CMS Books in Mathematics
Ouvrages de mathématiques de la SMC

1 HERMAN/KUČERA/ŠIMŠA Equations and Inequalities
2 ARNOLD Abelian Groups and Representations of Finite Partially Ordered Sets
3 BORWEIN/LEWIS Convex Analysis and Nonlinear Optimization
4 LEVIN/LUBINSKY Orthogonal Polynomials for Exponential Weights
5 KANE Reflection Groups and Invariant Theory
6 PHILLIPS Two Millennia of Mathematics
7 DEUTSCH Best Approximations in Inner Product Spaces
8 FABIAN ET AL. Functional Analysis and Infinite-Dimensional Geometry
9 KŘÍŽEK/LUCA/SOMER 17 Lectures on Fermat Numbers
10 BORWEIN Computational Excursions in Analysis and Number Theory
11 REED/SALES (Editors) Recent Advances in Algorithms and Combinatorics
12 HERMAN/KUČERA/ŠIMŠA Counting and Configurations
13 NAZARETH Differentiable Optimization and Equation Solving
14 PHILLIPS Interpolation and Approximation by Polynomials
15 BEN-ISRAEL/GREVILLE Generalized Inverses, Second Edition
16 ZHAO Dynamical Systems in Population Biology
17 GÖPFERT ET AL. Variational Methods in Partially Ordered Spaces
18 AKIVIS/GOLDBERG Differential Geometry of Varieties with Degenerate Gauss Maps
19 MIKHALEV/SHPILRAIN/YU Combinatorial Methods
20 BORWEIN/ZHU Techniques of Variational Analysis
21 VAN BRUMMELEN/KINYON Mathematics and the Historian's Craft: The Kenneth O. May Lectures

Glen Van Brummelen Michael Kinyon
Editors

Mathematics and the Historian's Craft

The Kenneth O. May Lectures

With 91 Figures

Springer

Glen Van Brummelen
Bennington College
Bennington, VT 05201
USA
gvanbrum@bennington.edu

Michael Kinyon
Department of Mathematical Sciences
Indiana University South Bend
South Bend, IN 46634-7111
USA
mkinyon@iusb.edu

Mathematics Subject Classification (2000): 01-06, 01A85

Library of Congress Control Number: 2005923503

ISBN-10: 0-387-25284-3 Printed on acid-free paper.
ISBN-13: 978-0387-25284-1

Printed in the United States of America. (EB)

9 8 7 6 5 4 3 2 1

springeronline.com

To Miriam May, in memory of Ken

Preface

1974 was a turning point for the history and philosophy of mathematics in North America. After years of planning, the first issue of the new journal *Historia Mathematica* was printed. While academic journals are born and die all the time, it was soon clear that *Historia Mathematica* would be a major factor in shaping an emerging discipline; shortly, it became a backbone for a global network of professional historians of mathematics. In the same year, the Canadian Society for History and Philosophy of Mathematics (CSHPM) was founded, adopting *Historia Mathematica* as its official journal. (In the 1990s, the CSHPM recognized its broader mission by naming *Philosophia Mathematica* as its official philosophical journal, rechristening *Historia Mathematica* as its historical journal.) Initially consisting almost entirely of Canadian members, the CSHPM has become in practice the North American society for the scholarly pursuit of history and philosophy of mathematics. The joint establishment of society and journal codified and legitimized the field, commencing what has become a renaissance of activity for the past 30 years.

These initiatives were begun by, and received much stimulus from, one man: Kenneth O. May, of the Institute for History and Philosophy of Science and Technology at the University of Toronto. May was a brilliant researcher, but he recognized that the viability of the fledging discipline required administrative leadership as well. In the introduction that follows, Amy Shell-Gellasch, CSHPM archivist, describes May's life and some of his achievements. Central to May's vision of the history of mathematics was the dichotomy between the role of the historian and the use that a mathematician might find for history. Mathematical practitioners, for reasons of pedagogy or in order to contextualize their own work, tend to focus on finding the antecedents for current mathematical theories in a search for how particular sub-disciplines and results came to be as they are today. On the other hand, historians of mathematics eschew the current state of affairs, and are more interested in questions that bear on the changing nature of the discipline itself. How, for instance, have the standards of acceptable mathematical practice differed through time and across cultures? What role do institutions and organizations play in the development of the subject? Does mathematics naturally align itself with the sciences or the humanities, or is it its own creature, and do these distinctions matter? The lead article in this volume, by Ivor Grattan-Guinness, is a strong statement on what makes history of mathematics unique, and reflects well May's own vision for our field.

May passed away, too early, in 1977. However, his legacy lives on partly through our thriving community; the continued prosperity of the CSHPM, *Historia Mathematica*, and *Philosophia Mathematica* are a resounding testament to that. In 2002, on the 25th anniversary of his passing, the CSHPM held a special meeting in May's honour. One of our actions at this meeting was to re-christen the keynote addresses at our annual general meetings as the "Kenneth O. May Lectures". Each of our annual meetings is a special occasion: while also providing a forum for presentations on all aspects of history and philosophy of mathematics, each meeting focuses on a specific theme, with activity revolving around an invited keynote address by a scholar of international repute. The diversity of these sessions over the years, witnessed in the table below, is a clear testament to the breadth and significance of the CSHPM's activities.

Since 1988 the CSHPM has preserved a record of the scholarly activities of the annual general meeting through the production of a volume of Proceedings, to which all speakers are invited to contribute. These Proceedings, distributed internally to Society members, are by now a repository of a great deal of valuable research. Some of these works have appeared elsewhere but many which deserve wider exposure have not; this volume represents our first attempt to correct this state of affairs. By printing the Kenneth May Lectures since 1990, we hope not only to choose some of the finest work presented at CSHPM meetings but also to present ourselves to the broader scholarly community. This volume represents by example who we are, how we approach the disciplines of history and philosophy of mathematics, and what we find important about our scholarly mission.

Many things happen over fifteen years. The editors attempted to reach all May lecturers since 1988, but were not wholly successful. Also, some of their lectures appeared later in formal scholarly journals (which the Proceedings is not), and some of these later versions incorporated improvements. In these cases we have chosen to reprint the polished final articles rather than the original lectures. One implication of this is that the bibliographic standards vary from article to article, reflecting the different sources in which the articles appeared. We are grateful to the following organizations that granted us permission to reprint articles free of charge from the pages of their journals and books: the Association for Symbolic Logic, the Canadian Mathematical Society (CMS), the Mathematical Association of America, and *Philosophia Mathematica*.

As editors of this volume, we have received a great deal of support from many people. The CSHPM, both its executive and its members, has been pivotal in working with us over the past year to produce the best possible public imprint for the Society. The authors of the papers in this volume and archivist Amy Shell-Gellasch have combined to produce a truly admirable body of work. The editors of the CSHPM Proceedings over the years, listed below, have moved mountains to produce these volumes. Jonathan and Peter Borwein, editors of the CMS Books in Mathematics, provided highly valued encouragement and advice. Ina Lindemann, Mark Spencer, and Anne Meagher of Springer Verlag helped tremendously in bringing this volume to fruition. Thanks also go to Dennis Richter for technical support. Our families have sacrificed in their own ways, putting up with late dinners and with occasionally absent parents; we thank them especially for their patience. Finally, our greatest gratitude is due to the man to whom this volume is dedicated. Ken, your vision lives and prospers in the 21st century. Without your insight and formative efforts, the CSHPM might not be here today. Thank you.

Glen Van Brummelen and Michael Kinyon

A note on the title. Ken May considered the practice of the history of mathematics to be a unique melding of the crafts of mathematician and historian. This entails sensitivity both to the mathematical content of the subject, and to the various contexts in which it can be understood. Our daily work is constantly informed by our attempts to achieve this delicate balance. In Ken's words:

"Clearly in historical work the danger in missing the mathematical point is matched by the symmetric hazard of overlooking a historical dimension. The mathematician is trained to think most about mathematical correctness without a time dimension, i.e., to think ahistorically. Of course it is interesting to know how a historical event appears when viewed by a twentieth century mathematician. But it is bad history to confuse this with what was meant at the time. The historian concentrates on significance in the historical context and on the historical relations between events. And this is equally interesting to the mathematician who wishes to understand how mathematics actually developed.

"One could continue indefinitely, but the essential point is that the best history requires sensitivity to both mathematical and historical issues, a respect for good practice of the crafts of both the historian and the mathematician. It may even be that the best mathematical research is aided by an appreciation of historical issues and results. I know of many instances and hope that the work of historians may contribute to increasing their frequency."[1]

[1]Kenneth O. May, "What is good history and who should do it?", *Historia Mathematica* 2 (1975), 453.

Annual Meeting Themes & Kenneth O. May Lecturers Since 1990

2003: Maritime Mathematics (Halifax, NS)
– Jim Bennett, *Geometry, Instruments and Navigation: Agendas for Research, 1500-1800*

2002: In Memory of Kenneth May (Toronto, ON)
– Ivor Grattan-Guinness, *History or Heritage? Historians and Mathematicians on the History of Mathematics*

2001: French Mathematics (Québec, PQ)
– Jean Dhombres, *The Applied Mathematics Origins of Lebesgue Integration Theory and Why it was Read as Pure Mathematics During the First Years of the 20th Century*

2000: History of Mathematics at the Dawn of a New Millennium (Hamilton, ON)
– Rüdiger Thiele, *Hilbert and his 24 Problems*

1999: Joint meeting with the British Society for History of Mathematics (Toronto, ON)

1998: Late 19th-Century Mathematics (Ottawa, ON)
– Volker Peckhaus, *19th-Century Logic: Between Philosophy and Mathematics*

1997: Science and Mathematics (St. John's, NF)
– Rüdiger Thiele, *The Mathematics and Science of Leonhard Euler*

1996: Ancient Mathematics (St. Catharines, ON)
– Alexander Jones, *Greek Applied Mathematics*

1995: Mathematics Circa 1900 (Montreal, PQ)
– Joseph W. Dauben, *Cantor and the Epistemology of Set Theory*

1994: History of Mathematics in the United States and Canada (Calgary, AB)
– Thomas Archibald (co-author Louis Charbonneau), *Mathematics in Eastern British North America in the Nineteenth Century: Some Preliminary Remarks*
– Karen Hunger Parshall, *The Emergence of the American Mathematical Research Community 1876-1900*

1993: Philosophy of Mathematics (Ottawa, ON)
– Stuart Shanker, *Turing and the Origins of Artificial Intelligence*

1992: Ethnomathematics (Charlottetown, PEI)
– Michael Closs, *The Ancient Maya: Mathematics and Mathematicians*

1991: Women in Mathematics (Kingston, ON)
– Ann Hibner Koblitz, *Women in Mathematics: Historical and Cross-Cultural Perspectives*

1990: History and Pedagogy (Victoria, BC)
– Judith Grabiner, *Was Newton's Calculus a Dead End? A New Look at the Calculus of Colin Maclaurin*

CSHPM/SCHPM Presidents

1974 – Charles V. Jones
1975, 1976 – Viktors Linis
1977, 1978 – J. L. Berggren
1979, 1980 – G. de B. Robinson
1981, 1982 – Wesley Stevens
1983, 1984, 1985 – Edward J. Barbeau
1986 – Marshall Walker
1987 – Louis Charbonneau
1988, 1989 – J. L. Berggren
1990, 1991 – Craig Fraser
1992, 1993, 1994, 1995 – Thomas Archibald
1996, 1997 – Robert Thomas
1998, 1999 – James J. Tattersall
2000, 2001 – Glen Van Brummelen
2002, 2003 – J. L. Berggren
2004, 2005 – Robert Bradley

CSHPM/SCHPM Proceedings Editors

1988, 1989 – Tasoula Berggren
1990 – Francine Abeles, Victor Katz, Robert Thomas
1991 – Hardy Grant, Israel Kleiner, Abe Shenitzer
1992-1999 – James J. Tattersall
2000, 2001 – Michael Kinyon
2002-present – Antonella Cupillari

Copyright Permissions

Thomas Archibald and Louis Charbonneau. Mathematics in Canada before 1945: A preliminary survey, in Peter Fillmore, ed., *Mathematics in Canada*, vol. I, Ottawa, ON: Canadian Mathematical Society, pp. 1-90. The article appears in both English and French; only the English version (pp. 1-43) is reprinted here.

Judith V. Grabiner. Was Newton's calculus a dead end? The continental influence of Maclaurin's treatise of fluxions, *American Mathematical Monthly* **104** (5) (1997), 393-410.

Ivor Grattan-Guinness. History or heritage? An important distinction in mathematics and for mathematics education, *American Mathematical Monthly* **111** (1) (2004), 1-12.

Ann Hibner Koblitz. Mathematics and gender: Some cross-cultural observations, in Gila Hanna, ed., *Towards Gender Equity in Mathematics Education*, Dordrecht: Kluwer, 1996, pp. 93-109.

Volker Peckhaus. 19th century logic between philosophy and mathematics, *Bulletin of Symbolic Logic* **5** (4) (1999), 433-450. Copyright held by the Association for Symbolic Logic.

Stuart Shanker. Turing and the origins of AI, *Philosophia Mathematica* **3** (1) (1995), 52-85.

Contents

Introduction: The Birth and Growth of a Community
Amy Shell-Gellasch 3

1 History or Heritage? An Important Distinction in Mathematics and for Mathematics Education
Ivor Grattan-Guinness 7

2 Ptolemy's Mathematical Models and their Meaning
Alexander Jones 23

3 Mathematics, Instruments and Navigation, 1600-1800
Jim Bennett 43

4 Was Newton's Calculus a Dead End? The Continental Influence of Maclaurin's ***Treatise of Fluxions***
Judith V. Grabiner 57

5 The Mathematics and Science of Leonhard Euler (1707–1783)
Rüdiger Thiele 81

6 Mathematics in Canada before 1945: A Preliminary Survey
Thomas Archibald, Louis Charbonneau 141

7 The Emergence of the American Mathematical Research Community
Karen Hunger Parshall 183

8 19th Century Logic Between Philosophy and Mathematics
Volker Peckhaus 203

9 The Battle for Cantorian Set Theory
Joseph W. Dauben 221

10 Hilbert and his Twenty-Four Problems
Rüdiger Thiele 243

11 Turing and the Origins of AI
Stuart Shanker 297

12 Mathematics and Gender: Some Cross-Cultural Observations
Ann Hibner Koblitz 329

Index 347

List of Contributors

Thomas Archibald
Dept. of Mathematics and Statistics,
Acadia University, Wolfville, N. S.
B4P 2R6 Canada
tom.archibald@acadia.ca

Jim Bennett
Museum of the History of Science,
University of Oxford,
Broad Street, Oxford. OX1 3AZ
England
jim.bennett@mhs.ox.ac.uk

Louis Charbonneau
Département de Mathématiques,
Université de Québec a Montréal,
Montréal, QC H2X 3Y7 Canada
charbonneau.louis@uqam.ca

Joseph W. Dauben
Department of History,
Herbert H. Lehman College, CUNY,
250 Bedford Park Blvd West,
Bronx, NY 10469-1589 USA

Ph.D. Program in History,
The Graduate Center,
City University of New York,
365 Fifth Ave,
New York, NY 10016-4309 USA
jdauben@att.net

Judith V. Grabiner
Pitzer College,
1050 North Mills Avenue,
Claremont, CA 91711 USA
jgrabiner@pitzer.edu

Ivor Grattan-Guinness
Middlesex University at Enfield,
Middlesex EN3 4SF England
eggigg@ghcom.net

Alexander Jones
Department of Classics,
University of Toronto,
97 St. George Street,
Toronto, ON M5S 2E8 Canada
alexander.jones@utoronto.ca

Ann Hibner Koblitz
Arizona State University
P.O. Box 873404,
Tempe, AZ 85287-3404 USA
koblitz@asu.edu

Karen Hunger Parshall
Departments of History and
Mathematics, University of Virginia,
P. O. Box 400137, Charlottesville,
VA 22904-4137 USA
khp3k@virginia.edu

Volker Peckhaus
Universität Paderborn, Fakultät für Kulturwissenschaften – Philosophie,
Warburger Str. 100, D–33098
Paderborn, Germany
volker.peckhaus@upb.de

Stuart Shanker
Departments of Philosophy and Psychology,
Atkinson College, York University,
North York, ON M3J 1P3 Canada
shanker@yorku.ca

Rüdiger Thiele
Karl-Sudhoff-Institut
für Geschichte der Medizin und der Naturwissenschaften,
University of Leipzig, D-04109
Leipzig, Germany
thieler@server3.medizin.uni-leipzig.de

Kenneth O. May (1915-1977)
Originally appeared in *Historia Mathematica* **5** (1) (1978), 2.

Introduction: The Birth and Growth of a Community

Amy Shell-Gellasch

CSHPM/SCHPM Archivist

The Canadian Society for History and Philosophy of Mathematics, or Société d'Histoire et de Philosophie des Mathématiques, affectionately known as CSHPM/SCHPM, is a society of those interested in the history or philosophy of mathematics. Our constitution states that "the aim of the society is to promote throughout Canada discussion, research, teaching and publishing in the history and the philosophy of mathematics. Any person with interest in the history or in the philosophy of mathematics is eligible to become a member." Those statements are clearly obvious and necessary; however, they do not convey the depth, breadth or quality of the society and its members.

Currently the society has over two hundred members in nineteen countries, including Brazil, Sweden, Bangladesh and Japan. Though most of our members are academics, some do not work in academia but are simply consumers of the subject, either personally, as educators, or through professional interest from other disciplines. The diversity of the CSHPM is also its strength: our different motives and perspectives combine to produce richer portraits of the history of mathematics than we could achieve individually.

Our primary goal is to provide our members with the means to both present and receive current research in the field. This is done primarily through our Annual Meetings and the resulting *Proceedings*, as well as through our official historical journal *Historia Mathematica* and philosophical journal *Philosophia Mathematica*. Our semi-annual newsletters allow members to keep abreast of events in the field as well as interact with one another. Occasionally we hold joint meetings with our sister organization, the British Society for the History of Mathematics, as well as with the Canadian Mathematical Society. Our underlying goal, possibly the more important of the two, is to establish a community of scholars, practitioners and consumers of the history and philosophy of mathematics, with all the qualities and interactions that the word "community" implies.

The groundwork for establishing that community was laid in 1972. In that year Kenneth May sent letters to several colleagues inquiring into the desire among practitioners to organize a society in the history and philosophy of mathematics. The responses that May received show an enthusiastic reception to the idea. In May 1973 the first meeting of the new organization took place at the Learned Societies Congress (often shortened to the "Learneds") at Queen's University in Kingston, Ontario. The society was officially formed the following summer when the society's constitution was approved.

Kenneth Ownsworth May (1915-1977), at the time of the founding of the society, was at the Institute for History and Philosophy of Science (IHPS) at the University of Toronto. In addition he was the editor of the journal *Historia Mathematica*, which he officially launched in 1974, having been in newsletter form for the previous two years. May was an accomplished mathematician, historian and educator. He studied mathematics and economics at the University of California at Berkeley, receiving his A. B. in 1936, and his M. A. in 1937. En route to his doctorate under Griffith Evans, his life took many turns. At the recommendation of Evans, May became a fellow of the Institute of Current World Affairs in 1937, studying economic, social, and political conditions in Europe. He traveled to England and Russia to conduct his research. The next year May married and resigned his fellowship since his position and its funding were unsure. He and his wife then studied at the Sorbonne in Paris and became active in the workers movement. By 1939, he returned to Berkeley to resume work on his doctorate in mathematics with applications to social theory. However, in 1940 he was dismissed from his teaching duties at the University due to his involvement in the Communist Party. In 1942 May ran unsuccessfully for State Treasurer of California on the Communist ticket, nevertheless gaining 44% of the vote.

Just before finishing his thesis, May's life changed again. With Russia allied with the U.S. for the war, May sought to join the service; however, married men were not accepted in the service at that time. When his wife filed for divorce in mid-1942, May was able to enlist. May joined the 87th Mountain Infantry (10th Mountain Division), and served in the Aleutians (1943) and in Italy (1945). In 1944 he remarried, and after the war he and his second wife stayed in Italy, where May taught mathematics at the Army University Study Center. He returned to California in 1946 and defended his thesis, "On the Mathematical Theory of Employment", under Evans. May then accepted an assistant professorship at Carleton College in Northfield, Minnesota. During the late 1940s he published and presented his research in mathematics and industrial theory with titles such as his 1947 "The Aggregate Effect of Technological Changes in a Two-Industry Model". Throughout the 1950s, May's research focused on election theory, in which he published extensively. In the 1960s May's interests were directed towards the history of mathematics. In 1966 he moved to the University of Toronto's Institute for the History and Philosophy of Science, where he promoted the history of mathematics and science through his involvement in the IHPS, his founding of *Historia Mathematica*, and the founding of the CSHPM in 1973.

At the last moment, May did not attend the 1973 Queen's University meeting at which the society was founded. Charles V. Jones of York University chaired the meeting. At that meeting, the name of the society was agreed upon and *Historia Mathematica* was selected as its official journal. Jones was elected President, along with Thomas Settle as Vice President and J. Lennart Berggren as Secretary-Treasurer. At this meeting a modest set of papers in the history and philosophy of mathematics was presented. During 1973 and early 1974, Jones and Settle drafted the original bylaws by which the society still operates (with only slight modifications).

The following year the first official meeting of the society was held at the Learneds, with sixty charter members. A more extensive program of papers was presented at this meeting, including invited papers from all three of the executive members. These were the first annual guest presentations, of which this volume contains a sampling. After May's death in 1977 the Kenneth O. May fund was established, which helps to bring noted historians to the annual meetings as guest speakers.

Over the years, the society has traditionally held its annual meeting at the Learneds Congress (now the Congress of the Humanities and Social Sciences) every spring. From time to time we sponsor joint sessions with the Canadian Society for the History and Philosophy of Science, the first in 1974. In 1996 reciprocal memberships between the two organizations were introduced. Our most recent meeting with the Canadian Mathematical Society occurred in 2000 in Hamilton; another is planned for the year 2005. On a grander scale, at about the same time CSHPM and the British Society for History of Mathematics (BSHM) became sister organizations, with joint meetings held in 1997 in Oxford, 1999 in Toronto and most recently, 2004 in Cambridge.

As our membership continues to grow and diversify, so does interest in history and philosophy of mathematics from the mathematical community at large. In the past few years, interest in using the history of mathematics in teaching to motivate learning at both the school and collegiate levels, and an interest in the subject in its own right, has increased dramatically. To facilitate this new interest from those outside of the specialty, the History of Mathematics Special Interest Group of the Mathematical Association of America (HOM SIGMAA) was formed in 2002. Initial discussions leading to the formation of this new organization occurred during the annual CSHPM meeting in Quebec, 2001. Two members of the society drafted the constitution of this new group. Though HOM SIGMAA and CSHPM are not officially affiliated, they maintain a close informal working relationship. The CSHPM focuses on scholarly activity in the history and philosophy of mathematics, and HOM SIGMAA focuses primarily on the pedagogical aspects of the history of mathematics. The goal is for a symbiotic relationship that will promote not competition but complementary pursuits. Currently, all the HOM SIGMAA executive members are also CSHPM members.

This volume represents the next major project undertaken by the CSHPM. Since 1973, a wide variety of original work in the history and philosophy of mathematics has been presented at our annual meetings. That work has been recorded since 1988 in our internally produced annual *Proceedings*. After sixteen years, it is time to present some of that material to a wider audience. Though all the papers presented in the *Proceedings* deserve wider attention, this volume will showcase the papers presented by the keynote speakers at the Annual Meetings. These papers are of a consistently high quality by known experts in the field.

On behalf of the Society, I would like to thank Michael Kinyon and Glen Van Brummelen for their time and energy in seeing this project to completion. I would also like to thank all the speakers over the years who have shared their research and interest in the history of mathematics with us at our Annual Meetings. Of course these meetings, and the Proceedings that result, could not happen without those who devote many hours to making sure that the meetings run smoothly and that the *Proceedings* are published. We hope to share our love of the history of mathematics with you through this sampling of CSHPM activities.

1

History or Heritage? An Important Distinction in Mathematics and for Mathematics Education*

Ivor Grattan-Guinness

Middlesex University at Enfield

To the fond memory of John Fauvel (1947–2001)

1.1 Interest and Disagreements

During recent decades there has been a remarkable increase in work in the history of mathematics, including its relevance to mathematics education. But at times considerable differences of opinion arise, not only about its significance but even concerning *legitimacy*–that is, whether or not an historical interpretation counts as history at all. In this paper I consider the latter issue, and also note some consequences for education.

The disagreements are general, in that they may arise for any branch of mathematics in any period or culture; so they need a general resolution. I offer one in the form of a distinction in the ways of interpreting a piece of mathematics of the past. Take such a mathematical notion N; it could be anything from one notation through a definition, proof, proof–method or algorithm to a theorem, a wide-ranging theory, a whole branch of mathematics, and ways of teaching it. By its 'history', which becomes a technical term, one considers the development of N during a particular period: its launch and early forms, its impact, and applications in and/or outside mathematics, and so on. It addresses the question 'What happened in the past?' by offering descriptions. Maybe some kinds of explanation will also be attempted to answer the companion question 'Why did it happen?'.

History should also regard as important two companion questions, namely 'What did not happen in the past?' and 'Why not?'. The reasons may involve the other side of this distinction, which I call 'heritage'. There one is largely concerned with the effect of N upon later work, during any relevant period including that of its launch. Some modernised versions of N are likely to be

*First published in the *American Mathematical Monthly* **111** (1) (2004) 1–12.

taken, for heritage is largely concerned with the question 'How did we get here?', that is, to some current version of the context in question.

The distinction between history and heritage is often sensed by people who study some mathematics of the past, and feel that there are fundamentally different ways of doing so. Hence the disagreements can arise; one man's reading is another man's anachronism, and his reading is the first one's irrelevance. The discords often exhibit the differences between the approaches to history usually adopted by historians and those often taken by mathematicians.

The claim put forward here is that *both history and heritage are legitimate ways of handling the mathematics of the past; but muddling the two together or asserting that one is subordinate to the other, is not.* Many consequences flow from this stance, which will be treated in sections 3 and 4; first let us take a simple and well-known example, from the distant past.

1.2 Pythagoras's Theorem, Euclid Style

One of the best-known theorems in Euclid's *Elements* (fourth century B.C.E.) concerns the sides of a right-angled triangle ABC in Figure 1.

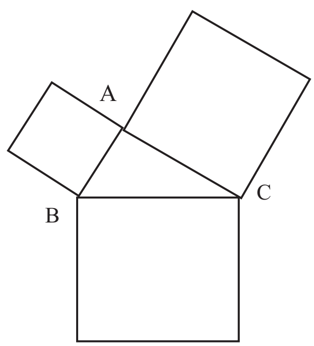

Fig. 1.1.

We recognise it as saying of the sides AB, AC, and BC that

$$AB^2 + AC^2 = BC^2; \tag{1.1}$$

but Euclid actually says something quite different [11, Book 1, Proposition 47J]: in right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle. There is an attached diagram of which Figure 1 is part, and the differences between it and (1) are basic. Not only is (1) algebraic whereas the figure is geometric: the diagram shows the squares outside the triangle, which (1) does not convey.

Were any of the squares to lie over the triangle, then both (1) and the theorem would still be true: but the complicated proof, not shown in the figure, could not be effected. The algebraic character of (1) emerges further when, as was and is commonly done, the letters 'a', 'b', and 'c' are used for the sides: for algebra is the branch of mathematics in which special words and especially symbols are used to a significant extent to represent constants, unknowns, variables, and operations.

Another important difference concerns the word 'on'. Euclid never used the phrase 'side squared', for in his geometrical Books he never multiplied geometric magnitudes together, either in the statement of theorems or (more importantly) in any proof. For example, he did not draw upon side-squaring when proving Pythagoras's theorem, either in the complicated proof just mentioned, which relies upon congruence, or in a more elegant one for the more general theorem about rectangles with the same ratio of sides set upon the sides of the triangle, where the proof deploys similar triangles and ratio theory [11, Book 6, Proposition 31]. Thus 'BC^2' is already a transgression from his geometry (and the frequent use in diagrams of small letters such as 'a^2' even more so). Instead Euclid constructed a square *on* a given line–indeed, in the proposition immediately preceding Pythagoras's theorem [11. Book 1, Proposition 46].

The issue is more profound than it may seem. Both here and everywhere else in the *Elements* Euclid works with *lines* rather than *lengths*, the latter being lines upon which some arithmetical measure has been imposed. Euclid presented geometry without *arithmetic* in the sense just explained; numbers are also present, but for other purposes, such as saying that this line is twice that line, or that the ratio of two lines is the same ratio as 5 : 7. In the same way he worked with planar regions but not (measured) areas, with solids but not volumes, with angles but not in degrees. By contrast, which is sometimes overlooked, in the arithmetical Books 7–9 multiplication of integers themselves occurs as usual [15].

These remarks concern the history of Euclid. When one moves to its heritage, then a quite different situation arises, in which (1) and many other such equations are prominent. For the *Elements* played a major role in the development of common algebra among some of its Arabic initiators, and a still greater one when Europe at last woke up during the twelfth century and began to elaborate that algebra with symbols introduced both for unknown quantities and for operations. *Both* (1) and Pythagoras's theorem as shown by the figure are legitimate readings of Euclid, but are quite different from each other.

The *Elements* is a particularly interesting historical example, because common algebra as in (1) became the dominating reading of Euclid (including in mathematics education) to such an extent that during the nineteenth century it also became the normal historical interpretation; apparently Euclid had been a 'geometric algebraist', talking geometry but really practising common algebra. A supporter of this reading was T. L. Heath, whose English edition and translation, first published in 1908, is still the most widely used, usually now in the second edition [11]. Greek specialists tell me that his translation is very reliable both to the language and to the mathematics; in particular, for Pythagoras's theorem and all other contexts he says there ‘square on the side’, not ‘square of the side’ as many earlier translations had rendered (the word ‘apo’ can admit both ‘on’ and ‘of’ as translations) but which can easily lead to the algebraic ‘side squared’. Nevertheless, Heath added to his translation many algebraic versions of the propositions without seeming to notice the differences entailed.

While some historians of that time did not follow the algebraic interpretation of Euclid—for example, the Dutchman E. J. Dijksterhuis [26, chap. 5]—the standard view came under severe challenge only from the 1960s onwards. In particular, in the mid 1970s the historian Sabatei Unguru attacked it strongly, to the opposition of some mathematicians interested in history. Unguru's charges of anachronism and ahistory are largely vindicated: his mathematician opponents were inheritors [20].

We shall take another Euclid example in section 8. First, though, let us explore some general consequences of the distinction.

1.3 Some Principal Differences between History and Heritage

The distinction between the history of a notion N and its heritage obviously involves its respective pre- and post-histories; but much more is at hand, for history has to use post-history also. To see this, let us consider the advice, which is quite often put forward for history of all kinds, about a way of being ‘history-minded’ about N (say, Pythagoras's theorem in Euclid); namely, forget everything that has happened since N was formed, and read Euclid with the eyes with which he wrote it. But this advice *begs the question at hand.* For in order to forget everything E that has happened since N, then one has to know E already; however, to do that one needs to be able to distinguish E from the history and pre-history of N; but this is the task to be attempted.

Thus the distinction between history and heritage rests in part upon the ways in which notions later than N are to be used. When they are determined to be later notions, the view urged here is this: by all means bring them to bear, and deploy them to understand the heritage from N, but *avoid* feeding them back to appraise its history (such-and-such did not happen). Further, when considering periods intermediate between that of N and some later ones such as now, apply the distinction carefully. Thus, in our example the equation (1) is not only part of the history of René Descartes and the heritage of Euclid but also belongs to the heritage from, among others, the algebraist François Viète in the sixteenth century, whose work also belongs to the history of Descartes. Note also that history is usually *a history of heritages*; it is a tale of mathematicians taking and modifying notions from the past (often pretty recent) without enquiring about the history of those notions.

Various other matters can be explored; a more detailed discussion, largely focussed upon history, is given in a companion paper [16]. The following table summarises the main features of handling past notions N in the two different ways suggested. An apparent contradiction between the third and fourth rows needs to be addressed. When the historian reconstructs past muddles, he will conflate notions that we now know to be different, a feature that the inheritor will stress. But the difference that the reconstruction exposes is that between past ignorance of the distinction, which is different from our (and the inheritor's) present knowledge of it.

Feature	History	Heritage
Motivation(s) to N	Important issue; maybe hard to find (for example, for Euclid's *Elements*)	Probably only of minor interest
Types of influence	Can be negative as well as positive; both should be noted	Likely to draw only upon the positive cases
Relationships of N to earlier and to later notions	Major issue; differences stressed as much as similarities, maybe more	Important issue: similarities stressed more than differences
Handling unclarities evident in N	Reconstruct them, and as clearly as possible	Recognise them, but clean them up
Successful developments	Very important: but also study failures, delays, missed opportunities, and late arrivals	Likely to be the main concern
Role of chronology	Usually important; can be hard to establish	Beyond broad details, not likely to matter so much
Historical consequences	May try to reconstruct the *foresight* (hopes, and so on) for N held by the historical figures	May try to construct *hindsight* and historical perspective of the developments after N

Determinism?	Preferably not claimed: the actual developments were so-and-so, but not necessarily so-and-so	May carry a determinist flavour; we *had* to get here (but see the history column!)
Foundations of a theory	Dig down to them, and build upon a swamp	Lay them down and build up from them, like on solid ground
Level of importance or popularity of N	Can vary over time, independently of content; should be noted (and maybe explained)	Not normally considered; current importance assigned

1.4 Changing Habits

A further type of issue, which is not susceptible to tabular expression, concerns the use of notions that have become standard and therefore are now used habitually. Such habits may well help in determining the heritage from N: but historical anachronism can easily arise, which needs to be controlled. I note three important examples.

First, after an interesting history of its own from the 1870s [9], Georg Cantor's set theory has been part of our mathematical furniture for just over a century; so for the mathematics of this period its use may well be faithful. Now collections of things have been handled in mathematics since at least Greek antiquity; but the earlier theory of so doing was part-whole theory, where (say) British women form part of the class of women, membership is not distinguished from inclusion, and an object is not distinguished from its unit class. The differences between part-whole and set theories are considerable, both technically and philosophically, and the historian needs to mark them carefully. By contrast, the inheritor can deploy set theory with little chance of deception.

Second, while the influence of Euclid was great in Western mathematics, his stress on axioms and common notions was rarely imitated (though to some extent Newton's *Principia* is an example). The axiomatisation of mathematical theories became more prominent only during the late nineteenth century, especially in connection with the axioms of Euclidian and non-Euclidian geometries, and the emergence of abstract algebras [8]. Both developments attracted the attention of David Hilbert, and led him to launch the wide-ranging use of axiomatisation during the first half of the twentieth century, an attitude that has now become pretty standard: a clear path of heritage can be traced up to present-day practises. But the historian should be careful when looking at the structure of earlier mathematical theories, for axiomatisation may well not be prominent beyond specifying basic principles or laws.

Cantor's set theory is a good example: while it too was developed during the late nineteenth century, he showed little interest in the axioms that it may require.

Third, vector and matrix theory have become standard fare in mathematics, though (especially in the second case) only from the l930s onwards and after rather scrappy historical developments in various contexts during the nineteenth century. Once again, care should be exercised in applying them to earlier work. For example, much of the mechanics developed by figures such as L. Euler, J. L. Lagrange, and P. S. Laplace can be rewritten in vectorial and matricial forms, but historical understanding will not profit. For none of these figures knew that their theories could be developed in terms of strings or arrays of scalar elements; they worked instead in terms of collections of simultaneous linear or differential equations, or quadratic and bilinear forms [14, chaps. 5–6]. The introduction of vectors or matrices is not merely a matter of changing notation; new theories are involved. It is of course nice to save such space, for one thing; but if the historian does deploy these theories, then a chronological health warning should be appended.

By contrast to all these cautions to the historians, the inheritors can execute all these reformulations of theory quite legitimately; indeed, much nice heritage mathematics may emerge. Further, some history of mathematics produced after the initial period under study might be created; for, as was mentioned in section 1, mathematicians normally read the past in a heritage spirit.

As an example, take Lagrange and others in mechanics. A major problem, which he formulated in the 1770s, was to prove mathematically that the planetary system was stable. (Previous figures such as Newton and Euler had relied on God to watch out for danger; that is, a religion influenced mathematics.) In terms of matrix theory, Lagrange's brilliant theory sought proof of the reality of all the eigenvalues and eigenvectors (to use modern terms) of a certain matrix. But he had no such theory, and worked with the corresponding quadratic forms; so did Laplace, who adapted his results to some extent; neither man found a watertight proof. The next major contribution came in 1829 from (surprisingly) A. L. Cauchy, and in 1829 he did formulate 'tableaux' of scalar entries in his own work on this problem [17]. Thus matrix theory may—indeed, should—be used to describe Cauchy's contribution, and thus to help us to grasp an important part of his heritage from his predecessors. And we also have a nice example of the 'What did not happen?' question; for Cauchy never realised the significance of his achievement and rarely used it later, so that unfortunately he was not an influential founder of the spectral theory of matrices.

1.5 Some Philosophical Background

It is obvious that this talk of earlier and later notions, the development of theories, and so on, is not confined to mathematics: such features occur also in the histories of other sciences (including technology, engineering, and medicine), and indeed elsewhere (for example and a nice one, practices to be adopted and avoided in the so-called authentic performance of older music). The main general principles that underlie the foregoing discussion are as follows.

First, history is *unavoidable*, whether one likes it or not. A mathematician who presents his theory without concern with history is not thereby immune from it. For example, an enthusiast for axiomatics mentioned in section 3 will lay out his theory in a very formal way without reference to predecessors or precedents; but they will be there, including previous formal theories laid out by preceding axiomatists without reference to their own predecessors or precedents. Thus the question of whether or not one can use history in mathematics is miscast: it is rather the question of whether it is done consciously or not. Indeed, independent of the content of this paper, it is useful to have some general historical idea of a topic of interest, whatever it may be.

Second, knowledge and ignorance go together. This symbiosis has not received the general philosophical attention that it deserves. In particular and of special significance for mathematics, there is knowledge of ignorance, especially when one formulates a problem. When, for example, J. P. G. Dirichlet studied the convergence problem of Fourier series in the late 1820s, he knew that he did not know sufficient conditions on a function to establish convergence to it: finding some was precisely his problem. Having done so, he knew that he did not know whether or not they could be weakened, thereby setting the next problem in this chain (to which the first answer was the Lipschitz condition, by the way). One can also have ignorance of ignorance, or unawareness, where people do not know that they do not know something because the required connections between notions have not yet been laid down. Thus Dirichlet did not know that he did not know how his proof bore upon the specification of function spaces, because that notion did not emerge until the late nineteenth century [21].

Third, and following from the preceding line of thought, knowledge of all kinds is stratified into theory, metatheory, ... For mathematics this means not only metamathematics of the technical kind that Hilbert launched, but also informal kinds. In particular, the history of notion N is one kind, its heritage is another, manners of its possible teaching a third, heuristic strategies to explain its significance a fourth, and there may well be others. The relationship between knowledge and ignorance just outlined lie in the metatheory of the notions involved. Similarly, metatheory requires metametatheory as its own forum for discussion, and so on upwards as far as is needed. An example of metametatheory is the history of the history of mathematics, an interesting story recently recorded in detail in [10]; the comments on Heath in section 2 form an example of it; and this paper itself is a self-referring example, with its heritage (if any) awaited!

The recognition of history and heritage as metatheoretic also releases both historians and inheritors from the need to like what they find in the past that they study. Why should they? After all, they were not there (as a rule). The point seems obvious enough; after all, one can be a good historian (or inheritor) of, say, military history without being a militarist. Yet not infrequently historians and inheritors become overly attached to their objects and figures of study, in any kind of history, and feel that they have to defend what they find. While of course such attachment can be felt if it arises naturally, no compunction to it should even be encouraged.

The generality of stratification is an insight forged in connection with symbolic logic in the early 1930s, thanks principally to Kurt Gödel and Alfred Tarski. In logic the distinction of (object-level) logic itself from metalogic is especially tricky but thereby all the more important; as was known already in Greek times, failure to make a distinction of some kind admits nasty paradoxes. Gradually stratification spread into other disciplines, especially mathematics and some types of philosophy. One follower, inspired by Tarski in the mid 1930s, was Karl Popper. Several parts of his philosophy of fallibilism are metaphilosophical; for example, his preference for indeterminism over determinism [19]. Of particular relevance to this paper is his essay 'On the Sources of Knowledge and Ignorance' [18, introduction], for it contains an insight largely missing from other kinds of philosophy; that ignorance is nice, for it is the site (in metatheory) of our problems when construed as knowledge of ignorance. In most other philosophies ignorance is a disease to be cured by the acquisition of knowledge however that acquisition is claimed to occur (see [25, chaps. 1–6] for the various forms of this view maintained within the sceptical tradition of philosophy). So far explicit use of stratification has not been widely canvassed among prevalent philosophies of history (which are well surveyed in [23]); but it seems worthy of further elaboration.

1.6 General Remarks about History in Mathematics Education

In recent decades a considerable and international increase has developed in the use of history in mathematics education, in order to temper and challenge the normal picture of mathematics as a human-free zone, all answers but no questions, all solutions but no problems. Several edited or authored books and special issues of journals have appeared containing material of various kinds: textbooks significantly informed by the relevant history; summary histories of particular developments; surveys of the lives and works of important historical figures; international and/or multicultural comparisons of the development of (more or less) the same theories; translations of original texts with commentary; and suggested strategies for using history in teaching practice, both in specific contexts and in general. The emphasis often falls upon motivation and context, on showing that mathematics is after all human activity despite appearances, and moreover that much of it is not Western in origin. The range of concerns is well captured in a recent volume [12].

Most attention seems to have fallen on teaching at school and college level, but the university level has also been addressed. Much more work has been done on pure mathematics than on applied or applicable mathematics, or on probability and statistics; a redress of balance would be most welcome. I do not attempt to review this literature here, but I consider the place and utility of the distinction between history and heritage in mathematics education in general.

As with researchers in history mentioned in section 1, there is an evident sense of the distinction in this kind of educational literature, or at least an intuition that the mathematics of the past can be used in different ways. Where is mathematics education to be found between history and heritage? My answer is that *that is exactly where it should be found*, so that it can profit from *both* sides. In particular, if notion N is to be taught, then both its history and its heritage can be used. Euclid's *Elements* is a good example, where the inherited use of algebra has been well used quite frequently. In addition, the historical Euclid deserves attention, with its geometry presented without arithmetic with lines rather than lengths, and the beautiful theory of ratios used in both his geometry and his arithmetic.

1.7 History-Satire and the Calculus

In the paper [13] I introduced long ago the term 'history-satire' to characterise a way in which history and also heritage can be used in mathematics education. Under it the broad features of the historical record are respected and used; but usually many detours and complications occur that, while they attract the historian, will impede teaching and so should be set aside or at most treated only in passing. The 'genetic method' of Otto Toeplitz, which he introduced initially in the late 1920s in connection with teaching the differential and integral calculus, is similar in sentiment [24]. More recently the Mathematical Association of America published a novel and important textbook in real-variable mathematical analysis by David Bressoud, in which he gives prominent places to the main developments, especially of the nineteenth century, such as Fourier series [5].

As Bressoud duly notes, a major innovation of the century was the founding of analysis in the 1820s by Cauchy. His approach was based upon a newly sophisticated theory of limits, not with limit left as an intuitive notion. Undoubtedly it was much superior to the preceding versions in the organisation of the subject and statements and proofs of the theorems; however the loss in heuristics was heavy, and both his colleagues and students objected forcefully to it [14, chaps. 10–11 passim, and 20.8].

For an explicit example, here is a use of history-satire that I found helpful in my own teaching. In a remarkable analogy, Cauchy adapted his real-variable analysis to complex variables and their functions and thereby introduced a major new subject into mathematics. But it seems a strange subject when