1.1.1)),
- various *constraints* ((1.1.3), (1.1.4)).

A fundamental hypothesis for the control problem (P) is that of *admissibility*. It can be stated in the following form:

$$\exists\, [\overline{y},\overline{u}] \in C \times U_{ad} \text{ such that } L(\overline{y},\overline{u}) < \infty \text{ and } A\overline{y} = B\overline{u} + f. \tag{1.1.6}$$

Without this assumption, the problem (P) may have an empty *admissible set* and be meaningless. For mathematical reasons, the case in which the admissible set of (P) is "rich" in some sense (typically, it has to be an open or a dense set with respect to some topology) is more interesting. Under such assumptions, we say that (P) is *nontrivial.* On the other hand, if (P) is "trivial," then its solution may be simple and thus not of mathematical interest.

Finally, let us mention that all the assumptions mentioned here can be relaxed in various ways; some of them may even be omitted. For instance, there is a rich literature on control problems without convexity hypotheses on L, E, C, U_{ad}, or allowing (1.1.1) not to be well-posed, and so on. One well-known alternative approach is to require various differentiability or generalized differentiability assumptions instead. In this connection, we refer to the monographs by Lions [1983] and Clarke [1983], where some extensions of this type are thoroughly examined. We shall study such topics in later sections of this monograph.

1.2 Motivating Examples

1.2.1 Cost Functionals

The cost functionals studied in this monograph will generally be of the form

$$L(y,u) = \theta(y) + \psi(u), \tag{1.2.1}$$

where $\theta : V \to]-\infty, +\infty]$, $\psi : U \to]-\infty, +\infty]$ denote some proper, convex, and lower semicontinuous functions. A standard instance of this type is the quadratic functional

$$L(y,u) = \frac{\alpha}{2}\,|y - y_d|_V^2 + \frac{\beta}{2}\,|u|_U^2, \quad \alpha, \beta \geq 0, \tag{1.2.2}$$

where $y_d \in V$ is given.

The interpretation of (1.2.2) in connection with the control problem (P) is the following: we seek an admissible control $u \in U_{ad}$ such that the associated state $y \in C$ given by (1.1.1) is as close as possible to the "desired state" y_d. In addition, this control has to obey a minimal expenditure of energy condition (or minimal expenses condition, in general) reflected by the second term in (1.2.2). In fact, a compromise between the two (usually conflicting) aims "y close to y_d" and "minimal expenses" has to be found, and the relative importance of the criteria with respect to each other is expressed by the choice of the *weight coefficients* $\alpha, \beta \geq 0$.

As an anecdotal observation, we remark that the coefficients in (1.2.2) are chosen in this special form (as very frequently in the scientific literature) just because this "simplifies" the writing of the gradient of L, which plays a central role and is frequently used.

Notice that while (1.2.1), (1.2.2), and (1.1.2) define convex or even strictly convex functionals, the composed functional characterizing the control problem (P),

$$J(u) = L(y(u), u), \tag{1.2.3}$$

may be nonconvex. In fact, the state $y = y(u)$ defined by (1.1.1) may depend nonlinearly on u. If the operator A is linear, then J remains convex (or strictly convex), and any *optimal control* u^* is *global* (unique) if it exists. That is, the minimization property is valid with respect to the whole admissible set. The set of the global optimal controls is then convex. Otherwise, J may admit many *local* minimum points, in general. The existence of *optimal pairs* $[y^*, u^*]$ will be discussed in the next chapter. Their characterization, the development of methods to recover additional information on them, and their numerical approximation are among our basic objectives in this monograph.

Another fundamental example for a quadratic cost functional is obtained in the following way: Suppose that another Banach space W is given, and let $D : V \to W$ denote a linear and bounded operator (which in this connection is usually called an *observation operator*). We then consider the cost functional

$$L(y,u) = \frac{\alpha}{2}\,|Dy - \overline{y}_d|_W^2 + \frac{\beta}{2}\,|u|_U^2, \tag{1.2.4}$$

where $\overline{y}_d \in W$ has the same significance as y_d above. This setting is of particular practical importance and typically arises in situations in which the state y cannot be directly or fully observed, but only indirectly or in parts through the observation Dy. Typically, if (1.1.1) is a partial differential equation in a smooth domain, the operator D may be some trace operator on the boundary of the domain, a restriction operator to some subdomain, a partial differential operator of lower order, or the like.

A general form for the mappings θ, ψ occurring in (1.2.1) is obtained using integral functionals having *convex integrands*. To introduce such functionals, let $\Omega \subset \mathbf{R}^d$, $d \in \mathbf{N}$, be (Lebesgue) measurable, and suppose that $g : \Omega \times \mathbf{R}^m \to]-\infty, +\infty]$, $m \in \mathbf{N}$, satisfies the following conditions:

(i) $g(x, \cdot)$ is proper, convex, and lower semicontinuous for a.e. $x \in \Omega$.

(ii) g is measurable with respect to the σ-field of $\Omega \times \mathbf{R}^m$ generated by the product of the Lebesgue σ-field in Ω and the Borel σ-field in $\mathbf{R}^m$.

Such mappings g are called *normal convex integrands* (see Rockafellar [1970], Ioffe and Tikhomirov [1974], Levin [1985]). They have the basic property that the function $x \mapsto g(x, y(x))$ is measurable on Ω for any measurable function $y : \Omega \to \mathbf{R}^m$ (cf. Appendix 1, Proposition A1.1). Conditions (i), (ii) generalize the classical Carathéodory condition that $g(\cdot\,, \cdot)$ be finite and measurable in the first variable and continuous in the second.

For $y \in L^p(\Omega)^m$, $p \geq 1$, we then define the integral cost functional θ on $V = L^p(\Omega)^m$ by

$$\theta(y) = \begin{cases} \int_\Omega g(x, y(x))\, dx\,, & \text{if } g(\cdot, y(\cdot)) \in L^1(\Omega), \\ +\infty\,, & \text{otherwise.} \end{cases} \tag{1.2.1$'$}$$

Under appropriate conditions, θ turns out to be proper, convex, and lower semicontinuous (cf. Appendix 1). For the mapping ψ occurring in (1.2.1), we can proceed in a similar way. Also, one may consider the case that y is replaced by some Dy in (1.2.1)$'$.

Finally, let us point out a simple trick that is very useful in the numerical solution of optimal control problems. Suppose that $[y_0, u_0]$ is an admissible pair for (P), i.e., satisfies (1.1.1), (1.1.3), (1.1.4). Then, we may slightly modify the form of (1.2.4) by setting

$$\widetilde{L}(y, u) = \frac{\alpha}{2}\, |Dy - Dy_0|_W^2 + \frac{\beta}{2}\, |u - u_0|_U^2. \tag{1.2.4$'$}$$

The advantage of this form is that $[y_0, u_0]$ is obviously a global minimum (even when A is nonlinear and the corresponding $\widetilde{J}(u) = \widetilde{L}(y(u), u)$ is nonconvex) for the control problem $(\widetilde{\mathrm{P}})$ defined by (1.1.1), (1.1.4), (1.1.3), (1.2.4)$'$, with the *optimal value* equal to zero. Moreover, $(\widetilde{\mathrm{P}})$ has a structure that is very similar to that of (P). This a priori knowledge is helpful if one wants to test numerical code for the solution of (P). In particular, this idea is simple to apply when no state constraint (1.1.4) is imposed ($C = H$). Otherwise, even the question of finding an admissible pair $[y_0, u_0]$ may be very difficult due to the implicit character of (1.1.4).

1.2.2 Partial Differential Equations Setting

Here, we formulate several examples of elliptic state systems and related optimization problems that are among the objectives of this monograph. To this end, let a bounded domain $\Omega \subset \mathbf{R}^d$ with smooth boundary $\Gamma = \partial\Omega$ be given, and let $a_{ij} \in L^\infty(\Omega)$, $i,j = 1,\dots,d$, define a (possibly nonsymmetric) coefficients matrix that satisfies with some fixed $a > 0$ the ellipticity condition

$$\sum_{i,j=1}^{d} a_{ij}(x)\,\xi_i\,\xi_j \geq a\sum_{i=1}^{d}\xi_i^2 \quad \text{for all } \xi \in \mathbf{R}^d \text{ and a.e. } x \in \Omega. \tag{1.2.5}$$

Example 1.2.1 Define the linear and bounded operator $A : V = H_0^1(\Omega) \to V^* = H^{-1}(\Omega)$ by

$$Ay = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial y}{\partial x_j}\Big) + a_0\,y, \tag{1.2.6}$$

where $a_0 \in L^\infty(\Omega)$ with $a_0 \geq 0$ a.e. in Ω is given, and where the derivatives are understood in the sense of distributions. Let $U = L^2(\Omega)$, and let $B : L^2(\Omega) \to H^{-1}(\Omega)$ be the canonical injection operator, $Bu = iu = u$, for any $u \in L^2(\Omega)$. Then the state system (1.1.1) becomes a boundary value problem of Dirichlet type:

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial y}{\partial x_j}\Big) + a_0\,y \;=\; u + f \quad \text{in } \Omega, \tag{1.2.7}$$

$$y \;=\; 0 \qquad \text{on } \partial\Omega, \tag{1.2.8}$$

where $f \in L^2(\Omega)$ is fixed, and where (1.2.7), (1.2.8) have to be understood in the weak sense (see Appendix 2, Example A2.6), i.e.,

$$\sum_{i,j=1}^{d}\int_\Omega a_{ij}\,\frac{\partial y}{\partial x_j}\,\frac{\partial v}{\partial x_i}\,dx + \int_\Omega a_0\,y\,v\,dx = \int_\Omega (u+f)\,v\,dx \quad \forall\, v \in H_0^1(\Omega). \tag{1.2.9}$$

We say that we have a *distributed control* (or action) since u is defined in the domain Ω. A related situation is obtained if ω is a measurable subset of Ω and $B : L^2(\Omega) \to H^{-1}(\Omega)$ is given by $Bu = u\chi_\omega$, with χ_ω being the characteristic function of ω in Ω. Then the control action is again distributed, namely in ω, and (1.2.7) becomes

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\,\frac{\partial y}{\partial x_j}\Big) + a_0\,y = u\,\chi_\omega + f \quad \text{in } \Omega. \tag{1.2.7$'$}$$

We indicate some possible choices for the cost functional (1.2.4) that are appropriate in this situation. If $W = L^2(\Omega)$ and $D : H_0^1(\Omega) \to L^2(\Omega)$ is the canonical injection, then

$$L(y,u) = \frac{\alpha}{2}\int_\Omega |y(x) - y_d(x)|^2\,dx + \frac{\beta}{2}\int_\Omega u^2(x)\,dx, \quad y_d \in L^2(\Omega), \tag{1.2.10}$$

and we have a *distributed observation*. If $Dy = \nabla y$, we have a distributed observation of the gradient of the solution:

$$L(y,u) = \frac{\alpha}{2}\int_\Omega |\nabla y(x) - \tilde{y}_d(x)|^2_{\mathbf{R}^d}\, dx + \frac{\beta}{2}\int_\Omega u^2(x)\, dx, \quad \tilde{y}_d \in L^2(\Omega)^d. \quad (1.2.11)$$

The domain Ω in (1.2.10), (1.2.11) may be replaced by some measurable subsets of Ω, at least in one of the integrals.

Let us assume now that the coefficients a_{ij}, a_0, $i,j = 1,\ldots,d$, are sufficiently regular to guarantee that the solution y of (1.2.7), (1.2.8) belongs to $H^2(\Omega)$ (i.e., is a strong solution, cf. Appendix 2). Then, by virtue of the trace theorem (Appendix 2, Theorem A2.1), the *outer conormal derivative*

$$\frac{\partial y}{\partial n_A} = \sum_{i,j=1}^{d} a_{ij}\frac{\partial y}{\partial x_j}\cos(n, x_i) \quad (1.2.12)$$

on Γ (n is the outer unit normal to Γ) satisfies $\frac{\partial y}{\partial n_A} \in H^{1/2}(\Gamma)$. Taking some (relatively) open part $\Gamma_0 \subset \Gamma$, we may then choose as cost functional

$$L(y,u) = \frac{\alpha}{2}\int_{\Gamma_0}\left|\frac{\partial y}{\partial n_A} - \hat{y}_d\right|^2(\sigma)\, d\sigma + \frac{\beta}{2}\int_\Omega u^2(x)\, dx, \quad \hat{y}_d \in L^2(\Gamma_0). \quad (1.2.13)$$

In this case, we say that we have a *boundary observation* (while the control remains distributed in Ω).

To complete the definition of the control problem (P) for this example, let us discuss some instances of possible constraints. The simplest case is of course the *unconstrained* one when $U_{ad} = U = L^2(\Omega)$, $C = H = L^2(\Omega)$. One rough classification of the constraints is to distinguish between *local* and *global* ones. *Pointwise* constraints like

$$U_{ad} = \left\{u \in L^2(\Omega) : -1 \le u(x) \le 1 \text{ for a.e. } x \in \Omega\right\}, \quad (1.2.14)$$

$$U_{ad} = \left\{u \in L^2(\Omega) : u(x) \ge \ell(x) \text{ for a.e. } x \in \Omega,\ \ell \in L^2(\Omega) \text{ given}\right\}, \quad (1.2.15)$$

$$C = \left\{y \in H^1(\Omega) : |\nabla y(x)|_{\mathbf{R}^d} \le 1 \text{ for a.e. } x \in \Omega\right\}, \quad (1.2.16)$$

are of local type. Standard examples for constraints of global type are *integral* constraints like

$$U_{ad} = \left\{u \in L^2(\Omega) : |u|_{L^2(\Omega)} \le \mu\right\}, \quad \mu > 0, \quad (1.2.17)$$

$$C = \left\{y \in L^2(\Omega) : \int_\Omega y(x)\, dx \le 0\right\}. \quad (1.2.18)$$

A simple example of *mixed pointwise* constraints is given by

$$E = \left\{[y,u] \in L^2(\Omega) \times L^2(\Omega) : y(x) \le u(x) \text{ a.e. in } \Omega\right\}. \quad (1.2.19)$$

Let us briefly return to the control constraint (1.2.14). In this case, it is possible to introduce a new control $w \in L^2(\Omega)$ satisfying

$$u(x) = \sin(w(x)) \quad \text{a.e. in } \Omega.$$

Making corresponding substitutions, the optimal control problem (P) can be transformed into a control problem without constraints for w. The price to be paid for this simplification is that in (1.1.1) the dependence of the state on the new control variable w (more precisely, the operator corresponding to B) becomes nonlinear and that the convexity is lost. However, such simple tricks may be very effective in applications. For further details, we refer to Banichuk [1983, Chapter I].

We conclude this example with the remark that the above discussion of cost functionals and of constraints applies to any type of elliptic control problem. In the subsequent examples we will therefore focus our attention on the analysis of the state equation and control action.

Example 1.2.2 Let us now concentrate on *boundary control* problems. We begin with control action via Neumann boundary conditions, by considering the state system

$$Ay = f \quad \text{in } \Omega, \tag{1.2.20}$$

$$\frac{\partial y}{\partial n_A} = u \quad \text{on } \partial\Omega, \tag{1.2.21}$$

where A is given by (1.2.6), and where we assume that $a_0(x) \geq \mu > 0$ a.e. in Ω. The variational (weak) formulation of (1.2.20), (1.2.21) is obtained using Green's formula:

$$\int_\Omega \sum_{i,j=1}^{d} a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \int_\Omega a_0\, y\, v\, dx = \int_\Omega f v\, dx + \int_{\partial\Omega} u\, v\, d\sigma \quad \forall\, v \in H^1(\Omega). \tag{1.2.22}$$

To recover the abstract setting (1.1.1), we fix some $f \in L^2(\Omega)$ and put $V = H^1(\Omega)$ and $U = H^{-1/2}(\partial\Omega)$. Moreover, $A : V \to V^*$ is generated by the left-hand side of (1.2.22) (cf. Appendix 2, Theorem A2.3), while $B : U \to V^*$ is defined by

$$(Bu, v)_{H^1(\Omega)^* \times H^1(\Omega)} = \int_{\partial\Omega} u\, v\, d\sigma \quad \forall\, v \in H^1(\Omega). \tag{1.2.23}$$

Obviously, the restriction of A to $H_0^1(\Omega)$ coincides with (1.2.6). Notice that also the choice $U = L^2(\partial\Omega)$ is possible with the same definition (1.2.23) of B.

Next, we turn our attention to control action via Dirichlet boundary conditions. It is known that the inhomogeneous Dirichlet boundary value problem does not admit a purely variational (weak) formulation and that a suitable translation has to be employed first in order to reduce the problem to the homogeneous case (Křížek and Neittaanmäki [1990]). In the setting of control problems the corresponding translation operator may be, roughly speaking, interpreted as the operator B. If the state system is described by (1.2.20) and

$$y = u \quad \text{on } \partial\Omega, \tag{1.2.24}$$

then we may fix $B : H^{-1/2}(\partial\Omega) \to L^2(\Omega)$ by $Bu = y_u$, where y_u satisfies (1.2.24) and

$$Ay_u = 0 \quad \text{in } \Omega. \tag{1.2.25}$$

We refer at this place to Appendix 2, Example A2.7, for the definition of a very weak solution of (1.2.24), (1.2.25), using the transposition method. We choose $V = V^* = L^2(\Omega)$, $U = H^{-1/2}(\partial\Omega)$, define a new operator $\tilde{A} : V \to V^*$, $\tilde{A}y = y$, and a new $\tilde{f} \in L^2(\Omega)$, given by

$$A\tilde{f} = f \quad \text{in } \Omega, \qquad \tilde{f} = 0 \quad \text{on } \partial\Omega. \tag{1.2.26}$$

If we write the abstract equation (1.1.1) in the form $\tilde{A}y = Bu + \tilde{f}$, then it is equivalent to (1.2.20), (1.2.24).

The operator B is called the *Dirichlet mapping* and plays an essential role in this formulation.

Example 1.2.3 Let us also address the *pointwise control* of linear systems. We take $V = H_0^k(\Omega)$, $k > \frac{d}{2}$, with d being the dimension of Ω. By virtue of the Sobolev embedding theorem (Appendix 2, Theorem A2.2), we have $V \subset C(\overline{\Omega})$, and the Dirac functional $\delta_{x_0} : V \to \mathbf{R}$, $\delta_{x_0}(v) = v(x_0)$, with some given $x_0 \in \Omega$, is linear and continuous on V, that is, $\delta_{x_0} \in V^*$.

Let us put $U = \mathbf{R}$, and let $B : U \to V^*$ be given by $Bu = u\,\delta_{x_0}$, which is a linear and bounded operator. We assume $A : V \to V^*$ in the form

$$Ay = \sum_{|\alpha| \le k} (-1)^{|\alpha|} D^\alpha (a_\alpha(x)\, D^\alpha y), \quad a_\alpha \in L^\infty(\Omega), \tag{1.2.6$'$}$$

where the multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}_0^d$, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, and $|\alpha| = \alpha_1 + \ldots + \alpha_d$ is its length, the derivatives are taken in the distributional sense, and the coercivity condition

$$(Ay, y)_{H^{-k}(\Omega) \times H_0^k(\Omega)} \ge c\, |y|^2_{H_0^k(\Omega)} \quad \forall y \in H_0^k(\Omega), \tag{1.2.5$'$}$$

with some $c > 0$, is assumed to hold. Then the state equation (1.1.1) becomes

$$\sum_{|\alpha| \le k} (-1)^{|\alpha|} D^\alpha (a_\alpha(x)\, D^\alpha y) = Bu + f \quad \text{in } \Omega, \tag{1.2.27}$$

$$y = 0, \quad \frac{\partial y}{\partial n} = 0, \quad \dots, \quad \frac{\partial^{k-1} y}{\partial n^{k-1}} = 0 \quad \text{on } \partial\Omega. \tag{1.2.28}$$

According to Appendix 2, this system admits a unique weak solution. Owing to the definition of B, the control u is concentrated in the point $x_0 \in \Omega$.

Example 1.2.4 We now examine *nonlinear* elliptic boundary value problems as state equations. We start with the semilinear case. Let A be defined as in (1.2.6) and consider a continuous mapping $\varphi : \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ having a continuous derivative φ_y with respect to its second argument variable, and the property that for any $u \in L^s(\Omega)$, $s \geq \max\{2, \frac{d}{2}\}$ (d = dimension of Ω), the mapping $\varphi(\cdot,\cdot,u(\cdot))$ is of Carathéodory type. Moreover, the following conditions are assumed:

$$|\varphi(x,0,v)| \leq M(x) + \hat{C}\,|v|, \quad \text{a.e. in } \Omega,\ v \in \mathbf{R}, \tag{1.2.29}$$

$$0 \leq \varphi_y(x,y,v) \leq (M(x) + \hat{C}\,|v|)\,\eta(|y|), \quad \text{a.e. in } \Omega,\ v,y \in \mathbf{R}, \tag{1.2.30}$$

with a function $M \in L^s(\Omega)$, a constant $\hat{C} > 0$, and a nondecreasing function $\eta : \mathbf{R}_+ \to \mathbf{R}_+$. The state equation has the form

$$Ay + \varphi(x, y(x), u(x)) = 0 \quad \text{in } \Omega, \tag{1.2.31}$$

$$y = 0 \quad \text{on } \partial\Omega. \tag{1.2.32}$$

If $a_{ij} \in C^1(\overline{\Omega})$, then (1.2.31), (1.2.32) has a unique strong solution $y \in W^{2,s}(\Omega) \cap H_0^1(\Omega) \cap L^\infty(\Omega)$; see Theorem A2.10 in Appendix 2 and the remark following it. In (1.2.31), (1.2.32), the control variable u appears implicitly. In order to fit this system into the formalism from Section 1.1, we put $B = 0$, and we allow $A = A(u)$, $u \in U = L^s(\Omega)$, to depend directly on the control parameter. One possible way to achieve this is to include the semilinear term $\varphi(x,\cdot,u)$ in the definition of $A(u)$ as a superposition (Nemytskii) operator (cf. Pascali and Sburlan [1978]).

One particular situation of interest is the *control in the coefficients* case. For instance, for $\varphi(x,y,u) = |u|y$ all the above assumptions are obviously fulfilled. The partial differential equation (1.2.31) then becomes linear with respect to y, but the dependence $u \mapsto y$, induced by it, is highly nonlinear. As a consequence, the associated optimization problems are nonconvex, and since they may have many local minima, they are stiff and hence difficult to solve numerically. An important application of this type arises in optimal shape design theory in connection with the so-called *mapping method.* For details, we refer to Pironneau [1984], Haslinger and Neittaanmäki [1988], as well as to the problems studied below in (1.2.51) and in §5.3.1.

Another important class of applications that may be described by control in the coefficients problems is given by the so-called *identification problems* to be discussed in Example 1.2.6 below.

Example 1.2.5 Let us assume for the moment that the symmetry condition $a_{ij} = a_{ji}$, $i,j = 1,2,\ldots,d$, is fulfilled. Then the Dirichlet principle shows that the solution $y \in V = H_0^1(\Omega)$ to (1.2.7), (1.2.8) (or, equivalently, to the weak formulation (1.2.9)) admits the alternative variational characterization

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial y}{\partial x_j} \frac{\partial y}{\partial x_i}\, dx + \int_\Omega a_0\, y^2\, dx - 2 \int_\Omega (u+f)\, y\, dx$$

$$= \operatorname*{Min}_{z \in V} \Big\{ \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial z}{\partial x_j} \frac{\partial z}{\partial x_i}\, dx + \int_\Omega a_0\, z^2\, dx - 2 \int_\Omega (u+f)\, z\, dx \Big\}. \quad (1.2.33)$$

Now let us consider the minimization problem when in (1.2.33) the full space V is replaced by a (nonempty) convex and closed set $S \subset V$. Again, there exists a unique minimizer $y_S \in S$ since the quadratic form in (1.2.33) is coercive and strictly convex (see Appendix 1). A straightforward computation shows that y_S is the unique solution to

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial y_S}{\partial x_j} \Big(\frac{\partial y_S}{\partial x_i} - \frac{\partial z}{\partial x_i} \Big) dx + \int_\Omega a_0\, y_S\, (y_S - z)\, dx \le \int_\Omega (u+f)\, (y_S - z)\, dx, \quad (1.2.34)$$

for all $z \in S$. Since, in turn, any solution to (1.2.34) is also a solution to the minimization problem, then (1.2.33) (with V replaced by S) and (1.2.34) are in fact equivalent problems. Relation (1.2.34) is called a *variational inequality* associated with the closed and convex set S. Notice that the symmetry condition is not essential for the existence of a unique solution to the variational inequality (1.2.34), as follows from the Lions–Stampacchia theorem (see Appendix 2, Theorem A2.3), which is a generalization of the classical Lax–Milgram lemma.

Now let $I_S : V \to]-\infty, +\infty]$ denote the (proper, convex, and lower semicontinuous) indicator function of S in V. Then (1.2.34) may be reformulated in the form

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial y_S}{\partial x_j} \Big(\frac{\partial y_S}{\partial x_i} - \frac{\partial z}{\partial x_i} \Big) dx + \int_\Omega a_0\, y_S\, (y_S - z)\, dx + I_S(y_S) - I_S(z)$$

$$\le \int_\Omega (u+f)\, (y_S - z)\, dx \quad \forall z \in V. \quad (1.2.34)'$$

More generally, let us consider for any proper, convex, and lower semicontinuous mapping $\Lambda : V \to]-\infty, +\infty]$ the variational inequality

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial y}{\partial x_j} \Big(\frac{\partial y}{\partial x_i} - \frac{\partial z}{\partial x_i} \Big) dx + \int_\Omega a_0\, y\, (y - z)\, dx + \Lambda(y) - \Lambda(z)$$

$$\le \int_\Omega (u+f)\, (y - z)\, dx \quad \forall z \in V. \quad (1.2.35)$$

Then it follows directly from the theory of maximal monotone operators (cf. Appendix 1, Theorem A1.7) that (1.2.35) admits a unique solution $y \in \operatorname{dom}(\Lambda)$.

Moreover, using the subdifferential $\partial\Lambda$ of Λ, we may rewrite (1.2.35) as a semilinear elliptic *inclusion*, namely as

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial y}{\partial x_j}\Big)+a_0\, y+\partial\Lambda(y) \ni u+f \text{ in } \Omega, \quad y=0 \text{ on } \partial\Omega. \qquad (1.2.35)'$$

Generally speaking, (1.2.35) or (1.2.35)$'$ may be viewed as extensions of the semilinear problem (1.2.31), (1.2.32) in the sense that the mapping φ is replaced by the nonsmooth and discontinuous (multivalued) subdifferential mapping $\partial\Lambda$.

In what follows, we give some important examples for possible sets S. We begin with the so-called *obstacle problem*:

$$S=\Big\{y\in H_0^1(\Omega)=V: y(x)\geq \mu(x) \text{ a.e. in } \Omega\Big\}, \qquad (1.2.36)$$

where $\mu\in H^2(\Omega)$ is a given function (called the *obstacle*) having the property that $\mu|_{\partial\Omega}\leq 0$, which ensures that S is nonempty.

Formally, the solution y of the *obstacle problem* (1.2.34), (1.2.36) will satisfy

$$\begin{aligned}
-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial y_S}{\partial x_j}\Big)+a_0\, y_S &= u+f \quad \text{a.e. in } \Omega^+=\{x\in\Omega: y_S(x)>\mu(x)\},\\
y_S &= \mu \quad \text{a.e. in } \Omega\setminus\Omega^+,\\
\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial y_S}{\partial x_j}\Big)+a_0\, y_S &\geq u+f, \quad y_S\geq\mu, \quad \text{a.e. in } \Omega.
\end{aligned}$$

The "surface" $\partial\Omega^+\setminus\partial\Omega$ separating Ω^+ from $\Omega\setminus\overline{\Omega}^+$ is a priori unknown and is called the *free boundary* of the obstacle problem. The region $\Omega\setminus\Omega^+$, where y_S is equal to the obstacle, is called the *coincidence set*.

Next, we consider the set S in (1.2.34) that characterizes the so-called *elasto-plastic torsion problem*,

$$S=\Big\{y\in H_0^1(\Omega)=V: |\nabla y(x)|\leq 1 \text{ a.e. in } \Omega\Big\}. \qquad (1.2.37)$$

Again, we may (formally) define two subregions of Ω,

$$\text{the \textit{plastic} region} \qquad \Omega_1=\Big\{x\in\Omega: |\nabla y_S(x)|=1\Big\},$$

$$\text{the \textit{elastic} region} \qquad \Omega_2=\Big\{x\in\Omega: |\nabla y_S(x)|<1\Big\},$$

such that (1.2.34) becomes an equality in one of the subregions (namely in Ω_2).

Let us mention that for choice $\mu(x)=d(x,\partial\Omega)$ the two problems (1.2.36), (1.2.37) are in fact equivalent (cf. Brézis and Sibony [1971]).

We also notice that the solution y_S of the variational inequality (1.2.34) satisfies $y_S\in S$, obviously. But this should be distinguished from a state constraint (although here the form is similar to (1.2.16)), since it is automatically fulfilled.

Indeed, it follows from the Lions–Stampacchia theorem mentioned above that a unique solution $y_S \in S \subset V$ exists for any $u \in U = L^2(\Omega)$.

Unilateral problems, that is, problems involving inequalities in place of equations, may also be formulated on $\partial\Omega$. For instance, consider the set

$$S = \left\{ y \in V = H^1(\Omega) : y|_{\partial\Omega} \geq 0 \right\}. \tag{1.2.38}$$

In this case, the (formal) interpretation of (1.2.34) can be deduced from the following chain of formal calculations: first, we insert $z = y_S + v \in S$ for all $v \in \mathcal{D}(\Omega)$ in (1.2.34). Then we obtain

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\,\frac{\partial y_S}{\partial x_j}\Big) + a_0\, y_S = u + f \quad \text{in } \mathcal{D}'(\Omega). \tag{1.2.39}$$

Next, multiplying (1.2.39) by any $z \in S$ and applying (formally) Green's formula, we find that

$$-\int_{\partial\Omega} \frac{\partial y_S}{\partial n_A}\, z\, d\sigma + \int_\Omega \sum_{i,j=1}^{d} a_{ij}\, \frac{\partial y_S}{\partial x_i}\, \frac{\partial z}{\partial x_j}\, dx + \int_\Omega a_0\, y_S\, z\, dx = \int_\Omega (u+f)\, z\, dx. \tag{1.2.40}$$

Then, we replace z in (1.2.34) by $z + y_S$, which is possible in view of (1.2.38), and use (1.2.40), to find that

$$\int_{\partial\Omega} \frac{\partial y_S}{\partial n_A}\, z\, d\sigma \geq 0 \quad \forall z \in S.$$

That is, we have

$$\frac{\partial y_S}{\partial n_A} \geq 0, \quad y_S \geq 0, \qquad \text{on } \partial\Omega.$$

Moreover,

$$\int_{\partial\Omega} y_S\, \frac{\partial y_S}{\partial n_A}\, d\sigma = 0,$$

which follows by using $z = y_S$ as test function in (1.2.39), (1.2.40), and by comparing with (1.2.34), where we put $z = 0$. Such boundary conditions are known as the *Signorini problem* and describe an elastic body Ω subject to volume forces $u + f$ and in contact with a rigid support body. This is an example of unilateral conditions on the boundary.

More generally, let $\Lambda : V = H^1(\Omega) \to]-\infty, +\infty]$ be defined by

$$\Lambda(y) = \begin{cases} \int_{\partial\Omega} j(y)\, d\sigma & \forall\, y \in V \text{ with } j(y) \in L^1(\partial\Omega), \\ +\infty & \text{otherwise}, \end{cases}$$

where $j : \mathbf{R} \to]-\infty, +\infty]$ is a proper, convex, and lower semicontinuous mapping. Then the variational inequality (1.2.35) (or, equivalently, the elliptic problem (1.2.35)′) has a unique solution $y \in V$ (see Barbu [1984]) that (formally) satisfies

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial y}{\partial x_j}\Big) + a_0\, y \;=\; u + f \quad \text{in } \mathcal{D}'(\Omega),$$
$$\frac{\partial y}{\partial n_A} + \partial j(y) \;\ni\; 0 \quad \text{on } \partial\Omega.$$

We remark that all the above formal arguments can be made rigorous provided that the solution y_S belongs to $H^2(\Omega)$ (strong solution). Boundary control action u may be studied as well.

1.2.3 Applications

We devote this paragraph to a first examination of some physically oriented applications. Further details and solutions of the problems will be provided later.

Example 1.2.6 We begin with a problem arising in the confinement of plasma in a *tokamak machine*. Let Ω be a smooth and bounded domain in $\mathbf{R}^2$ representing the cross section of the void chamber of a tokamak machine, and let $D \subset \Omega$ denote its (unknown) subdomain occupied by the confined plasma. Within the void region $\Omega \setminus \overline{D}$ the (unknown) poloidal flux ψ satisfies (cf. Blum [1989, Chapter V])

$$-\frac{\partial}{\partial x}\Big(\frac{1}{x}\frac{\partial\psi}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(\frac{1}{x}\frac{\partial\psi}{\partial y}\Big) = 0 \quad \text{in } \Omega \setminus \overline{D}, \tag{1.2.41}$$

which is a nonsingular second-order linear elliptic equation since the natural choice of coordinates, based on the symmetry of the torus representing the tokamak in $\mathbf{R}^3$, yields $x > c > 0$ in Ω for some constant c (see Figure 2.1). The boundary ∂D of the plasma region is an unknown of the problem and represents a *free boundary.* It is characterized as a level set by the relation

$$M \in \partial D \text{ if and only if } \psi(M) = \sup_{x\in F} \psi(x),$$

where F (see Figure 2.1) represents physical devices called *limitators* that may have various shapes.

The only available data are the measurements on the outer boundary $\partial\Omega$:

$$\psi \;=\; f \quad \text{on } \partial\Omega, \tag{1.2.42}$$

$$\frac{1}{x}\frac{\partial\psi}{\partial n} \;=\; g \quad \text{on } \partial\Omega. \tag{1.2.43}$$

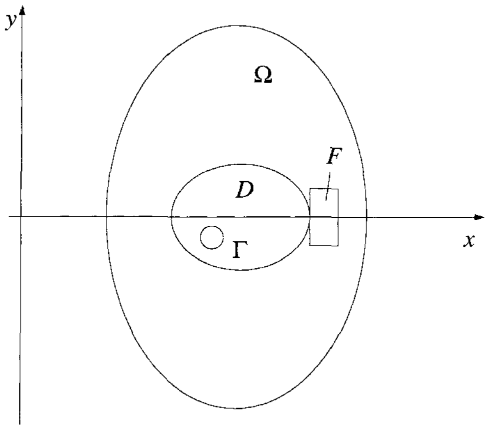

Figure 2.1. Schematic representation of the void chamber.

Thus, the problem to identify the subdomain D occupied by the plasma leads to an elliptic Cauchy problem ((1.2.41)–(1.2.43)) and is as such *ill-posed.* A *fictitious domain* approach to this problem consists in fixing some (artificial) smooth closed curve $\Gamma \subset D$ (see Figure 2.1), and defining the *least squares* boundary control problem in the domain Ω_0 limited by Γ and $\partial\Omega$,

$$\operatorname*{Min}_{u\in L^2(\Gamma)} \left\{ J(u) = \frac{1}{2} \left| \frac{1}{x}\frac{\partial\psi}{\partial n} - g \right|^2_{L^2(\partial\Omega)} \right\} \tag{1.2.44}$$

subject to

$$-\frac{\partial}{\partial x}\Big(\frac{1}{x}\frac{\partial\psi}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(\frac{1}{x}\frac{\partial\psi}{\partial y}\Big) = 0 \quad \text{in } \Omega_0, \tag{1.2.41$'$}$$

$$\psi = f \quad \text{on } \partial\Omega, \tag{1.2.42$'$}$$

$$\psi = u \quad \text{on } \Gamma. \tag{1.2.45}$$

In view of the lack of coercivity in (1.2.44), a *Tikhonov regularization* technique may be used. We choose some *regularization parameter* $\varepsilon > 0$ and replace the minimization problem (1.2.44) by

$$\operatorname*{Min}_{u\in L^2(\Gamma)} \left\{ J_\varepsilon(u) = \frac{1}{2} \left| \frac{1}{x}\frac{\partial\psi}{\partial n} - g \right|^2_{L^2(\partial\Omega)} + \frac{\varepsilon}{2} |u|^2_{L^2(\Gamma)} \right\}, \tag{1.2.46}$$

subject to (1.2.41)$'$, (1.2.42)$'$, and (1.2.45). This results in a standard boundary control problem with boundary observation and a linear state system. The convergence analysis for $\varepsilon \searrow 0$ was performed in Neittaanmäki and Tiba [1995], Neittaanmäki, Räisänen, and Tiba [1994]; see §5.2.3.1.

While the regularized problem (1.2.46) appears to be easy to solve, the sensitivity to measurement errors, which is intrinsic to all ill-posed problems, remains an important problem, and the interpretation of the results in terms of the original problem turns out to be a difficult task (see Falk [1990]).

Another category of problems that may be handled via control methods are the so-called *identification problems*. Suppose that some physical system (for instance, the equilibrium position of a clamped membrane) is described by the following mathematical model:

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial y}{\partial x_j}\Big) + a_0(x)\, y \;=\; f(x) \quad \text{in } \Omega, \qquad (1.2.7)''$$

$$y(x) \;=\; 0 \quad \text{on } \partial\Omega, \qquad (1.2.8)''$$

where $\Omega \subset \mathbf{R}^d$ is a smooth bounded domain and where the assumption (1.2.5) is fulfilled. In many applications it turns out to be difficult to measure or to have precise a priori knowledge of all the coefficient functions, which usually depend on the physical properties of the membrane or other parameters. On the other hand, it is natural to assume that some observation of the real state (the deflection of the membrane) $\tilde{y}$ of the system, denoted by $D\tilde{y}$, is available via measurements.
Suppose that $a_0 \in L^\infty(\Omega)$, $a_0 \geq 0$, is the unknown coefficient. Then the least squares approach to its determination leads to the problem

$$\operatorname*{Min}_{a_0 \in L^\infty(\Omega)_+} \Big\{ J(y) = \frac{1}{2}\,|Dy - D\tilde{y}|_W^2 \Big\} \qquad (1.2.47)$$

subject to (1.2.7)′, (1.2.8)′, and where W is the associated *observation space*, i.e., $D : H_0^1(\Omega) \to W$ is linear and continuous. We obtain a control in the coefficients problem (compare with Example 1.2.4), and one clear difficulty is its nonconvexity; in addition, it is also noncoercive. Therefore, a Tikhonov regularization technique is indicated also in this situation.

The above examples are special cases of *inverse problems*, an area of applications in which the optimal control approach is a standard method.

Example 1.2.7 Next, we describe some optimization problems involving geometric parameters, generally called *optimal shape design* problems. One such case, called the *optimal layout of materials*, is introduced as follows, starting from (1.2.7)″, (1.2.8)″ (in Example 1.2.6) and (1.2.47). We assume that $a_{ij}(x) = \delta_{ij} a(x)$ (δ_{ij} is the Kronecker symbol) and $a_0(x) = 1$ in Ω. The coefficient a can be interpreted as the thermal conductivity of the body given by Ω. We assume that the body consists of different materials having the thermal conductivities k_i, $i = \overline{1,m}$, that is,

$$a(x) = \sum_{i=1}^{m} \chi_i(x)\, k_i,$$

where χ_i is the characteristic function in Ω of the region occupied by the material indexed by i.

We then may ask the following question: If a fixed heat source f is given, what is the *optimal distribution* of the materials that maximizes the temperature y in a given subdomain $\omega \subset \Omega$ (or on some open part $\Gamma_0 \subset \partial\Omega$, etc.)?

To solve this problem, we may take one of the cost functionals

$$\operatorname*{Min}_{a}\left\{-\int_{\omega} y(x)\,dx\right\}, \tag{1.2.47$'$}$$

$$\operatorname*{Min}_{a}\left\{-\int_{\Gamma_0} y(\sigma)\,d\sigma\right\}. \tag{1.2.47$''$}$$

The minimization parameters are the subsets of Ω occupied by the various materials. Equivalently, one can use the characteristic functions χ_i, $i=\overline{1,m}$, as control unknowns. Apparently, we can interpret the problem as a control into coefficients problem, where 0 and 1 are the only admissible values for the controls. For a detailed discussion, we refer the reader to Tartar [1975], Pironneau [1984, §8.4], and §2.3.4, §5.2.2.1.

Let us now briefly comment on a stationary variant of the so-called *electrochemical machining process*. To this end, we consider the bounded domains $C \subset E \subset D \subset \Omega$ in $\mathbf{R}^3$ where D is variable (see Figure 2.2). In $D \setminus C$, we consider the obstacle problem (compare with (1.2.34), (1.2.36))

$$\int_{D\setminus C} \nabla y(x)\cdot\nabla(y-z)(x)\,dx \le \int_{D\setminus C} f(x)\,(y(x)-z(x))\,dx, \tag{1.2.48}$$

$$\forall\, z\in S=\left\{w\in H^1(D\setminus C): w|_{\partial C}=0,\ w|_{\partial D}=1,\ w\ge 0 \text{ a.e. in } D\setminus C\right\},$$

$$y\in S. \tag{1.2.49}$$

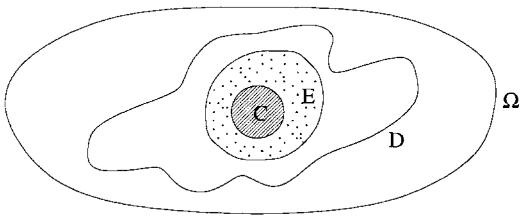

Figure 2.2. The electrochemical machining process.

The connection with the electrochemical machining process is the following: $D \subset \Omega$ represents the machine that contains a given core C (a hole, for instance) that cannot be influenced by the process. The sets ∂C and ∂D represent the electrodes, and the boundary condition $y=1$ on the boundary ∂D indicates that some fixed constant voltage is applied. If $\tilde{y}$ is the extension of y by 0 inside C, then the condition on ∂D in (1.2.49) should be understood in the sense $\tilde{y}-1\in H_0^1(D)$. The desired shape for the metallic workpiece to be