PLASTICITY AND GEOTECHNICS
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To Xiu-Li, Christina and Thomas
for inspiration, encouragement and forbearance

“What is new is the inter-relation of concepts,
the capacity to create new types of calculation,
and the unification of the bases for judgement.”
A.N. Schofield and C.P. Wroth (1968)
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Geomaterials are very complex and theoretical soil and rock mechanics involves making important idealisations of the real material - in order to analyse it is always necessary to idealise. The process of analysis may be thought of as an exercise in modelling and as such may be carried out at various degree of sophistication ranging from the purely intuitive through to the very complex depending on the problem.

The key to successful engineering modelling lies in having a clear grasp of the nature of the idealisations that have been made and their influence on the calculated solutions. A well known class of ideal material is the porous elastic solids. By modelling the ground from such a material it is possible to find solutions to a wide range of practical problems. The good engineer does not believe that the soil really is elastic but by adopting such an idealisation it is often possible to arrive at valuable approximations to, and insights into, the behaviour of the real material.

Plastic materials form a wide ranging class of ideal materials which are much more 'soil-like' or 'rock-like' in their behaviour than their elastic counterparts. The theory of plasticity as applied to metals has reached its maturity and many books are available to guide practising engineers and students alike in its applications. The application of plasticity theory to geomaterials has been developing rapidly in recent years. Even for an expert in the field of plasticity it is difficult to keep up with the many developments that have, and are taking place. Practising geotechnical engineers, researchers and students could be forgiven for feeling a sense of bewilderment at the range of models now available and the claims made for them.

It is most timely that Professor Yu should have provided this comprehensive review of plasticity theory for geomaterials and its applications to geotechnical modelling. The first part of the book deals with the fundamentals of the theory of plasticity and in particular the general elastic-plastic theorems and principles. In the words of Professor Yu: “It would be hard to overstate the importance of general theorems in the development and application of the theory of plasticity.” Formal proofs are developed for all the major theorems and principles that underpin the theory including the Principle of Virtual Work, the Uniqueness Theorems, Minimum and Variational Principles, Theorems of Plastic Collapse for Limit Analysis and the Shakedown Theorems. Readers who are relatively unsophisticated mathematically should not be put off by the formal mathematical treatment adopted as the physical
implications of the results are explained and some valuable historical insights are woven into the story.

The second part of the book traces the development of plasticity models from the classical perfect plasticity models of Mohr-Coulomb, Tresca, von Mises through a range of Critical State isotropic hardening models and multi-surface and bounding surface models to some more recent models involving non-coaxiality of stress and strain increment and "plasticity without a prior yield criterion." Of particular interest is the detailed development of Professor Yu’s unified Critical State Model known as CASM.

The third part of the book deals with solution techniques. These range from the rigorous analysis of some elastic-plastic problems, slip line analysis and limit analysis, moving on to less well known shakedown analysis and ending with a full chapter on the use of Finite Element Analysis. This final chapter places particular emphasis on the significant improvements in accuracy that can be achieved using new displacement interpolation functions developed by Professor Yu and which can be easily incorporated into standard codes.

Mention was made earlier of the bewildering number of plasticity models that are now available for tackling geotechnical problems. This book will serve as a most valuable source of reference to many of these models which are presented in a unified and rigorous manner, emphasising their strengths, weaknesses and applications in a balanced and objective way. It will prove invaluable to engineers, teachers and students working in this challenging and exciting subject.

J.B.B.
The theory of plasticity is concerned with the mathematical study of stress and strain in plastically deformed materials. In their Preface to the Proceedings of the Symposium on Plasticity and Soil Mechanics held in Cambridge on 13-15 September 1973, Peter Wroth and Andrew Palmer stated:

"Plasticity theory was developed by people who thought in terms of metals, and for about twenty years workers in soil mechanics have been looking at the theory, rather as outsiders, and asking whether it had anything to offer them."

Since that time, however, enormous progress has been made towards thoughtful and sensitive application and development of plasticity theory, backed by careful experiments, in geotechnical engineering. Although further progress is still expected, the whole framework of plasticity theory for geomaterials may have reached a good degree of maturity. At present, few, if any, would doubt the usefulness and importance of plasticity theory in geotechnical analysis and design.

This book arose from my belief that there is an urgent need for the geotechnical community to have a unified and coherent presentation of plasticity theory for geomaterials and its applications to geotechnical analysis. Accordingly, the book attempts to summarise and present, in one volume, the major developments achieved to date in the field of plasticity theory and its geotechnical applications. The book covers classical, recent and modern developments of appropriate constitutive theories of stress-strain relations for geomaterials and a wide range of analytical and computational techniques that are available for solving geotechnical design problems. My main concern in writing this book has been to bring out the key concepts behind the most useful theoretical developments, the inter-relation of these concepts and their use and implementation in analytical and numerical procedures that are needed for solving practical problems in geotechnical engineering.

The book is intended primarily as a reference book for civil, geotechnical and mining engineers, researchers and students who are concerned with plasticity theory and its engineering applications. It may also be adopted as a text book for postgraduate or advanced undergraduate courses in plasticity theory. Given plasticity theory has widespread applications in many disciplines, the book should be of direct interest to researchers and engineers in the fields of continuum mechanics, mechanical and chemical engineering who are concerned with granular materials.

As indicated by its table of contents, the book is divided into three parts. The first part, Chapters 1 to 4, covers fundamental elements of continuum mechanics and
classical plasticity theory. The second part, containing Chapters 5 to 9, describes all the major theories that have been developed for deriving non-linear stress-strain relations for geomaterials. The final part of the book, containing Chapters 10 to 14, presents a wide range of solution techniques that are available for solving boundary value problems in geotechnical engineering.

The book is the result of many years of study, research, teaching and reflection. I have benefited much from collaborations with many colleagues and students - their contribution speaks for itself. Those concerned will know that I appreciate their assistance and help with considerable gratitude. I would particularly like to thank, among many, the late Peter Wroth, Jim Mitchell, John Burland, Mike Jamiolkowski, Tony Spencer, Kerry Rowe, Mark Randolph, Ted Brown, Steve Brown, John Carter, Ian Collins, Scott Sloan, Guy Houltsby, Brian Simpson, Robert Mair, Antonio Gens, Harry Poulos, Wai-Fai Chen and Malcolm Bolton for their personal influence on shaping my approach to research and teaching in plasticity theory and geotechnical engineering. I also benefited from discussions with the late John Booker, Chandra Desai, Poul Lade, Yannis Dafalias, Roberto Nova, Gerd Gudehus, Alan Ponter, Peter Kleeman, Andrew Whittle, Pieter Vermeer, David Harris, Fernando Schnaid, Wei Wu, Glenn McDowell, Rodrigo Salgado, Len Herrmann, Colin Thornton, Ken Been, Radoslaw Michalowski, Patrick Selvadurai, Ian Moore, Harvey Burd and Andrew Abbo on various aspects of plasticity theory.

It should be mentioned that consulting the classic book on plasticity by Rodney Hill, Professor of Applied Mathematics at the University of Nottingham from 1953 to 1962, has always been a fruitful and inspirational experience for me. I am very grateful to Tony Spencer, Antonio Gens, Glenn McDowell, Nick Thom, Wei Wu, Rodrigo Salgado and Fernando Schnaid for reading through earlier versions of the book and for their valuable comments. My students Cuong Khong, Huaxiang Li, Xun Yuan, Yunming Yang, Jun Wang and Ringo Tan have also provided helpful suggestions. I wish to thank all members of the staff and students, past and present, at the Nottingham Centre for Geomechanics (NCG) for making it a supportive, scholarly and creative environment under which this book could be produced. Special thanks are extended to David Gao for his encouragement, and to John Martin-dale and Robert Saley of Springer for their support throughout this project.

Finally it is a pleasure to record my deep appreciation to Professor John Burland for writing a Foreword and for his support and encouragement over the years.

Hai-Sui Yu
West Bridgford, Nottingham
CHAPTER 1

INTRODUCTION

1.1 SCOPE AND AIMS

This book is concerned with plasticity theory of geomaterials (i.e. clay, sand, silt, rock etc) and its application to geotechnical analysis and design. In a classic book, Hill (1950) gives the following concise definition for plasticity theory:

"The theory of plasticity is the name given to the mathematical study of stress and strain in plastically deformed solids. It takes as its starting point certain experimental observations of the macroscopic behaviour of a plastic solid in uniform state of combined stress. The task of the theory is two-fold: first, to construct explicit relations between stress and strain agreeing with the observations as closely and as universally as need be; and second, to develop mathematical techniques for calculating non-uniform distributions of stress and strain in bodies permanently distorted in any way."

This definition is followed in the present book which will focus on developments of appropriate constitutive theories of stress-strain relations for geomaterials and various analytical and computational solution techniques that can be used to solve geotechnical design problems involving plastic deformation.

Since the subject now has a very wide scope and is still undergoing a steady development, it will be a difficult task to write a 'definitive' book that would cover every aspect of the development. Instead of covering the whole field cursorily, this book aims to bring together, in one volume, key concepts behind some of the most useful developments in plasticity theory for geomaterials and to discuss their applications to geotechnical analysis. The emphasis is on recent achievements, the inter-relation of key concepts together with their connections to classical metal plasticity, as well as the research work that I have been involved with over the past two decades. Despite this selective nature, the book still gives a comprehensive and unified account of plasticity theory for geomaterials. It is hoped that this publication will facilitate further development and application of plasticity theory in geotechnical engineering.

1.2 A BRIEF HISTORICAL OUTLINE

In this section, a very brief review is given on the development of the subject of plasticity theory of geomaterials. In view of the above discussion, it is instructive
to treat elastic-plastic stress-strain relations and plastic solution techniques sepa­

rately.

1.2.1 Elastic-plastic stress-strain relations

The foundation of classical plasticity theory was laid by the 1950s and 1960s after
several decades of theoretical and experimental research on plastic behaviour of
metals. A review of this early development can be found in Nadai (1950), Hill
(1950), Drucker (1950), Prager (1955) and Naghdi (1960). Key concepts of this
foundation include the assumption of coaxiality of the principal stress and strain
rate tensors by de Saint-Venant (1870), plastic potential theory by von Mises (1928)
and Melan (1938), maximum plastic work principle by Hill (1948), Drucker’s sta­

bility postulate (Drucker, 1952, 1958), and kinematic hardening laws by Prager
(1955) and Ziegler (1959).

The early development of plasticity theory of geomaterials has been built upon
this foundation achieved in metal plasticity. Unlike metal plasticity, however, vol­
ume changes during loading play a key role in modelling plastic behaviour of geo­

materials. The work on soil hardening by Drucker et al. (1957) and that on soil
yielding by Roscoe et al. (1958) laid the foundations for critical state theory, a con­
cept that underpins much of the later developments in plasticity theory for geomat­

erials (Schofield and Wroth, 1968; Roscoe and Burland, 1968; Wroth and Houlsby,

More recent developments of metal plasticity include important concepts such
as bounding surface plasticity (Dafalias and Popov, 1975; Krieg, 1975), multi-sur­
face plasticity (Mroz, 1967; Iwan, 1967), and endochronic theory (Valanis, 1971).
All these concepts have been applied with considerable success in modelling geo­

materials over the last two decades. Other notable concepts that have been used to
develop plastic stress-strain relations for geomaterials are double shearing theory
(Spencer, 1964; de Josselin de Jong, 1971; Harris, 1995; Yu and Yuan, 2005, 2006),
yield vertex theory (Rudnicki and Rice, 1975; Yang and Yu, 2006a,b), thermome­
chanical approach (Houlsby, 1982; Maugin, 1992; Collins and Houlsby, 1997),
mathematical theory of envelopes (Chandler, 1985), and hypoplastic theory
(Green, 1956; Kolymbas, 1991). Apart from stress space-based formulations,
Naghdi and Trapp (1975) and Yoder and Iwan (1981) show that plasticity models
can also be formulated in strain space. Although the strain space approach was used
by a few geotechnical researchers (Zheng et al., 1986; Simpson, 1992; Einav,
2004), its application in geotechnical engineering has so far been very rare.

Most stress-strain relations currently in use are developed based on experimental
observations of the macroscopic behaviour of geomaterials in a uniform state of
combined stress in the laboratory (e.g. Jamiołkowski et al., 1985; Mitchell, 1993). In recent years, however, there has been an increasing use of micromechanics and the discrete element method (DEM) (Cundall and Strack, 1979; Thornton, 2000; McDowell and Bolton, 1998; McDowell and Harireche, 2002; Jiang et al., 2005; Jiang and Yu, 2006) for validating or providing physical insights for continuum plasticity theories.

1.2.2 Plastic solution techniques

Once a suitable stress-strain relation is developed, it needs to be combined with equilibrium equations and compatibility conditions for solving geotechnical boundary value problems. In general, these governing equations are too complex to be solved analytically. Analytical solutions are possible only for problems with very simple geometry and boundary conditions such as cavity expansion problems solved by Hill (1950) and Yu (2000a). For most problems of practical interest, however, numerical methods (e.g. finite element methods, finite difference methods, boundary element methods, and discrete element methods) will have to be employed (e.g. Sloan and Randolph, 1982; Brown, 1987; Gens and Potts, 1988; Zienkiewicz et al., 1998; Carter et al., 2000; Yu, 2000b).

Many geotechnical designs rely on two key calculations: stability analysis and deformation analysis (Terzaghi, 1943; Wroth and Houlsby, 1985). The former is to ensure that geotechnical structures are safe and stable and the latter is to ensure that deformation experienced by a geotechnical structure under working loads is not excessively large. In the past, geotechnical stability analysis has been carried out largely based on a perfectly plastic material model. This is because that for perfectly plastic behaviour, the slip line method and bound theorems of limit and shakedown analysis developed in the classical plasticity theory allow the failure and stability calculations to be carried out in a relatively simple manner (Hill, 1950; Sokolovski, 1965; Koiter, 1960; Davis, 1968, Chen, 1975; Salencon, 1977).

With respect to deformation analysis, past practice has been based on elastic analysis (Poulos and Davis, 1974). This is now recognized to be inaccurate for many cases as experimental research suggests that behaviour of geomaterials is highly nonlinear and plastic, even at very small strain (Burland, 1989). Therefore an appropriate deformation analysis would need to be based on the use of nonlinear elasticity and accurate plastic stress-strain relations.

There is no doubt that a most important development over the last three decades in geotechnical analysis has been the widespread application of finite element methods in both stability and deformation calculations (Naylor et al., 1981; Chen and Mizuno, 1990; Zienkiewicz et al., 1998; Potts and Zdravkovic, 1999; Carter
et al., 2000). Finite element analysis is particularly popular because it is very general and is capable of incorporating any material stress-strain relations. The finite element method can easily account for both material and geometric nonlinearities, which are often present in boundary value problems facing the geotechnical engineer.

1.3 CONTINUUM VERSUS DISCRETE APPROACHES

Mechanics is the science that deals with the interaction between force and motion. In understanding the difference between continuum (i.e. macroscopic) and discrete (i.e. microscopic) approaches, the following remarks of Spencer (1980) may prove instructive:

"Modern physical theories tell us that on the microscopic scale matter is discontinuous; it consists of molecules, atoms and even smaller particles. However, we usually have to deal with pieces of matter which are very large compared with these particles; this is true in everyday life, in nearly all engineering applications of mechanics, and in many applications in physics. In such cases we are not concerned with the motion of individual atoms and molecules, but only with their behaviour in some average sense. In principle, if we knew enough about the behaviour of matter on the microscopic scale it would be possible to calculate the way in which material behaves on the macroscopic scale by applying appropriate statistical procedures. In practice, such calculations are extremely difficult; only the simplest systems can be studied in this way, and even in these simple cases many approximations have to be made in order to obtain results."

Continuum solid mechanics is concerned with the mechanical behaviour of solids on the macroscopic scale. It ignores the discrete nature of matter and treats material as uniformly distributed throughout regions of space. For reasons outlined above, continuum mechanics has been for many years and, in my view, will continue to be the main theoretical basis for modelling mechanical behaviour of geomaterials.

As noted earlier, we have seen an increasing use of discrete mechanics (i.e. the microscopic approach) in recent years. In geotechnical engineering, this trend stems from the development of the Discrete Element Method (DEM) by Cundall and Strack (1979) for modelling granular material. The discrete element method was originally intended as a research tool for investigating the micro-mechanics of granular materials in order to identify the appropriate physically sound continuum model that then might be used in finite element analysis of boundary value problems. As stated by Thornton (2000), however, little progress has been made towards achieving this long term goal. Despite this slow progress, DEM simulations have
provided useful insights into the behaviour of granular materials at the grain scale (Rothenburg and Bathurst, 1992; Cundall, 2000; Thornton, 2000; Jiang et al., 2005; Jiang and Yu, 2006). Thornton (2000) was right to point out that such information obtained from DEM simulations about what happens inside granular materials could lead to reassessment of the underlying concepts and assumptions embedded in traditional continuum mechanics.

In view of the above discussion, this book is mainly concerned with continuum theories of plasticity. However, microscopic information derived from analytical micro-mechanical studies or discrete element modelling will also be used to aid our development whenever possible.

1.4 SIGN CONVENTIONS

Much of the theory of plasticity was initially developed for metals for which tensile stresses are usually considered to be positive. Unfortunately the opposite sign convention is usually adopted in geomechanics because compressive normal stresses are more common than tensile ones. In general, this book adopts the conventional geomechanics sign notation. As in the book of Davis and Selvadurai (1996), however, there are exceptions to this convention particularly in some of the later chapters that deal with elastic-plastic solutions (e.g. Chapters 8, 10, 12 and 13). This should not cause confusion as we will make it clear at each stage of the text whenever a tension positive notation is employed.

REFERENCES


CHAPTER 1


INTRODUCTION


CHAPTER 2

ELEMENTS OF CONTINUUM MECHANICS

2.1 INTRODUCTION

This chapter reviews some of the key elements of continuum mechanics that are essential to both the understanding and development of the theory of plasticity. These concepts are mainly concerned with the analysis of stress and strain, equilibrium equations and compatibility conditions, as well as elastic stress-strain relations. The reader is referred to other texts such as Prager (1961), Fung (1965), Timoshenko and Goodier (1970), Spencer (1980) and Malvern (1969) for a detailed treatment of continuum mechanics.

2.2 STRESS STATE AND EQUILIBRIUM

2.2.1 Two-dimensional elements

As shown in Figure 2.1, the stress state for a two-dimensional element is defined by four stress components $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$ and $\sigma_{yx}$. The moment equilibrium demands that two shear stresses are equal in magnitude, namely $\sigma_{xy} = \sigma_{yx}$. Note that compressive stresses are treated as positive here.

These stress components can be displaced as elements of a square matrix:

$$
\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{bmatrix}
$$

Figure 2.1: Stress state for two-dimensional elements
The two most frequent cases of two-dimensional engineering problems are those of plane stress and plane strain. For the case of plane stress, the stresses normal to the $xy$ plane are assumed to be identically zero. On the other hand, the case of plane strain only has non-zero strain components in the $xy$ plane. In this case, the normal stress in the direction normal to the $xy$ plane may be determined from the stresses acting on the $xy$ plane $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ through elastic stress-strain relations that will be discussed later in this chapter. Whilst plane stress is a good assumption for simplifying many engineering problems in structural and mechanical engineering, plane strain is most relevant in geotechnical engineering. This is because many important geotechnical problems, such as embankments and tunnels, may be adequately analysed as a two-dimensional plane strain problem.

(a) Transformation of stresses and principal stresses

Now let us investigate the stress components at the point with respect to a new coordinate system $(x', y')$, which is obtained by rotating the original coordinate system $(x, y)$ anticlockwise by an angle of $\theta$ (see Figure 2.1). It can be easily shown that the stresses in these two coordinate systems are related by the following equations:

\[
\sigma'_{xx} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{yy} - \sigma_{xx}}{2} \cos 2\theta - \sigma_{xy} \sin 2\theta \quad (2.1)
\]

\[
\sigma'_{yy} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{yy} - \sigma_{xx}}{2} \cos 2\theta + \sigma_{xy} \sin 2\theta \quad (2.2)
\]

\[
\sigma'_{xy} = \sigma_{xy} \cos 2\theta + \frac{\sigma_{yy} - \sigma_{xx}}{2} \sin 2\theta \quad (2.3)
\]

The principal stresses are those acting on a principal plane where shear stress is zero. The principal planes can be determined by setting equation (2.3) to zero, which gives:

\[
\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \quad (2.4)
\]

Substituting the above solution into equations (2.1) and (2.2) leads to the expressions for the two principal stresses:

\[
\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \quad (2.5)
\]

\[
\sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \quad (2.6)
\]

where $\sigma_1$ and $\sigma_2$ are also known as the major and minor principal stresses respectively.
The transformation of stresses, analytically expressed by the above equations, can also be simply achieved by using a Mohr-circle construction. Assume that the positive plane normal to the x direction is denoted by A and the positive plane normal to the y direction by B. Whilst compressive normal stresses are regarded as positive, shear stresses acting clockwise are treated as positive.

![Figure 2.2: Transformation of stresses using a Mohr-circle construction](image)

Using the Mohr-circle construction shown in Figure 2.2, the stresses on the plane A and B are defined by the coordinates of points A and B in the Mohr-circle. The stresses for the corresponding planes $A'$ and $B'$ with respect to a new coordinate system $(x'oy')$ are equal to the coordinates of the points $A'$ and $B'$ in the Mohr-circle. It is noted that the points $A'$ and $B'$ are arrived by rotating the points A and B respectively by two times the angle between the coordinate systems $(xoy)$ and $(x'oy')$.

By definition, the principal stresses are the coordinates of the interaction points between the Mohr-circle and the normal stress axis.

(b) Equations of interior stress equilibrium

By accounting for stress variation with coordinates, equations of stress equilibrium can be established. It is instructive to first consider all the stresses in the x direction, as shown in Figure 2.3. The quantity $X$ is assumed to be the body force (i.e. force per unit volume). The equation of force equilibrium in the x direction leads to the following equation of stresses:

$$
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = X
$$

(2.7)