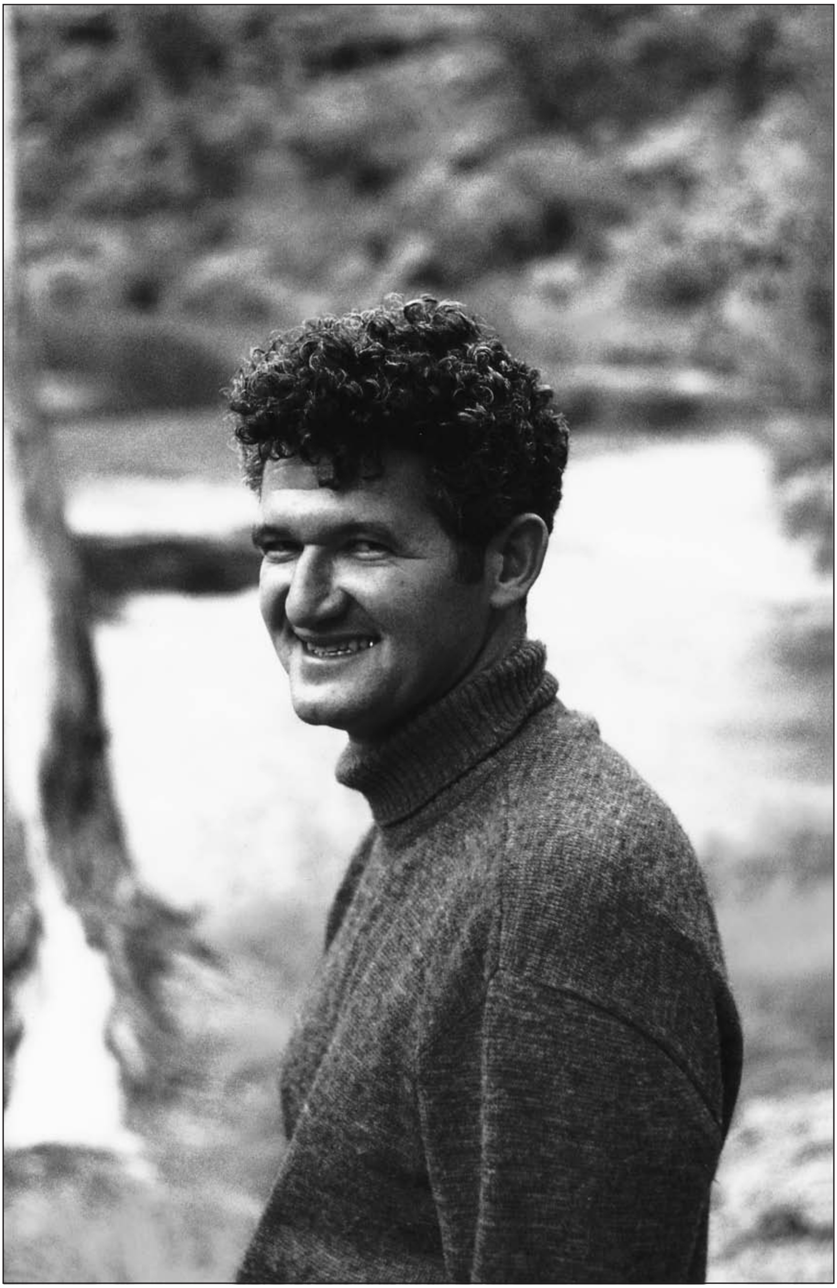
Selected Papers

Volume I



Arizona, 1968

Peter D. Lax

Selected Papers

Volume I

Edited by
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Springer

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New York, NY 10012
USA

Mathematics Subject Classification (2000): 11Dxx, 35-xx, 37Kxx, 58J50, 65-xx, 70Hxx, 81Uxx

Library of Congress Cataloging-in-Publication Data
Lax, Peter D.
[Papers. Selections]
Selected papers / Peter Lax ; edited by Peter Sarnak and Andrew Majda.
p. cm.
Includes bibliographical references and index.
ISBN 0-387-22925-6 (v. 1 : alk paper) — ISBN 0-387-22926-4 (v. 2 : alk. paper)
1. Mathematics—United States. 2. Mathematics—Study and teaching—United States. 3. Lax, Peter D. 4. Mathematicians—United States. I. Sarnak, Peter. II. Majda, Andrew, 1949- III. Title.
QA3.L2642 2004
510—dc22 2004056450

ISBN 0-387-22925-6 Printed on acid-free paper.

Printed in the United States of America. (EB)

9 8 7 6 5 4 3 2 1 SPIN 10950623

springeronline.com

Preface

Peter Lax's research spans many areas of pure and applied mathematics. It ranges from functional analysis, partial differential equations, and numerical methods to conservation laws, integrable systems, and scattering theory. Many of the papers in these volumes have become classics. They are a "must read" for any serious student of these topics, and their impact continues to be felt both explicitly and implicitly in current research. In terms of insight, depth, and breadth, Lax has few equals. The reader of these selecta will quickly appreciate his brilliance as well as his masterful touch. Having this collection of papers in one place allows one to follow the evolution of his ideas and mathematical interests and to appreciate how many of these papers initiated topics that developed a life of their own. It proves that even in today's highly specialized world of mathematics and science, it is still possible to work across disciplines at the very highest level.

The two volumes contain a selection from Lax's published papers. A complete list of these papers to date as well as a list of his books and monographs is provided in the text. The books are an outgrowth of years of research experience and far-reaching insights, an excellent example being his recent text on functional analysis. The selected papers are assembled according to topic and theme. After each paper, or collection of papers, is a commentary placing the paper in context and where relevant, discussing more recent developments. The volumes were edited together with Peter Lax.

We are very grateful to P. Deift, S. Friedland, F. Gesztesy, V. Guillemin, T.Y. Hou, S. Klainerman, H. McKean, B. Parlette and H. Widom for providing their expert commentaries on various papers. Thanks also to J. Heinze, I. Lindemann, and M. Spencer from Springer, who initiated and saw through this "Lax Selecta" project, and to Keisha Grady for her help in preparing the commentaries. The project took longer than was originally anticipated, but we believe that the result has been well worth the wait.

Andy Majda
and
Peter Sarnak
New York
May 2004

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[9] Reprinted from ***Contributions to the Theory of Partial Differential Equations***, edited by L. Bers, S. Bochner, and F. John, copyright 1954 by Princeton University Press. P. Lax and A. Milgram, Parabolic Equations, pp. 167–190, 1954, reprinted by permission of Princeton University Press.

Contents

Part III. Hyperbolic Systems of Conservation Laws

Part IV. Integrable Systems

PETER D. LAX
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1944

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PART I

Partial Differential Equations

ON THE EXISTENCE OF GREEN'S FUNCTION

PETER D. LAX

In this note we shall present a very short proof of the existence of Green's function for Laplace's equation for any domain with sufficiently smooth boundary in any number of independent variables. The proof is based on the continuous dependence of solutions of Laplace's equation on their boundary values. It is a modification of a proof given by Paul Garabedian, see [1]; the difference between the two approaches is that whereas Garabedian operates with a representation of harmonic functions in terms of their boundary data which he obtains by a variational argument, in our argument only the linear and bounded dependence of the solution on the boundary values figures.

1. In this section we shall treat the somewhat simpler two-dimensional case.

We consider a bounded domain D whose boundary C consists of a finite number of smooth curves (i.e., curves with continuous tangents).

B is the Banach space of all continuous functions defined on C, normed by the maximum norm.

B' is the submanifold of those elements of B for which the boundary value problem can be solved.[1]

Received by the editors December 13, 1951.

[1] It is easy to show that B' is closed, but this is not necessary for the argument.

Let P be any arbitrary point of D. Associated with P is a linear functional L_P defined over B':

$$L_P[\phi'] = h(P)$$

where h is the harmonic function whose value on C is ϕ'. L_P is clearly a linear functional, and by the maximum principle it is bounded, its bound being exactly one.

By the Hahn-Banach extension theorem, see [2], L_P can be extended as a bounded linear functional to the whole space B; and we imagine it so extended.[2]

Consider then the following two-parameter family of elements in B:

$$\psi(\xi, \eta) = \psi(Q) = \log |z - Q|, \qquad z \in C;$$

the point $Q = (\xi, \eta)$ may lie anywhere off the boundary C.

Clearly $\psi(\xi, \eta)$ has derivatives of second (in fact of all) order with respect to ξ and η, and is harmonic with respect to Q:

$$\psi_{\xi\xi} + \psi_{\eta\eta} = 0.$$

We construct the function

$$k_P(Q) = L_P[\psi(Q)].$$

The operations of differentiating with respect to ξ, η and applying the bounded linear functional L_P commute. Therefore $k_P(Q)$ is a harmonic function, i.e., harmonic in each of the components of the complement of C.

Let Q lie outside of D; the element $\psi(Q)$ lies in B', being the boundary value of the regular harmonic function $\log |z-Q|$. So

$$k_P(Q) = L_P[\psi(Q)] = \log |P - Q| \qquad \text{for } Q \text{ not in } D.$$

This explicit formula shows that $k_P(Q)$ is continuous up to the boundary in the exterior components. Next we shall show that the same is true as Q approaches the boundary from the interior, and in fact we shall show that $k_P(Q)$ is continuous across C. This would mean that $k_P(Q)$ in D is a harmonic function continuous up to the boundary and is equal to $\log |P-Q|$ there, i.e., $k_P(Q)$ is the regular part of Green's function.

As to the continuity of $k_P(Q)$ across C: Let Q be any point in D, Q' its mirror image with respect to the tangent at the nearest boundary point. If Q is near enough to the boundary, Q' will lie outside of D; furthermore on account of the smoothness of C the quotient

[2] It should be pointed out that since B is a separable Banach space, the extension can be accomplished without recourse to transfinite induction.

$$\frac{|z-Q|}{|z-Q'|}$$

tends to one uniformly for all z on C as the distance of Q from the boundary shrinks to zero. Hence

$$\max_{z \text{ on } C} \left| \log \frac{|z-Q|}{|z-Q'|} \right| = \| \psi(Q) - \psi(Q') \|$$

tends to zero as Q approaches C. Since L_P is a *bounded* linear functional,

$$k_P(Q) - k_P(Q') = L_P[\psi(Q) - \psi(Q')]$$

also tends to zero as Q approaches C.

This completes the proof for the plane; now we pass on to higher dimensions. The scheme of things remains the same; there is, however, some trouble in showing that $k_P(Q)$ is continuous across C for it is no longer true that

$$\psi(Q) - \psi(Q') = \frac{1}{|z-Q|^{n-2}} - \frac{1}{|z-Q'|^{n-2}}$$

tends to zero uniformly for all z on C as Q approaches the boundary. This difficulty will be overcome by using a more refined property of L_P than mere boundedness, namely, its monotonicity. According to the maximum theorem $L_P[\phi']$ is monotone over B', i.e., $L_P[\phi'] \geqq L_P[\phi'']$ if $\phi' \geqq \phi''$.[3] What we shall show is that L_P can be extended to B without losing this property of monotonicity. The proof of this relies on the full statement of the Hahn-Banach theorem:

Let $L[\phi]$ be a bounded linear functional defined on the submanifold B', $N(\phi)$ a subadditive, positive homogeneous functional defined over the whole space B which exceeds L over B':

$$L[\phi] \leqq N(\phi) \qquad \text{over } B';$$

Then there exists an extension of L to the full space which stays below N:

$$(1) \qquad L[\phi] \leqq N[\phi] \qquad \text{over } B.$$

Our choice of $N[\phi]$ is as follows:

$$(2) \qquad N[\phi] = \underset{\text{glb}}{\overset{\text{inf}}{\cancel{\text{lub}}}}\, L_P[\phi''], \qquad \phi'' \in B', \qquad \phi'' \underset{\geqq}{\cancel{\leqq}} \phi.$$

[3] The symbol $\phi' \geqq \phi''$ is defined to mean that $\phi' - \phi''$ is a non-negative function on C.

$N[\phi]$, ~~being the lub of linear functionals~~, is subadditive. It is clearly positive-homogeneous.

Since L_P is monotone over B', $L_P = N$ over B'. So, according to the Hahn-Banach theorem, $L_P[\phi]$ can be extended to the full space, linearly, so that it never exceeds $N[\phi]$.

To verify that L_P thus extended is monotone it is sufficient to show that $L_P[\phi]$ is nonpositive if ϕ is nonpositive on C. Now $N[\phi]$ itself is monotone, as is easily seen from its definition (2); for nonpositive ϕ, the handy sequence of inequalities

$$L_P[\phi] \leqq N[\phi] \leqq N[0] = 0$$

holds and constitutes a proof of the result desired.

We define as before

$$\psi(Q) = \frac{1}{|z - Q|^{n-2}}$$

and construct $k_P(Q) = L_P[\psi(Q)]$. $k_P(Q)$ is harmonic in D; what remains to be shown is that it is equal to $|Q-P|^{-n+2}$ on the boundary of D.

We have to impose a further restriction on the boundary: that through each boundary point we can pass two spheres of radius d, one lying entirely within D, the other outside of D, with the same constant d for all boundary points.[4]

Let Q be some point of D, R the nearest boundary point, and K' and K'' the interior and exterior tangent spheres through R. Let Q' and Q'' denote the inverted image of Q with respect to K' and K''. If Q is close enough to the boundary, Q' and Q'' are in the exterior.

Inverse points have the well known property that the quotient $|z-Q|/|z-Q'|$ has some constant value s' for all points z on the circumference of the circle K'. The value of s' approaches unity as Q tends to the boundary since the radius of K' does not change. Furthermore, if z is any point outside of K', $|z-Q|/|z-Q'|$ is greater than s'. This is in particular the case for z lying on C:

$$s' \leqq \frac{|z - Q|}{|z - Q'|} \qquad \text{for } z \text{ on } C$$

which implies that

$$\frac{s'^{n-2}}{|z - Q|^{n-2}} \leqq \frac{1}{|z - Q'|^{n-2}} \qquad \text{for } z \text{ on } C.$$

[4] This kind of restriction on domains is familiar in potential theory.

We rewrite the above as

$$\psi(Q) \leqq \frac{1}{s'^{n-2}} \psi(Q'). \tag{3}$$

The monotone character of L_P permits us to apply L_P to both sides of (3) and retain the inequality:

$$L_P[\psi(Q)] \leqq \frac{1}{s'^{n-2}} L_P[\psi(Q')].$$

By definition of the function k_P this can be written as

$$k_P(Q) \leqq \frac{1}{s'^{n-2}} k_P(Q').$$

As in the two-dimensional case $k_P(Q')$ can be evaluated explicitly since Q' is not in D:

$$k_P(Q') = \frac{1}{|P - Q'|^{n-2}}.$$

So our inequality becomes

$$k_P(Q) \leqq \frac{1}{s'^{n-2}} \frac{1}{|P - Q'|^{n-2}}.$$

The reverse inequality holds for Q'':

$$\frac{1}{s''^{n-2}} \frac{1}{|P - Q''|^{n-2}} \leqq k_P(Q).$$

As Q tends to some boundary point R, so does Q' and Q'', and s' as well as s'' tend to one. Thus the upper and lower bounds for $k_P(Q)$ both tend to $1/|P-R|^{n-2}$, and $k_P(Q)$ itself cannot do otherwise.

I should like to mention that C. Miranda [3] has made use of the extension theorem for this problem in a different manner; his reasoning is slightly more delicate but then it gives more: the solvability of the boundary value problem for all continuous boundary data, not just the existence of Green's function. A similar reasoning plays a role in his treatment [4] of the first boundary value problem for the biharmonic equation; in this case of course the bounded dependence of the solution on the boundary data lies much deeper.

G. Fichera has made use of an equivalent of the Hahn-Banach theorem in his investigations on boundary value problems, see [5] and [6], and so has J. Deny in [7]. M. Brelot, in [8], utilized the possibility of monotonic extension of monotonic linear transformations to prove that the generalized solution, in Wiener's sense, of the first boundary value problem for Laplace's equation can be characterized as the least (resp. greatest) harmonic majorant (resp. minorant) of harmonic functions which are continuous up to the boundary and whose boundary values are less (resp. greater) than the prescribed data.

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IX. PARABOLIC EQUATIONS

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§1. INTRODUCTION

This paper is about the initial-boundary value problem for the parabolic equation.

$$(1.1) \qquad u_t = - Lu$$

where L is a $2p^{th}$ order elliptic differential operator

$$(1.2) \qquad Lu = (-1)^p \sum_{\nu=0}^{2p} \sum_{i_1,\ldots,i_\nu=1}^{n} a_{i_1\ldots i_\nu} \frac{\partial^\nu}{\partial x_{i_1}\cdots \partial x_{i_\nu}} u$$

The coefficients a may depend on the space variables but not on time; they are supposed to have continuous derivatives of at least p^{th} order.

The value of u at $t = 0$ is prescribed on a bounded domain G of the Euclidean space E^n; on the boundary of G the function u and all its derivatives up to order $p - 1$ are prescribed to be zero.

We shall prove in this note that this problem has a unique solution. Our proof is an application of a theorem of Hille (see [11] and [13] and Yosida (see [23]) on unbounded operators which are infinitesimal generators of semigroups (i.e., for which the exponential function e^{At} can be defined). This theorem has been applied by Hille and Yosida to the case where L is a <u>second</u> order elliptic operator or a system of such (see [12], [24], [25]). The application presented here for the higher order case is made possible by the recently developed theory of higher order elliptic operators (see Gårding [10] and Browder [1] and [2]) in particular by Gårding's lemma, which asserts that such operators are bounded from below for functions satisfying the first boundary condition.

The operator (1.2), as written there, applies only to $2p$ times differentiable functions (which are required to satisfy the boundary condition); it has to be extended before the Hille-Yosida theorem can be applied to it. This is accomplished here by generalizing the Friedrichs extension of symmetric half-bounded operators to the non-symmetric case. The Friedrichs extension (see [4], [5] and [19]), we recall, assigns a unique self-adjoint extension to every halfbounded symmetric operator by means of the quadratic form induced by the operator; it has been used by Friedrichs to discuss formally self-adjoint second order elliptic operators. Both Browder and Gårding make use of it in treating the first boundary value problem for formally self-adjoint higher order elliptic operators.

It is easy to verify that the extended operator $\tilde{L}$ satisfies the hypotheses of the Hille-Yosida theorem, but all we can conclude is that the generalized equation $u_t = -\tilde{L}u$ has a unique solution with prescribed initial value. In section 5 we show however that these generalized solutions are genuine ones, at least if the coefficients of L are sufficiently differentiable. In the case of constant coefficients we show this by use of the fundamental solution of the parabolic equation (1.1), whose properties were investigated by Ladyzhenskaya (see [17]) and P. C. Rosenbloom (see [20]). For variable coefficients the differentiability properties can be deduced from the differentiability properties of solutions of elliptic equations; such theorems were obtained recently by Gårding [10], who refers to L. Schwartz, also by Browder, who uses the fundamental singularity (see [1]), by F. John who uses the method of spherical means (see [14], [15] and [16]), and by K. O. Friedrichs who employs estimates for the L_2 norms of higher derivatives and the mollifier (see [8]). In this paper we employ Friedrichs' version of these differentiability theorems.

The extension of unsymmetric operators is presented in section 2; some further properties of this extension (which are not needed in the rest of this paper) are discussed in section 4. The Hille-Yosida theorem is described in section 3, and differentiability properties of the solution in section 5.

The initial-boundary value problem for the parabolic equation (1.1) has also been treated by F. Browder by means of eigenfunction expansions (see [3]).

Our thanks are due to the Office of Naval Research and the Office of Ordnance Research for their support.

§2. POSITIVE BILINEAR FUNCTIONALS IN HILBERT SPACE AND ASSOCIATED OPERATORS

In this section we describe the Friedrichs extension of a nonsymmetric positive definite operator. The extension is based on the theory of linear transformations induced by bilinear forms.

The following theorem is a mild generalization of the Fréchet-Riesz Theorem on the representation of bounded linear functionals in Hilbert space.

THEOREM 2.1. Let H be a real Hilbert space, and $B(x,y)$ a (not necessarily symmetric) bilinear functional which is

a) bounded, i.e.,

$$|B(x,y)| \leqq C' \;\|x\| \cdot \|y\|$$

and

b) positive definite in the sense that there exists a positive constant C such that

$$C \;\|x\|^2 \leqq B(x,x)$$

for all x in H.

If a fixed element is substituted for either argument of $B(x,y)$, B becomes a bounded linear functional of the other. We claim that all linear functionals can be obtained in this way, i.e., to each bounded linear functional l defined over H there corresponds two unique elements x_l, x_l^* in H such that

$$l(x) \equiv B(x_l,x) \equiv B(x,x_l^*)$$

for all x in H.

PROOF. Let V be the subset of H consisting of those elements y to which there corresponds an element z in H for which

$$B(z,x) \equiv (y,x)$$

for all x in H.

We note that z is unique. For suppose that z and $\bar{z}$ both satisfy this condition; then $B(z,x) = B(\bar{z},x)$ for all x in H, that is $B(z-\bar{z},x) = 0$ for all x, in particular for $x = z-\bar{z}$. Because of the positiveness of B we can conclude that $\|z-\bar{z}\| = 0$, i.e., $z = \bar{z}$.

This reasoning shows us at the same time that the dependence of z on y is <u>bounded</u> with bound $1/C$. For

$$C\|z\|^2 \leqq B(z,\; z) = (y,\; z) \leqq \|y\|\,\|z\|$$

Our aim is to show that all elements of H belong to V. Clearly, V is a linear subspace; furthermore from the bounded dependence of z on

y and the continuity of the bilinear functional B it follows that the linear subspace V is <u>closed</u>. If V were not equal to all of H there would exist an element $z^o \neq 0$ orthogonal to all of V. Consider the linear functional $B(z^o,x)$. Since this is a bounded linear functional, by the Fréchet-Riesz Theorem there exists y such that $B(z^o,x) \equiv (y,x)$. Hence y lies in V. But taking x to be z^o in this relation yields $B(z^o,z^o) = (y,z^o)$, and this is zero, because of the orthogonality of z_o to V; it follows that V is all of H.

The above reasoning shows that every ordinary scalar product can be represented as a scalar product with respect to B; since according to the Riesz-Fréchet Theorem all bounded linear functionals can be so represented this proves Theorem 2.1.

COROLLARY 2.1. If U is a proper closed linear submanifold in H, there exists x_U and x_U^* in H such that

$$B(x_U,x) \equiv B(x,x_U^*) = 0$$

for all x in U.

PROOF. Let $\overline{x}_U$ be orthogonal to U. By Theorem 2.1, there exist x_U, x_U^* such that

$$B(x_U,x) \equiv B(x,x_U^*) \equiv (\overline{x}_U,x)$$

In the applications to follow, this representation Theorem 2.1 will be applied in the case where H is a subset of another Hilbert space H_o.

We denote by (x,y), $(x,y)_o$ and $\|x\|$, $\|x\|_o$ the respective inner products and norms in H and H_o. In addition we suppose that

(2.1) H is a dense subset of H_o

(2.2) there exists a constant k such that

$$\|x\| \geq k \|x\|_o$$

for all x in H.

We shall suppose as before that we are given a bilinear functional $B(x,y)$ defined on H, which, in terms of the metric in H, is bounded and positive. We shall show that this bilinear form induces two closed transformations, which are adjoints of each other: