LERAY–SCHAUDER TYPE ALTERNATIVES, COMPLEMENTARITY PROBLEMS AND VARIATIONAL INEQUALITIES
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LERAY–SCHAUDER TYPE ALTERNATIVES, COMPLEMENTARITY PROBLEMS AND VARIATIONAL INEQUALITIES

By

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Springer
To my family

The motion of Truth is cyclical,
The way of Truth is pliant

The Works of Lao Zi
Truth and Nature
CONTENTS

PREFACE .................................................................................................................. xi

1 PRELIMINARY NOTIONS ...................................................................................... 1

1.1 Topological spaces. Some fundamental notions ................................................. 1
1.2 Metric spaces ....................................................................................................... 4
1.3 Some classes of topological vector spaces ......................................................... 6
1.4 Compactness and compact operators ................................................................. 13
1.5 Measures of noncompactness and condensing operators ................................. 14
1.6 Topological degrees .......................................................................................... 19
1.7 Zero-epi mappings ............................................................................................ 29
1.8 Convex cones ...................................................................................................... 35
1.9 Projection operators ........................................................................................... 40

2 COMPLEMENTARITY PROBLEMS AND VARIATIONAL INEQUALITIES ............... 49

2.1 Complementarity problems ............................................................................... 49
2.2 Variational inequalities ....................................................................................... 59
2.3 Complementarity problems, variational inequalities, equivalences and equations ........................................................................................................... 62

3 LERAY–SCHAUDER ALTERNATIVES ................................................................. 71

3.1 The Leray–Schauder alternative by topological degree ........................................ 72
3.2 The Leray–Schauder alternative by the fixed point theory ................................. 74
3.3 The Leray–Schauder alternative by the topological transversality theory ........... 76
3.4 Some classes of mappings and Leray–Schauder type alternatives ................... 81
3.5 An implicit Leray–Schauder alternative ............................................................. 90
3.6 Leray–Schauder type alternatives for set-valued mappings ............................. 95

4 THE ORIGIN OF THE NOTION OF EXCEPTIONAL FAMILY OF ELEMENTS .......... 109

4.1 Exceptional family of elements, topological degree and nonlinear complementarity problems in $\mathbb{R}^n$ ................................................................. 109
4.2 Exceptional family of elements, topological degree and implicit complementarity problems in $\mathbb{R}^n$ ................................. 118
4.3 A general notion of an exceptional family of elements for continuous mappings ......................................................... 121
4.4 An exceptional family of elements, zero-epi mappings and nonlinear complementarity problems in Hilbert spaces ............ 127
4.5 Two applications ........................................................................... 131

5 LERAY–SCHAUDER TYPE ALTERNATIVES. EXISTENCE THEOREMS ................................................................. 137

5.1 Nonlinear complementarity problems in arbitrary Hilbert spaces ................................................................. 138
5.2 Implicit complementarity problems .............................................. 171
5.3 Set-valued complementarity problems .......................................... 180
5.4 Exceptional family of elements and monotonicity .......................... 194
5.5 Semi-definite complementarity problems ....................................... 201
5.6 Feasibility and an exceptional family of elements ......................... 203
5.7 Path of $\varepsilon$-solutions and exceptional families of elements .......... 215

6 INFINITESIMAL EXCEPTIONAL FAMILY OF ELEMENTS ........................................................................... 225

6.1 Scalar derivatives ........................................................................... 225
6.2 Infinitesimal exceptional family of elements ................................. 326
6.3 Applications to complementarity theory ......................................... 238
6.4 Infinitesimal interior-point-$\varepsilon$-exceptional family of elements ...... 244

7 MORE ABOUT THE NOTION OF EXCEPTIONAL FAMILY OF ELEMENTS ................................................................. 247

7.1 $EFE$-acceptable mappings ............................................................. 247
7.2 Skrypnik topological degree and exceptional families of elements ......................................................................................... 256
7.3 A necessary and sufficient condition for the non-existence of an exceptional family of elements for a given mapping ........ 266
7.4 Exceptional family of elements. Generalization to Banach spaces ......................................................................................... 271
8 EXCEPTIONAL FAMILY OF ELEMENTS AND VARIATIONAL INEQUALITIES ........................................279

8.1 Explicit Leray–Schauder type alternatives and variational inequalities .......................................................... 279
8.2 Implicit Leray–Schauder type alternatives and variational inequalities .......................................................... 292
8.3 Asymptotic Minty’s variational inequalities and condition $(\partial)$ ................................................................. 303
8.4 Complementarity problems and variational inequalities with integral operators ........................................... 306
8.5 Comments ....................................................................................................................................................... 312

BIBLIOGRAPHY ................................................................................................................................................. 313

INDEX ............................................................................................................................................................... 335
PREFACE

This book deals with the Leray–Schauder Principle, the study of complementarity problems and the study of variational inequalities. The first is given by the following classical result.

**Theorem 1 [Leray–Schauder Principle].** Let \( (E, \|\|) \) be a Banach space, \( \Omega \subset E \) an open bounded set such that \( 0 \in \Omega \) and \( f: \overline{\Omega} \to E \) a continuous compact mapping. If \( f(x) \neq \lambda x \) for all \( x \in \partial \Omega \) and \( \lambda > 1 \), then \( f \) has a fixed point.

From **Theorem 1** we deduce the following result.

**Theorem 2 [Leray–Schauder alternative].** Let \( (E, \|\|) \) be a Banach space, \( \Omega \subset E \) an open bounded subset such that \( 0 \in \Omega \) and \( f: \overline{\Omega} \to E \) a continuous compact mapping. Then:

1. either \( f \) has a fixed point in \( \Omega \) or
2. there exists an element \( x^* \in \partial \Omega \) and a real number \( \lambda^* \in [0, 1[ \) such that \( x^* = \lambda^* f(x^*) \).

**Theorems 1 and 2** are considered to be the most important results in nonlinear analysis and lead to applications in the study of nonlinear functional equations.

**Complementarity theory** is a relatively new domain in applied mathematics with deep connections with several aspects of fundamental mathematics. The main goal of complementarity theory is the study of complementarity problems from several points of view. Complementarity problems represent a wide class of mathematical models related to optimization, economics, mechanics and engineering. In many mathematical models the complementarity condition is used to determine the equilibrium as used in physics or in economics. There exist few books dedicated to the study of complementarity problems: Some of these are (Cottle, R. W., Pang, J. S. and Stone, R. E. [1]), (Isac, G. [12] and [26]), (Hyers, D. H., Isac, G. and Rassias, Th. M. [1]) and (Isac, G., Bulavski, W. A. and Kalashnikov, V. V. [2])

The study of variational inequalities is another domain of applied mathematics. Variational inequalities have many applications to the study of certain problems with unilateral conditions, and there are many papers and books dedicated to this subject. A complementarity problem is associated
with a mapping and a closed convex cone, whereas a variational inequality is associated with a mapping and a closed convex set. It is known that a variational inequality associated with a mapping and a closed convex cone is equivalent to a complementarity problem. Until now all applications of the Leray–Schauder Principle [Theorem 1] have been exclusively dedicated to the study of existence of fixed points or of existence of solutions of nonlinear equations. See, for example, the books (O’Regan, D. and Precup, R. [1] and (Precup, R. [1]).

Considering these applications from the point of view of the Leray–Schauder alternative [Theorem 2], we observe that the authors considered only the conclusion (1) of Theorem 2. In this book we show that conclusion (2) of Theorem 2 has also interesting applications. By using this conclusion we introduce the notion of an exceptional family of elements for a mapping. This notion is related to a complementarity problem or to a variational inequality. The property of being without an exceptional family of elements is a kind of coercivity property, which is more general than the classical notion of coercivity.

The notion of an exceptional family of elements introduced in this book by the Leray–Schauder alternative is the same notion that was introduced in 1997 in our paper, (Isac, G., Bulavski, V. A. and Kalashnikov, V. V. [1]), by using the family’s topological degree. In this book we replace the topological degree by Leray–Schauder alternatives, because in this way we can define the notion of exceptional family of elements for classes of mappings for which the topological degree is not defined. The investigation method based on this notion is simpler and elegant.

Our notion of exceptional family of elements contains as a particular case the notion of “exceptional sequence of elements” which was introduced with respect to $\mathbb{R}^n$ in (Smith, T. E. [1]) and has no relation with the main result proved in (Eaves, B. C. [2]). Moreover, the main result proved by Eaves is strongly related to the fact that the convex cone $\mathbb{R}^n$ has a bounded base; his result can not be extended to an arbitrary cone in a Hilbert space or in a Banach space.

The notion of exceptional family of elements presented in this book has deep relations with fundamental notions of nonlinear analysis and
shows promise of other new developments. In particular, research shows that the investigation method based on this notion is a remarkable method for complementarity theory and for the theory of variational inequalities with respect to unbounded closed convex sets. The study of existence of solutions for complementarity problems and for variational inequalities is unified by this method.

Now, let us briefly describe the content of this book.

Chapter 1 is dedicated to the preliminary notions that are used systematically in this book.

Chapter 2 defines the complementarity problems and the variational inequalities used in this book and their equivalences.

Chapter 3 presents the Leray–Schauder type alternatives. The alternatives are given by their proofs.

Chapter 4 contains several results and facts considered as the origin of the notions of exceptional family of elements presented in Chapters 5–8.

Chapter 5 is dedicated to the results obtained for complementarity problems by the topological method based on the notion of exceptional family of elements.

Chapter 6 introduces the notion of infinitesimal exceptional family of elements. Here we apply scalar derivatives to the study of complementarity problems.

Chapter 7 presents several special notions and results related to the notion of exceptional family of elements. In this chapter we show that the notion of exceptional family of elements can be defined for more general classes of mappings and for this definition the Leray–Schauder alternatives are not necessary. In this chapter we give also a necessary and sufficient condition for the non-existence of an exceptional family of elements. This result is the starting point for new and interesting results.

Finally, Chapter 8 is dedicated to the study of variational inequalities by the method presented in this book. The last subject of this chapter is the study of variational inequalities with integral operators.
We note that the Bibliography contains not only the cited papers but other papers related to this subject.

The goal of many books is to present a collection of the most significant results on some subjects, obtained in a period of time, but the main goal of our book is to show a new method, applicable to the study of complementarity problems and variational inequalities. We would like the reader to consider this book as a starting point of a new topological method applicable to the study of complementarity problems and of variational inequalities. Certainly, this method can be improved, and many new developments based on ideas presented in this book are possible. In particular, the study of order complementarity problems by the method presented in this book is a completely open subject. Considering the fact that mathematics is a collective work, perhaps other authors will improve and develop our method.

It is impossible to finish this preface without to say many, many thanks to my wife Viorica, for her excellent work. She has carefully prepared the manuscript of this book with unlimited and constant enthusiasm. I will keep in my heart her real support.

To conclude, I would like to say that I appreciated very much the excellent assistance offered me by the staff of Springer Publishers.

June 1, 2005

Prof. G. Isac
The reader of this book must have a minimum background of a course in general topology and a course in functional analysis. However, to facilitate the lecture we recall in this chapter several preliminary notions. Certainly, other special notions related to the results presented in this book will be introduced in each chapter.

1.1 Topological spaces. Some fundamental notions

Let $X$, $Y$ be arbitrary sets. We use the standard notations $x \in X$ for “$x$ is an element of $X$”, $X \subset Y$ for “$X$ is a subset of $Y$” and $X = Y$ for “$X \subset Y$ and $Y \subset X$”. The complement of $X$ relative to $Y$ is the set $C_Y X = \{x \in Y : x \notin X\}$. The set of all subsets of $X$ is denoted by $\mathcal{P}(X)$. Let $\{X_i\}_{i \in I}$ be a family of sets. For the union of this family we use the notation $\bigcup_{i \in I} X_i$ and for intersection the notation $\bigcap_{i \in I} X_i$. If $I = \mathbb{N}$ we have a sequence of sets and we use respectively the notations $\bigcup_{n=1}^{\infty} X_n$ and $\bigcap_{n=1}^{\infty} X_n$. A mapping $f$ of $X$ into $Y$ is denoted by $f : X \to Y$. The domain of $f$ is $X$ and the image of $X$ under $f$ is called the range of $f$. For any $A \subset X$, we write $f(A)$ to denote the set $\{f(x) : x \in A\} \subset Y$. For any $B \subset Y$, $f^{-1}(B) = \{x \in X : f(x) \in B\}$. If
f : X → Y and g : Y → Z are mappings, the composition mapping \( x \mapsto g(f(x)) \) is denoted by \( g \circ f \). We denote the empty set by \( \emptyset \).

**DEFINITION 1.1.1.** Let \( X \) be any non-empty set. A subset \( \tau \) of \( \mathcal{P}(X) \) is said to be a topology on \( X \) if the following axioms are satisfied:

(i) \( X \) and \( \emptyset \) are members of \( \tau \),

(ii) the intersection of any two members of \( \tau \) is a member of \( \tau \),

(iii) the union of any family of members of \( \tau \) is again in \( \tau \).

We say that the couple \((X, \tau)\) is a topological space. If \( \tau \) is a topology on \( X \) the members of \( \tau \) are then said to be \( \tau \)-open subsets of \( X \), or merely open subsets of \( X \) if no confusion may result. The subset \( \tau_o = \{\emptyset, X\} \) of \( \mathcal{P}(X) \) is a topology on \( X \) called the trivial topology. It is easy to show that \( \tau^d = \mathcal{P}(X) \) is a topology on \( X \) called the discrete topology. The topologies \( \tau_t \) and \( \tau_o \) are not interesting. An interesting topology \( \tau \) on \( X \) must be such that \( \tau_t \subset \tau \subset \tau^d \).

**DEFINITION 1.1.2.** In a topological space \((X, \tau)\), we say that a subset \( F \) of \( X \) is \( \tau \)-closed (or merely closed) if \( F \supseteq U \), where \( U \) is a \( \tau \)-open set.

The closed subsets of \( X \) have the following properties:

(1) \( X \) and \( \emptyset \) are closed subsets of \( X \),

(2) the union of any two closed subsets of \( X \) is again a closed subset of \( X \),

(3) the intersection of any family of closed subsets of \( X \) is again a closed subset of \( X \).

**Remark.** There exist subsets that are not open and not closed.

Given a non-empty subset \( A \subset X \), the open set \( \text{int}A \) which is the union of all open subsets of \( A \), is called the interior of \( A \). The interior of a set may be empty. The closed set \( \overline{A} \), the intersection of all closed sets containing \( A \), is called the closure of \( A \). An element \( x \in \text{int}A \) is called an interior point of \( A \). An element \( x \in \overline{A} \) is called an adherent point of \( A \). We say that a subset \( V \) of \( X \) is a \( \tau \)-neighborhood (or merely neighborhood) of a point \( x \in X \) if there exists an open set \( U \) such that \( x \in U \subseteq V \). Let \((I, \leq)\) be any partially ordered set. It is said to be a directed set if given any \( i \) and
any \( j \) in \( I \) there is \( k \in I \) such that \( i \leq k \) and \( j \leq k \). Note that any totally ordered set is directed. In particular the set \( \mathbb{N} \) of natural real numbers is a directed set.

Let \((X, \tau)\) be a topological space and \( I \) be a directed set. A function \( x \) from \( I \) into \( X \) is said to be a net in \( X \). The expression \( x(i) \) is usually denoted by \( x_i \), and the net itself is denoted by \( \{x_i\}_{i \in I} \).

**Definition 1.1.3.** A net \( \{x_i\}_{i \in I} \) is said to be convergent to a point \( x^* \in X \) if for any neighborhood \( V \) of \( x^* \), there exists an index \( i_v \in I \) such that for any \( i \in I \) satisfying \( i_v \leq i \), we have that \( x_i \in V \).

If a net \( \{x_i\}_{i \in I} \) is convergent to \( x^* \), we write \( \lim_{i \in I} x_i = x^* \). It is known that a subset \( A \) of \( X \) is closed, if and only if for any net \( \{x_i\}_{i \in I} \) in \( A \) the condition \( \lim_{i \in I} x_i = x_0 \) implies \( x_0 \in A \).

**Definition 1.1.4.** We say that a topological space \((X, \tau)\) is a Hausdorff space, if and only if given any two distinct points \( x \) and \( y \) of \( X \), there are open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \).

It is known that a topological space \((X, \tau)\) is Hausdorff, if and only if given any convergent net \( \{x_i\}_{i \in I} \) in \( X \) the limit of \( \{x_i\}_{i \in I} \) is unique. In this book we will consider only Hausdorff topological spaces.

Let \((X, \tau_1), (Y, \tau_2)\) be topological spaces and let \( f : X \to Y \) be a mapping.

**Definition 1.1.5.** We say that \( f \) is continuous at a point \( x \in X \), if for each \( \tau_2 \)-neighborhood \( V \) of \( y = f(x) \), \( f^{-1}(V) \) is a \( \tau_1 \)-neighborhood of \( x \).

If \( f \) is continuous at any \( x \in X \), then in this case we say that \( f \) is continuous on \( X \).

The following statements are equivalent:

1. \( f \) is continuous on \( X \),
Leray–Schauder Type Alternatives

(2) for any subset $A$ of $X$, we have $f\left(\bar{A}\right) \subseteq \overline{f(A)}$.

(3) if $F \subseteq Y$ is $\tau_2$-closed, then $f^{-1}(F)$ is $\tau_1$-closed in $X$.

(4) if $U \subseteq Y$ is $\tau_2$-open, then $f^{-1}(U)$ is $\tau_1$-open in $X$.

Convergent nets can characterize the continuity of a mapping. In this sense we have the following classical result.

A mapping $f: X \to Y$ is continuous on $X$ if and only if for every net $(x_i)_{i \in I}$ in $X$ such that $(x_i)_{i \in I}$ is convergent to $x$, the net $(f(x_i))_{i \in I}$ in $Y$ converges to $f(x)$.

1.2 Metric spaces

First, we note that a metric space is a set in which we have a measure of the closedness or proximity of two arbitrary elements in the set. This measure is obtained by a "distance".

**DEFINITION 1.2.1.** Let $X$ be an arbitrary non-empty set. We say that a function $d: X \times X \to \mathbb{R}$ is said to be a metric (distance) on $X$ if:

1. $d(x, y) \geq 0$, for all $x, y \in X$,
2. $d(x, y) = d(y, x)$, for all $x, y \in X$,
3. $d(x, y) = 0$ if and only if $x = y$,
4. $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

The couple $(X, d)$ is said to be a metric space. If $(X, d)$ is a metric space, we can define on $X$ a topology by the following method. For any $x \in X$ and any positive real number $\varepsilon$, the $d - \varepsilon$-ball is the set:

$$B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$

Consider the following collection of subsets of $X$,

$$\tau_d = \{U \subseteq X : \text{for any } x \in U \text{ there exists } \varepsilon > 0 \text{ such that } B(x, \varepsilon) \subseteq U\}.$$
Obviously $X$, $\phi \in \tau_d$. Indeed, if $x \in X$ and $\varepsilon > 0$, then $B(x, \varepsilon) \subseteq X$. Since $\phi$ contains no points, it is true that for each $x \in \phi$ (there is no such $x$) and any $\varepsilon > 0$, $B(x, \varepsilon) \subseteq \phi$. We can prove that $\tau_d$ is a topology on $X$, named the topology defined by the metric $d$. Therefore, any metric space is a topological space, but the converse is not true. Moreover, any metric space is a Hausdorff topological space.

Let $(X, d)$ be a metric space and $\{x_n\}_{n \in \mathbb{N}}$, be a sequence in $X$.

**DEFINITION 1.2.2.** The sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be convergent to a point $x^*$ of $X$ if given any positive real number $\varepsilon$, there is a natural number $n_\varepsilon$ such that if $n > n_\varepsilon$, then $d(x^*, x_n) < \varepsilon$.

If $\{x_n\}_{n \in \mathbb{N}}$ converges to $x^*$, then we write $\{x_n\}_{n \in \mathbb{N}} \rightarrow x^*$, or $x_\infty = \lim_{n \rightarrow \infty} x_n$. The element $x^*$ is said to be the limit of $\{x_n\}_{n \in \mathbb{N}}$. In a metric space the limit of a sequence is unique.

Let $(X_1, d_1)$ and $(X_2, d_2)$ be metric spaces.

**DEFINITION 1.2.3.** A mapping $f : X_1 \rightarrow X_2$ is said to be continuous at a point $x_0 \in X_1$ if, given any positive real number $\varepsilon > 0$, there is a positive real number $\delta_\varepsilon$ such that if $d_1(x_0, x) < \delta_\varepsilon$, then $d_2(f(x_0), f(x)) < \varepsilon$.

Let $(X, d)$ be a metric space and $\{x_n\}_{n \in \mathbb{N}}$ a sequence in $X$.

**DEFINITION 1.2.4.** The sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be a Cauchy sequence if given any positive real number $\varepsilon$ there is a natural number $n_\varepsilon$ such that if $m, n$ are natural numbers and $m, n > n_\varepsilon$, then $d(x_m, x_n) < \varepsilon$.

**DEFINITION 1.2.5.** A metric space $(X, d)$ is said to be complete if every Cauchy sequence in $X$ converges to a point of $X$.

It is known that any incomplete metric space can be densely immersed in a complete metric space.
1.3 Some classes of topological vector spaces

In this book we will use only real vector spaces. Given a real vector space $E$ and a topology $\tau$ on $E$, the pair $(E, \tau)$ (or often denoted by $E(\tau)$) is called a topological vector space if the following axioms are satisfied:

1. $(x, y) \rightarrow x + y$ is continuous on $E \times E$ into $E$,
2. $(\lambda, x) \rightarrow \lambda x$ is continuous on $\mathbb{R} \times E$ into $E$.

An important class of topological vector spaces is the class of normed vector spaces.

**Normed vector spaces**

A real vector space $E$ is said to be a normed space if to every $x \in E$ there is associated a non-negative real number $\|x\|$, called the norm of $x$ such that the following axioms are satisfied:

1. $\|x + y\| \leq \|x\| + \|y\|$, for all $x$ and $y$ in $E$,
2. $\|\lambda x\| = |\lambda|\|x\|$, for all $x \in E$ and $\lambda \in \mathbb{R}$,
3. $\|x\| = 0$ if and only if $x = 0$.

**Remark.** From axiom (n$_3$) we have that $\|x\| > 0$ if $x \neq 0$.

A normed vector space will be denoted by $(E, \|\|)$. Every normed vector space $(E, \|\|)$ may be regarded as a metric space, in which the distance between $x$ and $y$ is $d(x, y) = \|x - y\|$. The topology defined on $E$ by this distance is called the topology of the normed vector space $(E, \|\|)$.

The sets $B(0, 1) = \{x \in E : \|x\| < 1\}$ and $\overline{B}(0, 1) = \{x \in E : \|x\| \leq 1\}$ are the open unit ball and the closed unit ball of $E$, respectively.

**Banach space**

A Banach space is a normed vector space, which is complete in the
metric defined by its norm, that is, every Cauchy sequence is convergent.

Many of the best-known function spaces, used in practical problems are Banach spaces. We mention just a few types: spaces of continuous functions on compact spaces, the well-known $L_p$-spaces, certain spaces of differentiable functions, spaces of continuous linear mappings from one Banach space into another etc.

Let $(E_1, \| \cdot \|_1)$ and $(E_2, \| \cdot \|_2)$ be Banach spaces. A mapping $L : E_1 \to E_2$ is called a linear mapping if $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$ for all $x, y \in E_1$ and all real numbers $\alpha, \beta$. The linear mapping $L$ is called continuous at $x_0 \in E_1$ if for any sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of $E_1$ such that $\|x_n - x_0\|_1 \to 0$ we have that $\|L(x_n) - L(x_0)\|_2 \to 0$. If $L$ is continuous at every $x \in E_1$, then we say that $L$ is continuous on $E_1$. A mapping $L : E_1 \to E_2$ is called bounded if there exists a number $\rho$ such that $\|L(x)\|_2 \leq \rho \|x\|_1$, for all $x \in E_1$. We denote by $\mathcal{L}(E_1, E_2)$ the set of all continuous mappings from $E_1$ into $E_2$. It is known that a linear mapping from $E_1$ into $E_2$ is continuous, if and only if it is bounded.

If we take $(E_2, \| \cdot \|_2) = (\mathbb{R}, |\cdot|)$, where $|x|$ is the absolute value of $x \in \mathbb{R}$, then we denote by $E_1^* = \mathcal{L}(E_1, \mathbb{R})$ and we say that $E_1^*$ is the topological dual of $E_1$. If for any $L \in \mathcal{L}(E_1, E_2)$ we define $\|L\| = \sup_{|x|=1} \|L(x)\|$, then we have that $L \to \|L\|$ is a norm on $\mathcal{L}(E_1, E_2)$ and it is known that $(\mathcal{L}(E_1, E_2), \| \cdot \|)$ is a Banach space. Consequently for any Banach space $(E, \| \cdot \|)$, its topological dual $E^*$ is also a Banach space.

**Hilbert space**

The class of Hilbert spaces is an important subclass of Banach spaces. In this book, we will consider only real Hilbert spaces. D. Hilbert (in his paper: Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Leipzig, 1912) initiated the theory of Hilbert spaces. After many years, John von Neumann (1903-1957) became the first to formulate an axiomatic theory of Hilbert spaces.
Let $E$ be a real vector space.

**DEFINITION 1.3.1.** We say that a mapping $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{R}$ is an inner-product in $E$ if for any $x, y, z \in E$ and $\alpha, \beta \in \mathbb{R}$ the following axioms are satisfied:

1. $\langle x, y \rangle = \langle y, x \rangle$,
2. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,
3. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

A real vector space with an inner-product is called an *inner-product space*, or a pre-Hilbert space.

**Examples**

I. The real field $\mathbb{R}$ is an inner-product space. The inner-product is defined by $\langle x, y \rangle = x \cdot y$.

II. The $n$-dimensional real vector space $\mathbb{R}^n$, with the inner-product defined by $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, is an inner-product space.

III. The space $l^2$ of all sequences $(x_1, x_2, \ldots, x_n, \ldots)$ of real numbers such that $\sum_{k=1}^{\infty} |x_k|^2 < +\infty$, with the inner-product defined by $\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k$, is an infinite dimensional inner-product space. This space is between the most important examples of inner-product spaces.

IV. Let $E$ be the real vector space of sequences $(x_1, x_2, \ldots, x_n, \ldots)$ of real numbers such that only a finite number of terms is non-zero. This is an inner-product space with the inner-product defined by $\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k$, where $x = (x_1, x_2, \ldots, x_k, \ldots)$ and $y = (y_1, y_2, \ldots, y_k, \ldots)$. 
V. The real vector space $C([a, b], \mathbb{R})$ of all continuous real-valued functions on the interval $[a, b] \subset \mathbb{R}$, with the inner-product
\[ \langle f, g \rangle = \int_a^b f(x) g(x) \, dx \]
is an inner-product space.

VI. The real vector space $L^2(\Omega)$ with the inner-product defined by
\[ \langle f, g \rangle = \int_{\Omega} f(x) g(x) \, dx \]
is a very important inner-product space.

VII. Let $E$ be the Cartesian product of Hilbert spaces $(E_1, \langle \cdot, \cdot \rangle_1), \ldots, (E_n, \langle \cdot, \cdot \rangle_n)$, i.e.,
\[ E = E_1 \times E_2 \times \cdots \times E_n = \{ (x_1, x_2, \ldots, x_n) : x_i \in E_i, \ldots, x_n \in E_n \} \]
is a Hilbert space with the inner-product defined by:
\[ \langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \langle x_1, y_1 \rangle_1 + \langle x_2, y_2 \rangle_2 + \cdots + \langle x_n, y_n \rangle_n. \]

Let $(E, \langle \cdot, \cdot \rangle)$ be an inner-product space. Two vectors $x$ and $y$ in $E$ are called orthogonal, denoted by $x \perp y$, if $\langle x, y \rangle = 0$. Any inner-product space $(E, \langle \cdot, \cdot \rangle)$ is a normed vector space with the norm defined by
\[ \|x\| = \sqrt{\langle x, x \rangle}, \text{ for any } x \in E. \]
The norm of any inner-product space $(E, \langle \cdot, \cdot \rangle)$ satisfies the following important properties:

Schwartz’s inequality. For any two elements $x$ and $y$ in $E$ we have
\[ |\langle x, y \rangle| \leq \|x\| \cdot \|y\|. \]
The equality $|\langle x, y \rangle| = \|x\| \cdot \|y\|$ holds, if and only if $x$ and $y$ are linearly dependent.

Parallelogram law. For any two elements $x$ and $y$ in $E$ we have
\[ \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \]

A consequence of the parallelogram law is the Pythagorean Formula: if $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

**Definition 1.3.2.** A complete inner-product space is called a Hilbert space. (By the completeness of an inner-product space $(E, \langle \cdot, \cdot \rangle)$, we mean the completeness of $E$ as a normed space).
The examples I-III and VI-VII are Hilbert spaces. The examples described in IV and V are not Hilbert spaces, since these spaces as normed vector spaces are not complete. We will denote a Hilbert space by \((H, \langle \cdot, \cdot \rangle)\).

The topological dual of a Hilbert space \((H, \langle \cdot, \cdot \rangle)\) can be identified (by an isomorphism) with \(H\). We recall also that a Hilbert space is called separable if it contains a complete orthonormal sequence. (An orthonormal sequence \(\{x_n\}_{n \in \mathbb{N}}\) in a Hilbert space \((H, \langle \cdot, \cdot \rangle)\) is said to be complete if for every \(x \in H\) we have \(x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n\).

Let \(E(\tau)\) be a topological vector space.

**DEFINITION 1.3.3.** A subset \(D\) of \(E\) is called bounded if for each 0-neighborhood \(U\) in \(E\), there exists \(\lambda \in \mathbb{R}\) such that \(D \subseteq \lambda U\).

For example, in a normed vector space \((E, \langle \cdot, \cdot \rangle)\), the sets \(B(0, 1)\) and \(\overline{B}(0,1)\) are bounded sets. The following notions are also useful. We say that a subset \(D_1 \subset E\) absorbs a subset \(D_2 \subset E\) if there exists \(\lambda_0 \in \mathbb{R}\) such that \(D_2 \subseteq \lambda D_1\) whenever \(|\lambda| \geq |\lambda_0|\). A subset \(D \subset E\) is called radial (absorbing), if \(D\) absorbs every finite subset of \(E\). A subset \(D \subset E\) is circled if \(\lambda D \subseteq D\), whenever \(|\lambda| \leq 1\). If \(A \subset E\) the circled hull of \(A\) is the intersection of all circled subsets of \(E\) containing \(A\).

**DEFINITION 1.3.4.** We say that a subset \(D\) of \(E\) is convex if \(x \in D\) and \(y \in D\) imply that \(\lambda x + (1-\lambda) y \in D\) for all scalars satisfying \(0 < \lambda < 1\).

It is known that the sets
\[
\{ \lambda x + (1-\lambda) y : 0 \leq \lambda \leq 1 \} \quad \text{and} \quad \{ \lambda x + (1-\lambda) y : 0 < \lambda < 1 \}
\]
are called the closed and open line segments. It is easy to show that convexity of a subset \(D \subset E\) is preserved under translation, i.e., \(D\) is convex if and only if \(x_0 + D\) is convex for every \(x_0 \in E\). If \(A, B\) are convex subsets of the space \(E\), then \(\text{int}(A), \overline{A}, A + B\) and \(\lambda A (\lambda \in \mathbb{R})\) are convex.

The union of two convex sets generally is not a convex set, but the intersection of any family of convex sets is a convex set.
Let $A$ be a subset of the space $E$. The convex hull of $A$, denoted by $\text{conv}(A)$, is the intersection of all convex sets containing the set $A$. It is known that

$$\text{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1 \text{ and } n \in \mathbb{N} \right\}.$$ 

**Definition 1.3.5.** If $D \subset E$ is any radial subset, the non-negative real function on $E$

$$x \mapsto p_D(x) = \inf \{ \lambda > 0 : x \in \lambda D \}$$

is called the gauge, or Minkowski functional of $D$.

A semi-norm on $E$ is the gauge of a radial, circled and convex subset of $E$. The analytical description of semi-norms is given by the following definition.

**Definition 1.3.6.** A real-valued function $p$ on $E$ is a semi-norm if and only if

1. $p(x + y) \leq p(x) + p(y)$ for any $x, y \in E$,
2. $p(\lambda x) = |\lambda| p(x)$ for any $\lambda \in \mathbb{R}$ and $x \in E$.

Obviously if $p$ is a semi-norm on $E$ then $p(0) = 0$ and $p(x) \geq 0$ for any $x \in E$. If $D \subset E$ is a radial, convex, circled set, then the semi-norm $p$ on $E$ is the gauge of $D$ if and only if $D_0 \subset D \subset D_1$ where $D_0 = \{ x \in E : p(x) < 1 \}$, $D_1 = \{ x \in E : p(x) \leq 1 \}$. It is known also that if $p$ is a semi-norm on $E$, then $p$ is continuous at $0 \in E$ if and only if $D_0 = \{ x \in E : p(x) < 1 \}$ is open in $E$, and also, if and only if $p$ is uniformly continuous on $E$. The following two notions are also useful in this book.

Let $E(\tau)$ be a topological vector space. A subset $D$ of $E$ is said to be star-shaped if there is at least one $x_0 \in D$ such that $(1 - \lambda)x_0 + \lambda x \in D$ for all $x \in D$ and $0 < \lambda < 1$. The point $x_0 \in D$ is said to be the star centre of $D$. Every convex set is star shaped but not conversely.

A subset $D$ of $E$ is called contractible if there is a continuous mapping $h: D \times [0,1] \to D$ such that $h(x,0) = x$ for all $x \in D$ and $h(x,1) = x_0$ for some $x_0 \in D$. Every star shaped set $D \subset E$ is contractible.
since the mapping \( h(x,t) = tx_0 + (1-t)x, \quad (x,t) \in D \times [0,1] \) where \( x_0 \) is the star centre of \( D \).

**Locally convex spaces**

A topological vector space \( E \) over \( \mathbb{R} \) will be called *locally convex* if it is a Hausdorff space such that every neighborhood of any \( x \in E \) contains a convex neighborhood of \( x \). We can show that \( E \) is a *locally convex topological vector space* if the convex neighborhoods of \( 0 \) form a base at \( 0 \) with intersection \( \{0\} \).

Analytically, a locally convex topology on \( E \) is determined by an arbitrary family \( \{p_\alpha\}_{\alpha \in \mathcal{A}} \) of semi-norms as follows: for each \( \alpha \in \mathcal{A} \), let

\[
U_\alpha = \{x \in E : p_\alpha(x) \leq 1\}
\]

and consider the family \( \left\{ \frac{1}{n} U \right\} \), where \( n \in \mathbb{N} \) and \( U \) ranges over all finite intersections of sets \( U_\alpha (\alpha \in \mathcal{A}) \). This family \( U \) satisfies the conditions indicated above and hence is a base at \( 0 \) for a locally convex topology \( \tau \) on \( E \), called *the topology generated by the family \( \{p_\alpha\}_{\alpha \in \mathcal{A}} \) ;* equivalently, \( \{p_\alpha\}_{\alpha \in \mathcal{A}} \) is said to be a *generating family of semi-norms for \( \tau \).* We denote a locally convex space by \( E(\tau) \) or \( (E(\tau), \{p_\alpha\}_{\alpha \in \mathcal{A}}) \).

Conversely every locally convex topology on \( E \) is generated by a suitable family of semi-norms; it suffices to take the gauge functions of a family of convex, circled 0-neighborhoods whose positive multiples form a subbase at \( 0 \). Obviously, every member of a generating family of semi-norms is continuous for \( \tau \).

We note that, we can prove that \( \tau \) is Hausdorff if and only if for each \( x \in E \), \( x \neq 0 \) and each family \( \mathcal{P} \) of semi-norms generating \( \tau \), there exists \( p \in \mathcal{P} \) such that \( p(x) > 0 \). Any Banach space is a locally convex vector space, but the converse is not true. There exist topological vector spaces that are not locally convex spaces. The general topological vector spaces are not very much used in mathematical modeling of practical problems, but the notion of topological vector space is a fundamental notion in mathematics.
1.4 Compactness and compact operators

Let \((X, \tau)\) be a topological space. We say that a family \(\{U_i\}_{i \in I}\) of open subsets of \(X\) is an open cover of \(X\) if \(X = \bigcup_{i \in I} U_i\). Let \(\{U_i\}_{i \in I}\) be an open cover of the space \(X\). A collection \(\{V_j\}_{j \in J}\) is said to be an open subcover of \(\{U_i\}_{i \in I}\) if \(\{V_j : j \in J\} \subseteq \{U_i : i \in I\}\), (that is, each \(V_j\) is a \(U_i\) and \(\{V_j\}_{j \in J}\) is itself an open cover of \(X\).

**Definition 1.4.1.** A topological space \((X, \tau)\) is said to be compact if given any open cover \(\{U_i\}_{i \in I}\) of \(X\), there is a finite subcover of \(\{U_i\}_{i \in I}\).

Let \((X, \tau)\) be a topological space and \(A \subseteq X\) a non-empty subset. An open cover of \(A\) is a collection \(\{U_i\}_{i \in I}\) of open subsets of \(X\) such that \(A \subseteq \bigcup_{i \in I} U_i\). Equivalently, \(\{U_i\}_{i \in I}\) is an open cover of \(A\) if \(\{U_i \cap A\}_{i \in I}\) is an open cover of the subspace \(A\). We say that \(A\) is a compact subset of \(X\) if every open cover of \(A\) has a finite subcover. Equivalently, \(A\) is a compact subset if the topological subspace \((A, \tau_A)\) is compact, where \(\tau_A\) is the topology on \(A\) induced by the topology \(\tau\). The following theorem is a classical result.

**Theorem 1.4.1.** Let \((X, \tau)\) be a topological space. The following statements are equivalent:

1. \((X, \tau)\) is a compact topological space,
2. for any family \(\{F_i\}_{i \in I}\) of closed subsets of \(X\) such that the intersection of any finite number of the \(F_i\) is non-empty we have that \(\bigcap_{i \in I} F_i \neq \emptyset\),
3. every net in \(X\) has a subnet convergent to an element of \(X\).

If \(A\) is a subset of topological space \((X, \tau)\), we say that \(A\) is relatively compact in \(X\), if \(\overline{A}\) is compact. Suppose we are given two topological spaces \((X_1, \tau_1)\) and \((X_2, \tau_2)\).
DEFINITION 1.4.2. We say that a mapping $f : X_1 \to X_2$ is compact with respect to a non-empty subset $A$ of $X_1$ if $f(A)$ is compact in $X_2$ (i.e., if $f(A)$ is relatively compact in $X_2$).

Remark. For nonlinear mappings (when $X_1$, $X_2$ are Banach spaces) the continuity is not a consequence of compactness. This is true only for linear mappings.

Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two Banach spaces. Let $T : E_1 \to E_2$ be a mapping (linear or nonlinear).

DEFINITION 1.4.3. We say that $T$ is completely continuous if and only if the following two properties are satisfied:

1. $T$ is a continuous mapping,
2. for any bounded subset $A$ in $E_1$, the set $T(A)$ is relatively compact in $E_2$.

Remark. If $T$ is linear then, in this case property (2) implies property (1), but for nonlinear operators this implication is not true.

Compactness and complete continuity are two fundamental notions in topology and in functional analysis (linear and nonlinear). Complete continuity will be very much used in this book.

DEFINITION 1.4.4. We say that a mapping $f : E_1 \to E_1$ is a completely continuous field if and only if there is a completely continuous operator $T : E_1 \to E_1$ such that $f$ has the representation $f(x) = x - T(x)$, for any $x \in E_1$ (or shortly, $f = I - T$, where $I : E_1 \to E_1$ is the identity mapping).

The notion of a completely continuous field is related to the notion of Leray–Schauder degree.

1.5 Measures of noncompactness and condensing operators

In this section we consider the basic notions connected with measures of noncompactness and condensing mappings. We give on this subject only the elementary properties necessary in this book.
The first notion of "measure of noncompactness" was introduced by K. Kuratowski in 1930, (Kuratowski, K. [1]). The theory of measures of noncompactness and of condensing operators received a new impetus after the work of G. Darbo (Darbo, G. [1]). Now, there exist some expository articles and books on this subject [(Sadovskii, B. N., [1]), (Danes, J., [1]), (Banaś, J. and Goebel, K., [1]), (Akhmerov, R. R., Kamenskii, M. I., Potapov, A. S., Rodkina, A. E. and Sadovkii, B. N., [1]).

We give the notions of noncompactness in a general Banach space. Let \((E, \|\cdot\|)\) be a Banach space and let \(\Omega\) be a subset of \(E\). Let \(A\) be a non-empty subset of \(E\). We recall that by diameter of \(A\), (denoted by \(\text{diam}(A)\)) one means the number \(\sup \{\|x - y\|: x, y \in A\}\). We use \(B = B(0, 1)\) to denote the open unity ball in \(E\).

**DEFINITION 1.5.1.** The Kuratowski measure of noncompactness \(\alpha(\Omega)\) of the set \(\Omega\) is the number \(\inf \{d > 0 : \Omega \text{ admits a finite covering of sets of diameter smaller than } d\}\).

We say that a set \(D \subset E\) is an \(\varepsilon\)-net of \(\Omega\) if
\[
\Omega \subset D + \varepsilon B = \{x + \varepsilon b : x \in D, b \in B\}.
\]

**DEFINITION 1.5.2.** The Hausdorff measure of noncompactness \(\chi(\Omega)\) of the set \(\Omega\) is the number \(\inf \{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon - \text{net in } E\}\).

Now, we indicate some of the properties of the Kuratowski and Hausdorff measures of noncompactness (denoted below by \(\psi\)).

**Property 1 (Regularity).** \(\psi(\Omega) = 0\) if and only if \(\overline{\Omega}\) is compact.

**Property 2 (Nonsingularity).** \(\psi\) is equal to zero on every one-element set.

**Property 3 (Monotonicity).** \(\Omega_1 \subset \Omega_2\) implies \(\psi(\Omega_1) \leq \psi(\Omega_2)\).

**Property 4 (Semi-additivity).** \(\psi(\Omega_1 \cup \Omega_2) = \max \{\psi(\Omega_1), \psi(\Omega_2)\}\).
Property 5 (Lipschitzianity). \( |\psi(\Omega_1) - \psi(\Omega_2)| \leq L_\psi \rho(\Omega_1, \Omega_2) \), where \( L_\chi = 1, L_\sigma = 2 \) and \( \rho \) denotes the Hausdorff semi-metric, i.e.,
\[
\rho(\Omega_1, \Omega_2) = \inf \{ \varepsilon > 0 : \exists \Omega_1 + \varepsilon \overline{B} \supset \Omega_2 \text{ and } \Omega_2 + \varepsilon \overline{B} \supset \Omega_1 \}.
\]

Property 6 (Continuity). For any \( \Omega \subset E \) and any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( |\psi(\Omega) - \psi(\Omega_1)| < \varepsilon \) for all \( \Omega_1 \) satisfying \( \rho(\Omega, \Omega_1) < \delta \).

Property 7 (Semi-homogeneity). \( \psi(\lambda \Omega) = |\lambda| \psi(\Omega) \) for any real number \( \lambda \).

Property 8 (Algebraic Semi-additivity). \( \psi(\Omega_1 + \Omega_2) \leq \psi(\Omega_1) + \psi(\Omega_2) \).

Property 9 (Invariance under Translations). \( \psi(\Omega + x_0) = \psi(\Omega) \) for any \( x_0 \in E \).

The following properties are important but the proof of each requires some technicalities.

**Theorem 1.5.1.** The Kuratowski and Hausdorff measures of noncompactness are invariant under passage to the closure and to the convex hull, i.e., \( \psi(\overline{\Omega}) = \psi(\Omega) = \psi(\text{conv}(\Omega)) \).

**Corollary 1.5.2.** We have the following useful formula:
\[
\alpha\left( \bigcup_{0 < \lambda < \lambda_0} \lambda \Omega \right) = \lambda_0 \alpha(\Omega).
\]

**Proof.** This formula is a consequence of properties (1), (3), (4) of Theorem 1.5.1 and of the fact that \( \bigcup_{0 < \lambda < \lambda_0} \lambda \Omega \subset \text{conv}(\lambda_0 \Omega \cup \{0\}) \). \( \square \)

**Theorem 1.5.3.** Let \( B = B(0,1) \) be the unit ball in \( E \). Then \( \alpha(B) = \chi(B) = 0 \) if \( \dim(E) < \infty \) and \( \alpha(B) = 2, \chi(B) = 1 \) if \( E \) is an infinite dimensional Banach space.
**THEOREM 1.5.4.** The Kuratowski and Hausdorff measures of noncompactness are related by the inequalities \( \chi(\Omega) \leq \alpha(\Omega) \leq 2\chi(\Omega) \).

Now, we give the definition and some properties of condensing operators. A condensing operator is a mapping under which the image of any set is in a certain sense "more compact" than the set itself. The degree of noncompactness of a set is estimated by a measure of noncompactness. Contractive maps and the completely continuous maps are condensing.

Let \((E_1, \|\cdot\|_1), (E_2, \|\cdot\|_2), (E_3, \|\cdot\|_3)\) be Banach spaces. Suppose we are given on each space a measure of noncompactness denoted respectively by \(\mu_1, \mu_2, \mu_3\). We denote by \(B_{E_i}\) the bounded sets in \(E_i\) (i = 1, 2, 3).

**DEFINITION 1.5.3.** We say that a mapping \(f : E_1 \rightarrow E_2\) satisfies the Darbo condition with a constant \(k \geq 0\), with respect to the measures of noncompactness \(\mu_1, \mu_2\) if for any \(D \in B_{E_1}\) we have \(f(D) \in B_{E_2}\) and \(\mu_2(f(D)) \leq k\mu_1(D)\).

**Remark.** We note that if \(f\) satisfies the Darbo condition with a constant \(k\), we say also that \(f\) is a \(k\)-set Lipschitz mapping.

If \(0 \leq k < 1\) and \(f\) satisfies the Darbo condition with the constant \(k\), then in this case we say that \(f\) is a \(k\)-set contraction. If \(f\) satisfies the Darbo condition, the smallest constant \(k\) such that \(\mu_2(f(D)) \leq k\mu_1(D)\) will be denoted by \(k(\mu_1, \mu_2, f)\). In the case \(E_1 = E_2\) and \(\mu_1 = \mu_2 = \mu\), we shall write \(k(\mu, f)\), instead of \(k(\mu_1, \mu_2, f)\).

The following propositions describe the basic properties of mapping satisfying the Darbo property.

**PROPOSITION 1.5.5.** If \(f : E_1 \rightarrow E_2\) and \(g : E_2 \rightarrow E_3\) satisfy the Darbo condition with respect to \((\mu_1, \mu_2)\) and \((\mu_2, \mu_3)\) respectively, then we have \(k(\mu_1, \mu_3, g \circ f) \leq k(\mu_1, \mu_2, f) \cdot k(\mu_2, \mu_3, g)\).
PROPOSITION 1.5.6. If $f_1, f_2 : E_1 \to E_2$ satisfy the Darbo condition, then $f = \lambda f_1 + (1 - \lambda) f_2$, with $0 \leq \lambda \leq 1$ also satisfy the Darbo condition and we have $k(\mu_1, \mu_2, f) \leq \lambda k(\mu_1, \mu_2, f_1) + (1 - \lambda) k(\mu_1, \mu_2, f_2)$.

PROPOSITION 1.5.7. If $f, g : E_1 \to E_2$ and $\mu_2$ is semi-homogeneous and semi-additive, then we have
\[ k(\mu_1, \mu_2, f + g) \leq k(\mu_1, \mu_2, f) + k(\mu_1, \mu_2, g) \]
and for any $\lambda \in \mathbb{R}$, also we have the formula,
\[ k(\mu_1, \mu_2, \lambda f) = |\lambda| k(\mu_1, \mu_2, f). \]

Let $(E_1, \|\cdot\|), (E_2, \|\cdot\|)$ be Banach spaces. Suppose that on $E_1$ (resp. on $E_2$) is defined a measure of noncompactness $\mu_1$ (resp. $\mu_2$), with values in some partially ordered set $(Q, \leq)$.

DEFINITION 1.5.4. A continuous mapping $f : E_1 \to E_2$ is said to be $(\mu_1, \mu_2)$-condensing if $\Omega \subset E_1$ and $\mu_2 \left[ f(\Omega) \right] \geq \mu_1(\Omega)$ imply that $\Omega$ is relatively compact.

Remarks.
1. In Definition 1.5.4 we can have $f : D(f) \to E_2$, where $D(f) \subset E_1$ is the domain of definition of $f$ and $D(f)$ is such that $D(f) \neq E_1$. In this case we must take $\Omega \subset D(f)$.
2. The mapping $f$ is said also to be $(\mu_1, \mu_2)$-condensing in the proper sense if $\mu_2 \left[ f(\Omega) \right] < \mu_1(\Omega)$ for any $\Omega \subset D(f)$ with the property that $\overline{\Omega}$ is not compact. We note that in a partially ordered set $(Q, \leq)$ the strict inequality $\alpha < \beta$ means $\alpha \leq \beta$ and $\alpha \neq \beta$. If the set $Q$ is totally ordered by the ordering “$\leq$", then in this case the two notions of condensing mapping coincide. Obviously, a completely continuous mapping and a contractive mapping, both are condensing with respect to the Kuratowski measure of noncompactness and any continuous and compact mapping is also condensing with respect to the Hausdorff measure of noncompactness.

Suppose that $(Q, \leq)$ is an ordered convex cone in a Banach space.