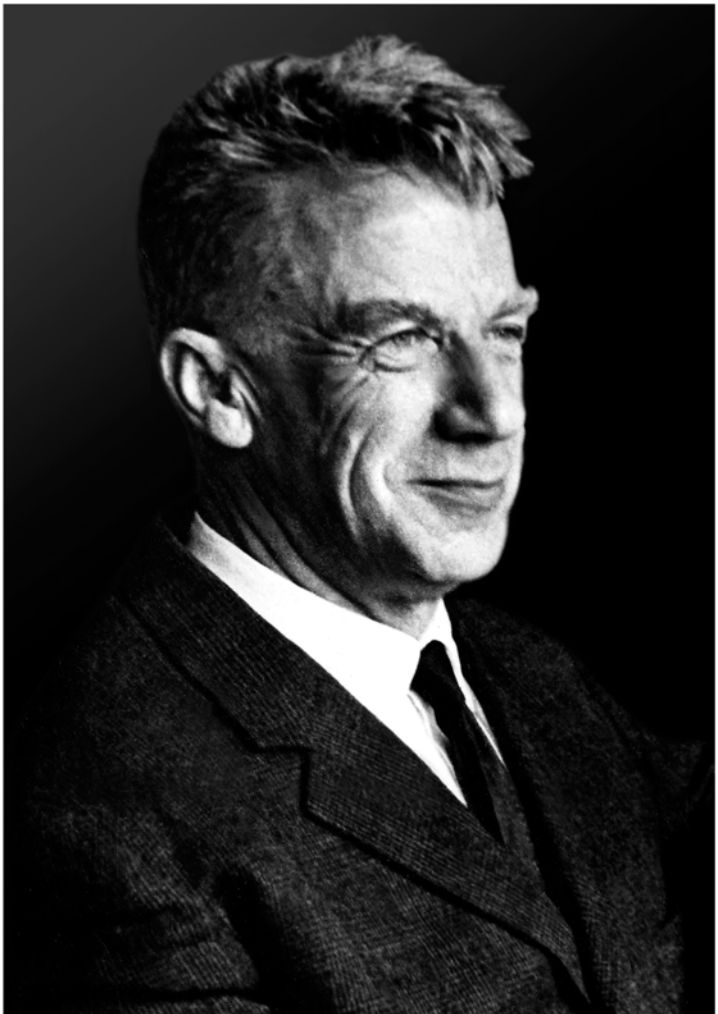
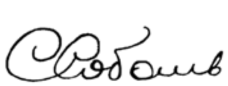
SELECTED WORKS OF S.L. SOBOLEV

Volume I: Mathematical Physics, Computational Mathematics, and Cubature Formulas





SELECTED WORKS OF S.L. SOBOLEV

Volume I: Mathematical Physics, Computational Mathematics, and Cubature Formulas

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Springer

Library of Congress Control Number: 2006924828

ISBN-10: 0-387-34148-X e-ISBN: 0-387-34149-8

ISBN-13: 978-0-387-34148-4

Printed on acid-free paper.

AMS Subject Classifications: 01A75, 35-XX, 65D32, 46N40

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Printed in the United States of America.

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Preface

The Russian edition of this book was dated for the 95th anniversary of the birth of Academician S. L. Sobolev (1908–1989), a great mathematician of the twentieth century. It includes S. L. Sobolev's fundamental works on equations of mathematical physics, computational mathematics, and cubature formulas.

S. L. Sobolev's works included in the volume reflect scientific ideas, approaches, and methods proposed by him. These works laid the foundations for intensive development of modern theory of partial differential equations and equations of mathematical physics, and were a gold mine for new directions of functional analysis and computational mathematics.

The book starts with the paper "Academician S. L. Sobolev is a founder of new directions of functional analysis" by Academician Yu. G. Reshetnyak. It was written on the basis of his lecture delivered at the scientific session devoted to S. L. Sobolev in the Institute of Mathematics (Novosibirsk, October, 2003).

The book consists of two parts. Part I includes selected articles on equations of mathematical physics and Part II presents works on computational mathematics and cubature formulas. All works are given in chronological order.

Part I consists of 11 fundamental works of S. L. Sobolev devoted to the study of classical problems of elasticity and plasticity theory, and a series of hydrodynamic problems that arose due to active participation of S. L. Sobolev in applied investigations carried out in the 1940s.

The first mathematical articles by S. L. Sobolev were written during his work in the Theoretical Department of the Seismological Institute of the USSR Academy of Sciences (Leningrad). Five articles from this cycle are included in this book (papers [1–5] of Part I). These works are devoted to solving a series of important applied problems in the theory of elasticity.

In the first paper included in the volume, S. L. Sobolev solves the classical problem posed in the famous article by H. Lamb (1904) on propagation of elastic vibrations in a half-plane and a half-space. At first, he considers H. Lamb's plane problem, then for this case studies reflection of longitudinal and transverse elastic plane waves from the plane. Using the theory of functions of complex variable, he proposes a method for finding plane waves falling at different angles on the boundary. In particular, he points out a method for finding the Rayleigh waves. Then, using H. Lamb's formulas and applying the method of superposition of plane waves, he gets integral formulas for longitudinal and transverse waves at any internal point of the medium. With these results he studies H. Lamb's space problem.

The next two papers by S. L. Sobolev and his teacher V. I. Smirnov are devoted to more general problems of H. Lamb type. In these articles the authors propose a new method for the study of problems of the theory of elasticity. Using the method, the authors get totally new results in the theory of elasticity and point out a series of problems which can be solved by the method. In the literature the method is known as the method of functionally invariant solutions. The main advantage of the method is that there is no need to use Fourier integrals as did H. Lamb. The method has visual geometric character and allows one to apply the theory of functions of a complex variable. The set of functionally invariant solutions contains important solutions of the wave equation (the Volterra solution, plane waves). This set is closed with respect to reflection and refraction. Using functionally invariant solutions, the authors solve H. Lamb's generalized problem on vibrations of an elastic half-space under the action of a force source inside the half-space. In these papers V. I. Smirnov and S. L. Sobolev obtain formulas for components of displacements at arbitrary point of the space. The authors give a physical interpretation of the obtained formulas. In particular, they conclude that, at infinity, elastic vibrations cause a wave of finite amplitude, and the wave moves with the velocity of the Rayleigh waves.

It should be noted that the first three works are practically unknown to readers because they were published in sources which are difficult to access.

In the paper [4] of Part I the problem on propagation of elastic vibrations in a half-plane and an elastic layer is considered. Unlike all preceding investigations, S. L. Sobolev studies the problem in the case of arbitrary initial conditions. For solving this problem he applies the Volterra method and the method of functionally invariant solutions. The main result of the author is integral formulas for components of displacements at arbitrary points of the medium at any point of time. In particular, the formulas clarify the reason for appearance of the Rayleigh space waves in the general case.

The Smirnov–Sobolev method found numerous applications in subsequent investigations. A review of results obtained by the method at the Seismological Institute of the USSR Academy of Sciences (Leningrad) is given in the paper [5] of Part I.

The paper [6] contains an exhaustive explanation of the Smirnov–Sobolev method of functionally invariant solutions for the wave equation. S. L. Sobolev proves that all functionally invariant solutions to the two-dimensional wave equation can be obtained by this method.

The paper [7] of Part I is devoted to the theory of diffraction of waves on Riemann surfaces. Solving the problem, the author comes to the necessity of using functions which are solutions to the wave equation

$$\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

in a generalized sense. S. L. Sobolev introduces a notion of weak solution of the wave equation. He says that a function u is called a weak solution of the wave equation in a domain D, if the function is the limit of a sequence of classical solutions of the equation in L_1. S. L. Sobolev studies properties of weak solutions and elaborates the method of average functions. Using properties of weak solutions, the author proposes a method for solving the problem of diffraction of waves on Riemann surfaces.

In his subsequent works S. L. Sobolev developed the notion of weak solution, introduced a notion of generalized derivative, defined functional spaces W_p^l called Sobolev spaces, and proved embedding theorems. These works laid the foundations of the modern theory of generalized functions. A series of works devoted to the subject will be included in the next volume of selected works of S. L. Sobolev.

In the paper [8] of Part I, S. L. Sobolev solves the important problem of propagation of a plastic state in an infinite plane, with a circular hole, exposed to the action of symmetrical forces causing displacements on the boundary. S. L. Sobolev indicates the method of computation of all quantities characterizing the motion, i.e., the displacement components at any point of time in the plastic and elastic zones, the stress tensor components in both zones, and the flow lines in the plastic zone.

The last three papers [9–11] of Part I are devoted to the problem of small oscillations of a rotating fluid. The problem is classical. The study of this problem began with the famous article “Sur l'equilibre d'une masse fluide animée d'un mouvement de rotation” by H. Poincaré (1885). Papers [9, 10] contain results of investigations carried out by S. L. Sobolev in the 1940s.

In the paper [9] S. L. Sobolev considers a system of partial differential equations of the form

$$\begin{aligned} &\frac{\partial \overrightarrow{v}}{\partial t} - [\overrightarrow{v} \times \mathbf{k}] + \nabla p = \overrightarrow{F}, \\ &\operatorname{div} \overrightarrow{v} = g. \end{aligned} \tag{1}$$

This system arises when studying small oscillations of a rotating ideal fluid. The main aim of the author is to research the Cauchy problem, the first and second boundary value problems for system (1) in a bounded domain. Using methods of functional analysis developed by him, S. L. Sobolev proves well-posedness of the problems, and proposes a method for construction of solutions. He establishes also a close connection between system (1) and the non-classical equation

$$\varDelta \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} = f. \tag{2}$$

By the method of potentials, S. L. Sobolev obtains explicit formulas of solutions to the Cauchy problem for system (1) and for equation (2).

System (1) can be written as

$$A_0 \frac{\partial}{\partial t}\begin{pmatrix} \overrightarrow{v} \\ p \end{pmatrix} + A_x \frac{\partial}{\partial x}\begin{pmatrix} \overrightarrow{v} \\ p \end{pmatrix} + A_y \frac{\partial}{\partial y}\begin{pmatrix} \overrightarrow{v} \\ p \end{pmatrix} + A_z \frac{\partial}{\partial z}\begin{pmatrix} \overrightarrow{v} \\ p \end{pmatrix} + B\begin{pmatrix} \overrightarrow{v} \\ p \end{pmatrix} = \begin{pmatrix} \overrightarrow{F} \\ g \end{pmatrix},$$

where the matrix A_0 is singular, i.e., system (1) is not a Cauchy–Kovalevskaya system. Probably, equations and systems not solvable with respect to the highest-order derivative were first studied by H. Poincaré (1885). Subsequently, they were considered in a number of articles by mathematicians and mechanicians. This was connected initially with research into certain hydrodynamics problems. Particularly, the most intense interest in equations and systems not solvable with respect to the highest-order derivative arose in connection with the investigation of the Navier–Stokes system by C. W. Oseen (1927), F. K. G. Odqvist (1930), J. Leray and J. Schauder (1934), E. Hopf (1950) and the study of the problem on small oscillations of a rotating fluid by S. L. Sobolev. The paper [9] was one of the first deep investigations of equations and systems not solvable with respect to the highest-order derivative. This paper originated intense research into such equations and systems. At present, system (1) is called the Sobolev system, equation (2) is called the Sobolev equation in the literature.

The paper [10] was written by S. L. Sobolev in 1943, but it was published only in 1960. In the work he considers the problem of stability of motion of a heavy symmetric top with a cavity filled with a fluid. It is assumed that the top rotates around its axis, and its foot is immovable. The author reduces the research into stability of motion to solving the differential equation

$$\frac{dR}{dt} = iBR + R_0,$$

where B is a linear operator self-conjugate with respect to a Hermitian form Q. This form depends on parameters characterizing mechanical properties of the shell and the fluid. It is interesting that the form Q can be positive definite or indefinite depending on values of the parameters. Since solutions to the equation are written by means of the resolvent of the operator B, the author studies the solutions in a space with indefinite metric.

It should be noted that the theory of differential equations in spaces with indefinite metric began to develop in the 1940s. Therefore the paper [10] is one of the first works in this direction.

The main results of the paper [10] follow from established properties of the resolvent of the operator B in a space with inner product defined by the form Q. In particular, if the form Q is positive definite, then the motion is stable; if the form Q is indefinite, then the motion can be unstable. S. L. Sobolev studies in detail the cases when the cavity filled with the fluid has the form of an ellipsoid or a cylinder. The author points out angular velocities under which the motion is stable, and he describes the cases when the resonance phenomenon is to be observed.

The paper [11] is a survey of S. L. Sobolev's lecture delivered at the International Symposium on Applied Analysis and Mathematical Physics (Cagliari-Sassari, Italy, 1964). In the paper he discusses mathematical problems connected with the research into system (1) and equation (2) arising when studying small oscillations of a rotating ideal fluid. The study of properties of solutions of system (1), equation (2) and more general equations began with appearance of S. L. Sobolev's famous article (1954; see [9] of Part I). In his lecture he gives a survey of results obtained in this direction for the past 10 years. In particular, he notes a series of unexpected results on spectral properties of operators generated by the problems. From the results it follows that, as a rule, solutions to the boundary value problems have non-compact trajectories. S. L. Sobolev points out that asymptotic properties of solutions depend essentially on domain geometry. On the other hand, in the case of a boundary time interval, S. L. Sobolev proves that solutions of many boundary value problems depend continuously on deformations of the domain boundary. He notes also some connections of many boundary value problems with various problems of mathematical analysis and other problems of partial differential equations. He emphasizes that the class of the problems under discussion is at an initial stage of study.

Part II of the book includes 29 articles on computational mathematics and cubature formulas. It starts with an early paper which is devoted to the Schwartz method for approximate solution of boundary value problems for partial differential equations of elasticity theory. The next five works were written by S. L. Sobolev as part of his active participation in applied investigations carried out in the Soviet Union in the 1940-50s. These articles are devoted to computational methods in difference and integral equations, and problems of approximation of linear operators. In these works S. L. Sobolev actively advocated the use of functional analysis in computational mathematics, and pointed out close interconnections between computational mathematics, differential equations and functional analysis. He emphasized that the use of computers for solving complex applied problems will be more effective under active collaboration of mathematicians and engineers.

A noticeable place in the scientific legacy of S. L. Sobolev is occupied by his contributions to the theory of approximate multidimensional integration which were accomplished during his stay of 25 years in Novosibirsk. His first article in this direction was published in 1961 and the last in 1985 and there are two dozen of these papers in this volume. In these papers S. L. Sobolev mainly pursue a functional-analytical approach. This implies that, first, the integrands are combined in a Banach space and, second, the difference between the integral and the approximative combination of the values of the integrand is treated as the result of applying some linear functional. This functional, called the error of a cubature formula, is usually continuous. Knowledge of the value of its norm allows us to derive guaranteed estimates for the accuracy of the cubature formula under study on the elements of the chosen space. In addition to describing the construction of the formulas under consideration, i.e., indicating their nodes and weights or algorithms for their determination, the functional-analytical approach implies the study of the norms of the respective errors in a chosen Banach space. In particular, two-sided estimates for these norms are derived. In papers [7, 8, 9] of Part II S. L. Sobolev addresses the main problems of the theory of cubature formulas and the theory of interpolation.

In the theory of cubature formulas, a term coined by S. L. Sobolev, four principal directions are specified. All are exposed in the present edition.

The first in chronological order of the directions consists in studying the cubature formulas in three-dimensional space which possess high polynomial degree and are invariant under the action of the rotation group of some regular polyhedron. The requirement that a cubature formula with fixed nodes be exact for polynomials up to a certain degree reduces the problem of constructing the weights of the formula to solution of a system of linear equations. The higher the desired order is and the larger are the number of nodes, the greater becomes the size of this system. However, in the case when the integration domain possesses some symmetry and we use an invariant cubature formula for approximate integration, it is possible to diminish substantially the size of the system to be solved. Papers [10, 11] of Part II address the question of how to achieve this.

The second direction in the theory, which seems to be most advanced, consists in studying asymptotically optimal cubature formulas on the spaces of functions of finite smoothness (papers [12, 14, 16, 18, 23] of Part II). In this respect S. L. Sobolev himself considered the Hilbert $L_2^{(m)}$ spaces. The construction of a regular boundary layer which he proposed makes it possible to find the weights of a cubature formula with arbitrarily many nodes by solving only a few standard systems of linear equations of size depending only on the order m (papers [13, 18, 21] of Part II). The central place in this direction is occupied by derivation of an asymptotic expansion of the $L_2^{(m)*}$ norm of an error with regular boundary layer. The expansion contains two summands. The first is written explicitly via the so-called generalized Bernoulli numbers, whereas the second is negligible as compared with the first, provided that the small mesh-size h of the lattice of integration is sufficiently small. The expansion implies that the norm of an error with regular boundary layer decreases like h^m as $h \to 0$. It is a rather deep analytical fact enabling us to give not an algebraic but rather a functional-analytical definition of the order of a cubature formula on some function class (paper [29] of Part II).

The expansion of the $L_2^{(m)*}$ norm of an error with regular boundary layer gives solid grounds for choosing a numerical integration formula with nodes comprising a lattice. Indeed, given N nodes, we may pose the problem of finding a cubature formula whose error has $L_2^{(m)*}$ norm minimal, with the minimum taken over not only the weights but also the nodes of the formula.

However, the ratio of the $L_2^{(m)*}$ norm of the error of such an optimal formula to the $L_2^{(m)*}$ norm of the error with regular boundary layer and the same number N of nodes is bounded from below by a positive quantity independent of N. This is immediate from the Bakhvalov Theorem (paper [14] of Part II). Increasing N, we could however hardly expect large gain from using formulas with arbitrary disposition of nodes instead of those with nodes comprising a parallelepipedal lattice. Moreover, to optimize a formula over nodes is a difficult problem involving solution of simultaneous nonlinear equations of high order. This is in sharp contrast to the formulas with regular boundary layer whose nodes are explicit and need no calculation at all.

Note that the theory of formulas with regular boundary layer actually presents the function summation problem pertinent to the calculus of finite differences. From this point of view, every cubature formula with regular boundary layer is a multidimensional analog of the classical quadrature formula of Gregory. Constructing such a cubature formula, we thus take account of the behavior of an integrand near to the boundary of the integration domain by especially selecting the weights of the formula at the nodes belonging to some boundary layer. All remaining weights coincide.

Remarkable is the method proposed by S. L. Sobolev for finding the norm of an error $l(x)$ and his use of the concept of extremal function $u(x)$ (papers [7, 12, 14, 15] of Part II). Such function is considered as a weak solution to the many-dimensional polyharmonic equation with a special right side

$$\Delta^m u(x) = (-1)^m l(x).$$

A solution to this equation on the real axis is a piecewise-polynomial function of the class $W_2^{(m)}$, i.e., a spline. In many dimensions, this approach enabled S. L. Sobolev to apply the methods he invented in the theory of partial differential equations to study of the classical problems of analysis.

The third direction of the theory comprises the S. L. Sobolev contribution to cubature formulas on the classes of infinitely differentiable functions (papers [17, 22, 29] of Part II). As such he considered the spaces of periodic functions of many variables with prescribed behavior of the integral norms in the $L_2^{(m)}$ spaces as m tends to infinity. The classification he proposed embraces the conventional spaces of entire functions of given type and order, spaces of analytic functions and the Gevrey classes containing quasianalytic functions. Considering the action on this space of the error of a lattice formula with equal weights, S. L. Sobolev obtained an asymptotic expansion of the logarithm of the norm of the error. In exact analogy with the case of the spaces of finite smoothness, the respective formula comprises two summands. One of them is explicitly expressed through the parameters of the initial class, whereas the other is negligible as compared with the first at a small mesh-size h. This research demonstrated in particular that a noteworthy effect accompanies the transition from functions of finite smoothness to infinitely differentiable functions. Namely, the norm of the error of a cubature formula, decreasing not faster than some power of the lattice mesh-size in the first case, decreases exponentially in the second case. S. L. Sobolev suggested that in the second case the order of a cubature formula be assumed infinite. More exactly, a cubature formula possesses infinite order in a Banach space provided that the norm of the corresponding error in the dual space vanishes faster than any degree of the mesh-size of the integration lattice. S. L. Sobolev exhibited one example of the sort in the case of many dimensions.

Finally, the fourth direction of the theory comprises the S. L. Sobolev research in $L_2^{(m)}$-optimal lattice cubature formulas (papers [20] and [24] of Part II). A central place is occupied here by description of some analytic algorithm for determining weights of such formulas. To this end, S. L. Sobolev defined and studied a special finite-difference operator whose action on a function of a discrete argument may be written as convolution with a special kernel in analogy with the action of the polyharmonic operator Δ^m on a continuously differentiable function (paper [19] of Part II).

The problem of calculating the convolution kernel for an arbitrary m turns out rather involved. It was partly solved in the one-dimensional case: here a formula is available expressing the desired values through the roots of the Euler–Frobenius polynomials of degree $2m$. The weights of optimal formulas are conveniently treated as the values at the appropriate points of some compactly-supported function of a many-dimensional discrete argument. This function happens to satisfy a linear finite-difference equation with a special right side. Applying to this right side a discrete convolution analog of the polyharmonic operator, S. L. Sobolev obtained an analytical formula for the sought weights (paper [24] of Part II). To use it in the one-dimensional case, he revealed many properties of the roots of the Euler–Frobenius polynomials (papers [25–28] of Part II). In particular, he obtained asymptotic formulas for the roots of these polynomials. The results by S. L. Sobolev on the weights of optimal cubature formulas generalized some results by A. Sard, I. Meyers, I. Schoenberg and S. Silliman derived by the method of splines.

The method of S. L. Sobolev for studying cubature formulas is deeply rooted in such fields of theoretical mathematics as mathematical analysis, the theory of differential equations and functional analysis. At the same time, the specific subject of research, a cubature formula for approximate integration, is traditionally ascribed to numerical analysis which the modern computational mathematics stems from. As a result, a theory has emerged which has undeniable import for applications. This order of events seems by far not random but rather an inevitable phenomenon of modern mathematics.

We would like to say a few words about selected works of S. L. Sobolev. In 2001 the Scientific Council of the Sobolev Institute of Mathematics of the Siberian Division of the Russian Academy of Sciences (Novosibirsk) made a decision to publish selected works of Academician S. L. Sobolev in many volumes. An editorial board was formed, consisting of Academician Yu. G. Reshetnyak, Prof. G. V. Demidenko, Prof. S. S. Kutateladze, Prof. V. L. Vaskevich, and Prof. S. K. Vodop'yanov. As mentioned above, the

Russian edition of the first volume came out in 2003. Prof. G. V. Demidenko and Prof. V. L. Vaskevich are the editors of this volume. The second volume will be published in Russian in 2006. It will include fundamental works of S. L. Sobolev on functional analysis and differential equations. The editors of the second volume are Prof. G. V. Demidenko and Prof. S. K. Vodop'yanov.

Selecting S. L. Sobolev's works for the first volume, the editors used the chronology of his works. It was composed by V. M. Pestunova and published in the Sobolev Institute of Mathematics in 1998. A big help in search of early works of S. L. Sobolev was given by the employees of the library of the Sobolev Institute of Mathematics: L. G. Gulyaeva, L. A. Mikuta, and V. G. Ponomarchuk.

Many people actively participated in the preparation of the manuscript: members of the Sobolev Institute of Mathematics L. V. Alekseeva and Dr. I. I. Matveeva; members of the Lavrentiev Institute of Hydrodynamics Prof. N. I. Makarenko and Dr. A. E. Mamontov; students of Novosibirsk State University L. N. Buldygerova, V. G. Demidenko, Yu. E. Khropova, T. V. Kotova, A. A. Kovalenko, M. A. Kuklina, A. V. Mudrov, A. M. Popov, and E. A. Samuilova.

The editors are much indebted to each of the contributors mentioned above.

The Russian edition was supported by the Federal Special Program "Integratsiya" (grant number C0015), by the Russian Foundation for Basic Research (grant number 03-01-14016), and by the Siberian Division of the Russian Academy of Sciences.

The editors are very grateful to Prof. H. G. W. Begehr for useful advice in regard to the English edition of this book.

The editors would like to take this opportunity to thank J. Martindale and R. Saley. The English edition became possible due to fruitful cooperation with them.

The editors would like to express their deep gratitude to Dr. V. V. Fokin for his huge work in the translation of this book into English.

Novosibirsk,
December 2005

Gennadii Demidenko
Vladimir Vaskevich

Academician S. L. Sobolev is a Founder of New Directions of Functional Analysis

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S. L. Sobolev – one of the most prominent mathematicians of the 20th century – was born on October 6, 1908 in Petersburg. His father, Lev Alexandrovich Sobolev, was a public attorney. Lev Alexandrovich studied at Petersburg University, but was expelled because of his participation in the revolutionary movement and sent to the army as a soldier. Afterwards, he passed, as external student, the state examinations at the Law Department of Kharkov University. Sergei Sobolev's paternal grandfather was a hereditary Siberian kazak.

It was in his early youth when Sergei Sobolev lost his father; he was brought up by his mother, Natalia Georgievna, a most educated woman, teacher of literature and history. Natalia Georgievna also had a second specialty: she graduated from a Medical Institute and worked as associate professor at the First Leningrad Medical Institute. She inculcated in Sergei Sobolev such personality features as fidelity to principle, honesty and purposefulness, which characterized him as scientist and person.

Sergei Sobolev mastered the high school program by himself, being particularly fond of mathematics. In the years of the Civil War, he lived in Kharkov with his mother. There he studied for one semester at preparatory courses to a labor technical night school. By 15 years of age, he knew the complete course of mathematics, physics, chemistry, and other sciences according to the high school curriculum, had read many books of classic Russian and foreign literature as well as books on philosophy, medicine, biology, etc. Having moved from Kharkov to Petrograd in 1923, Sergei Sobolev was enrolled in the final school year of School 190 and finished it with excellence in 1924. After finishing school, he could not enter a university because of his young age (he was under 16), so he began to study at the First State Art Studio, in a piano class.

In 1925, Sergei Sobolev entered the Physics and Mathematics Department of Leningrad State University, proceeding with his studies at the Art Studio. In Leningrad State University, he attended lectures by Professors N. M. Gyunter, V. I. Smirnov, G. M. Fikhtengol'ts and others. He wrote his diploma thesis on analytical solutions of a system of differential equations with two independent variables under the academic supervision of Prof. N. M. Gyunter.

In those years, Leningrad University was a major scientific mathematical center, which retained the remarkable traditions of the Petersburg mathematical school. It was famous for its great discoveries in mathematics and connected with the names of P. L. Chebyshev, A. M. Lyapunov, and A. A. Markov.

After graduating in 1929 from Leningrad University, Sergei Sobolev started to work in the Theoretical Department of Leningrad Seismological Institute under the direction of V. I. Smirnov. In that period, in close cooperation with V. I. Smirnov, they solved a number of fundamental mathematical problems in the wave transmission theory.

Since 1932, S. L. Sobolev worked in V. A. Steklov Mathematical Institute in Leningrad, and since 1934 – in Moscow. On February 1, 1933, when he was not yet 25 years old, he was elected a corresponding member of the USSR Academy of Sciences. He became a full member of the USSR Academy of Sciences on January 29, 1939. In 1941, for his works in mathematical theory of elasticity, S. L. Sobolev was awarded with the State Prize of the 1st Degree. During the Great Patriotic War, V. A. Steklov Mathematical Institute was evacuated in Kazan, and for a short period, from 1941 to 1943, S. L. Sobolev was the director of this institute. Since 1943, he worked in the institute headed by I. V. Kurchatov, which was then called the Laboratory of Measuring Instruments of the USSR Academy of Sciences (now I. V. Kurchatov Institute of Atomic Energy). He kept on working as research fellow at this institute before leaving for Novosibirsk.

S. L. Sobolev is known worldwide as a prominent mathematician and author of outstanding research works on the theory of differential equations, computational mathematics, and functional analysis. He gave rise to the wave transmission theory. He developed the theory of generalized functions as functionals on a set of smooth compactly-supported functions. On the basis of this theory, he defined the concept of a weak solution of a partial differential equation. S. L. Sobolev introduced new function spaces and proved embedding theorems for them (Sobolev spaces, Sobolev embedding theorems). He laid the foundations of the spectral theory for operators in spaces with indefinite metric in connection with studying solutions of hydrodynamic systems of rotating fluid. He made a significant contribution to the development of computational mathematics: he introduced the important concept of computational algorithm closure and constructed the theory of cubature formulas. He organized at Moscow University the country's first Chair of Computational Mathematics.

S. L. Sobolev was a forward-thinking man and a socially active person. For example, he vigorously supported cybernetics and mathematical economics when these schools of thought were victimized; he advocated protection of the unique ecosystem of the Baikal Lake. It is hard to enumerate all the important achievements that he attained.

S. L. Sobolev was involved in applied scientific projects that were highly important state matters – he developed mathematical support for the USSR nuclear project while working as deputy director for I. V. Kurchatov in the Measuring Instrument Laboratory.

S. L. Sobolev had great authority in world-wide science. He was elected an international member of the French Academy of Sciences, Accademia Nazionale dei Lincei in Roma, Berlin Academy of Sciences, Edinburgh Royal Society, honorary doctor of Charles University in Prague, honorary doctor of Humboldt University in Berlin, honorary doctor of Higher School of Architecture and Construction in Weimar, honorary member of Moscow and American Mathematical Societies.

The services S. L. Sobolev rendered to science and our country were highly valued and he was awarded with numerous orders and prizes even before his arrival in Novosibirsk – in the Siberian Division of the USSR Academy of Sciences. For the works done at the I. V. Kurchatov Institute of Nuclear Power, S. L. Sobolev was conferred the honorary title of Hero of Socialist Labor, decorated with several Lenin Orders and many other decorations of the Soviet government.

In 1957, Academician S. L. Sobolev together with Academicians M. A. Lavrentiev and S. A. Khristianovich became one of the three founders of the Siberian Division of the USSR Academy of Sciences.

S. L. Sobolev was the founder and director of the Institute of Mathematics of the USSR Academy of Sciences. He held the position of director from 1957 to 1983 when, after celebration of his 75th birthday, he left to go to Moscow to work at the Steklov Mathematical Institute. In 1988, he was put forward for a M. V. Lomonosov Gold Medal of the USSR Academy of Sciences.

In the last years of his life, S. L. Sobolev was seriously ill, and he passed away on January 3, 1989. The M. V. Lomonosov Gold Medal of the USSR Academy of Sciences was awarded to him posthumously in 1989.

One of the main achievements of S. L. Sobolev in mathematics was construction of the theory of generalized functions, one of the most important directions of modern functional analysis, and creation of the theory of functions with generalized derivatives. In the literature these spaces are called Sobolev spaces. These two directions in the scientific research of S. L. Sobolev appear as one whole.

As a separate direction of mathematics, functional analysis had been formed at the end of the 19th, beginning of the 20th centuries. The creation of set theory and based on it general (set theoretic) topology and the theory of functions of a real variable created favorable circumstances for functional analysis. The appearance of functional analysis was an answer to certain questions of theoretical mathematics, possibly even implicitly stated, and its applications. In applications it is often important to know the conditions not only in the particular example, but rather for all problems of a certain class.

The need for development of research methods, not for particular functions or equations, but for entire classes of functions and equations, had led to the creation of functional analysis. The role of functional analysis in modern mathematics is by no means complete with this description.

The applications of functional analysis to problems of the theory of partial differential equations were already known before works of S. L. Sobolev. In this connection, we can indicate, for example, the famous D. Hilbert's works devoted to the validation of the Dirichlet principle for the Laplace equation. By virtue of S. L. Sobolev's investigations, functional analysis become a universal method for solving problems of mathematical physics.

In the 1920–30s, many scientists working in the theory of partial differential equations concentrated their efforts in order to understand what is a weak solution of a differential equation, and, in particular, how to extend the notion of the derivative of a function, so it would satisfy all needs of the theory of partial differential equations.

The most effective and, I would say, the most spectacular way of solving this problem was indicated by S. L. Sobolev. He noticed that any locally summable function of n variables generates a certain functional on the space of smooth compactly-supported functions. If one identifies the function with this functional, then it becomes possible to extend on locally integrable functions various operations performed on smooth functions by means of an adjoint operator.

The basics of the theory of generalized functions were presented briefly by S. L. Sobolev in his note in the journal "Doklady Akademii Nauk SSSR" (1935). The complete presentation was given in the article of S. L. Sobolev "Méthode nouvelle à résoudre le problème de Cauchy pour les équations linéaires hyperboliques normales" (A new method of solving the Cauchy problem for linear normal hyperbolic equations. Mat. Sb., **1**, 39–72 (1936)). The Russian translation of this article is also given in the last edition of the book by S. L. Sobolev "Some Applications of Functional Analysis in Mathematical Physics", edited by O. A. Oleinik and published in 1988, with comments by V. I. Burenkov and V. P. Palamodov.

The basic ideas and constructions of the theory of generalized functions contained in S. L. Sobolev's articles appear in the modern theory practically without any changes. Let us point out the most important ideas.

1. A generalized function is defined as a functional on the space of smooth compactly-supported functions.

2. Linear differential operators in the space of generalized functions are introduced in the form of adjoints to the corresponding linear differential operators on the space of smooth compactly-supported functions.

3. The generalized functions are classified in the order of their singularity (in terms of S. L. Sobolev, by a class).

4. The regularization of generalized functions by means of convolution and approximation of an arbitrary generalized function by infinitely differentiable functions.

5. The flexible manipulation of spaces of test and generalized functions, defined by various conditions imposed on supports of test and generalized functions.

6. Reducing the Cauchy problem to a problem with a nontrivial right-hand side without initial conditions by transforming the initial conditions into sources of delta function type.

Let Ω be a domain, i.e., a connected open set in the space R^n. The function φ defined in Ω is called *compactly-supported*, if there exists a compact set $S_\varphi \subset \Omega$ such that $\varphi(x) = 0$ for $x \notin S_\varphi$. There is the smallest set among compact sets satisfying this condition. It is called the *support of the function* φ. Further we assume that S_φ is the support of the function φ. We say that the function $\varphi : \Omega \to R$ belongs to the class $C_0^r(\Omega)$, if it is compactly-supported and has all partial derivatives of order r in Ω, and all these derivatives are continuous. The symbol $C_0^\infty(\Omega)$ denotes the set of all functions φ belonging to the class $C_0^r(\Omega)$ for any $r \geq 1$.

The class $C_0^r(\Omega)$ is a vector space. We will consider linear functionals on the spaces $C_0^r(\Omega)$. The value of a functional f on a function $\varphi \in C_0^r(\Omega)$ is denoted by the symbol $\langle f, \varphi \rangle$. In the space $C_0^r(\Omega)$, a certain topology is introduced (I do not describe it in detail, referring instead to the book by S. L. Sobolev "Some Applications of Functional Analysis in Mathematical Physics"). A generalized function is a functional continuous in this topology.

In the work of S. L. Sobolev mentioned above (Mat. Sb., **1**, 39–72 (1936)) the generalized functions are simply called functionals. The term "generalized function" appeared later. French mathematician Laurent Schwartz used the term "distribution" to denote this object.

Let us present certain examples. They are significant for the theory of generalized functions.

1. Suppose that $f : \Omega \to R$ is an arbitrary measurable function in $L_{1,loc}(\Omega)$. The function f for every r defines on the space $C_0^r(\Omega)$ the linear functional $\widetilde{f}$ by the formula

$$\langle \widetilde{f}, \varphi \rangle = \int\limits_\Omega f(x)\varphi(x)\,dx.$$

The functional $\widetilde{f}$ is continuous in $C_0^r(\Omega)$ in the sense of the definition given above, and, hence, it is a certain generalized function.

The functional $\widetilde{f}$ defines the function f uniquely up to values on the set of measure zero. (This statement is known from the calculus of variations under the name of the Du Bois–Reymond lemma.) After S. L. Sobolev, in what follows, we identify the function $f \in L_{1,loc}(\Omega)$ with the functional $\widetilde{f} \in \mathcal{D}(\Omega)$. Therefore, I simply write f instead of $\widetilde{f}$. Thus, we obtain an embedding of $L_{1,loc}(\Omega)$ to the space $\mathcal{D}^r(\Omega)$ of linear functionals over the vector space $C_0^r(\Omega)$ for each integer $r > 0$. Thus, any function from the class $L_{1,loc}(\Omega)$ is a generalized function.

Similarly to this example, the notation $f(x)$ is used in the literature for any generalized function. According to this, instead of $\langle f, \varphi \rangle$ one uses the expression

$$\int_{\Omega} f(x)\varphi(x)\, dx.$$

2. Let $\Omega = R^n$ and let a be an arbitrary point in R^n. By the symbol $\delta(x-a)$ we denote the generalized function such that for any function $\varphi \in C_0^r(R^n)$ the following equality holds:

$$\int_{\Omega} \delta(x-a)\varphi(x)\, dx = \varphi(a).$$

We say that $\delta(x-a)$ is a δ-function concentrated at the point a of the space R^n. The notion of δ-function was introduced by Dirac and used in theoretical physics before the work of S. L. Sobolev.

Dirac defined $\delta(x-a)$ as the usual function such that $\delta(x-a) = 0$ for $x \neq a$, $\delta(0) = \infty$ and

$$\int_{R^n} \delta(x-a)\, dx = 1.$$

From a mathematical standpoint, the definition of Dirac is nonsense, even though its physical content is absolutely clear. For example, the Dirac δ-function is the unit mass concentrated in an arbitrarily small domain.

3. Let Ω be a domain in R^n. The symbol $\mathcal{B}_0(\Omega)$ denotes the union of all Borel sets $A \subset \Omega$, whose closures are compact and also contained in Ω. Let $\mu : \mathcal{B}_0(\Omega) \to R$ be a countably additive set function defined on the union of the sets $\mathcal{B}_0(\Omega)$. Then for any function $\varphi \in C_0^r(\Omega)$, $r \geq 1$, the following integral is defined:

$$\langle d\mu, \varphi \rangle = \int_{\Omega} \varphi(x) d\mu(x).$$

The set function μ is defined uniquely by the functional $d\mu$. Obviously, the notion of the δ-function is a particular case of the given example.

The generalized function $f(x)$ is called nonnegative in the domain Ω if for any nonnegative function $\varphi \in C_0^r(\Omega)$ the following inequality holds:

$$\int_{\Omega} f(x)\varphi(x)\, dx \geq 0.$$

The following statement can be easily proved: if the generalized function $f(x)$ is nonnegative, then $f = d\mu$, where μ is a nonnegative countably additive set function defined in Ω.

Let us show how the operations on usual functions are extended onto generalized functions. We use the example of differentiation for this.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be an n-dimensional multiindex, i.e., the vector in R^n, whose components are nonnegative integers. We set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and denote by the symbol D^α the operator of differentiation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

First we consider the case $n = 1$. Let Ω be an interval $(a, b) \subset R$, and let $f(x)$ be a function defined in Ω from the class C^r, i.e., it has a continuous derivative of order r at every point of this interval. Applying the rule of integration by parts, we obtain that for any function $\varphi \in C_0^r(\Omega)$ the inequality holds,

$$\int_a^b f^{(r)}(x)\varphi(x)\,dx = (-1)^r \int_a^b f(x)\varphi^{(r)}(x)\,dx.$$

Hence, by applying the Fubini theorem, we conclude that if Ω is a domain in R^n, and the function f belongs to the class $C^m(\Omega)$, $m = |\alpha|$, then for any function φ from the class $C_0^r(\Omega)$, $r \geq m$, the following equality holds:

$$\int_\Omega D^\alpha f(x)\,\varphi(x)\,dx = (-1)^m \int_\Omega f(x) D^\alpha \varphi(x)\,dx.$$

This equality presents a scheme to show how one can define the notion of a generalized derivative for arbitrary generalized functions.

If $f(x)$ is a generalized function in a domain Ω of the space R^n, then its derivative $D^\alpha f(x)$ is a linear functional, whose action on smooth functions is defined by the rule

$$\int_\Omega D^\alpha f(x)\,\varphi(x)\,dx = (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \varphi(x)\,dx.$$

By this definition, any generalized function has any derivative of any order.

Let us consider the simplest wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Any solution of this equation can be represented in the form

$$u(x, t) = f(x - at) + g(x + at). \tag{2}$$

To substitute the function $u(x, t)$ defined by (2) in equation (1) the functions f and g must have second order derivatives.

Each term in (2) has certain physical meaning. By (2), the function $u(x, t)$ is represented as a sum of two waves, one wave moves in one direction, and the other one moves in the opposite direction. The requirement of second order differentiability of the functions f and g is not much justified physically. The question of how to understand the solution of the wave equation was the subject of discussions among mathematicians already in the 18th century. In particular, they suggested to take any function of form (2) as a solution of the equation for any functions f and g.

For any locally integrable functions f and g, the function $u(x,t)$ defined by (2) always satisfy the wave equation under the condition that the derivatives in this equation are understood in the sense of the theory of generalized functions.

The definition given by S. L. Sobolev allows one to correct also "transgressions" of physicists, related to the δ-function, namely, to give a rigorous definition of the derivative of the δ-function. According to the definition of S. L. Sobolev, the derivative $D^\alpha\delta(x-a)$ is the generalized function such that the equality

$$\langle D^\alpha\delta, \varphi\rangle = (-1)^{|\alpha|}D^\alpha\varphi(a)$$

holds for any compactly-supported function from the corresponding class of smoothness.

It is necessary also to note the ingenious construction invented by S. L. Sobolev in order to smooth functions and generalized functions. This method allows one to approximate an arbitrary generalized function by functions from the class C^∞.

To illustrate this, let us indicate certain simple applications of the notions introduced by S. L. Sobolev.

The criterion of the monotonicity of a function, defined on a certain interval of the real line, is usually formulated in courses of differential calculus in the following way. If the function $f : (a,b) \to R$ is differentiable at each point of the interval (a,b), then it is increasing if and only if its derivative is always nonnegative. The theory of generalized functions allows one to remove the requirement of differentiability, more precisely, to replace it by a significantly weaker requirement of local integrability.

A locally integrable function $f : (a,b) \to R$ is increasing if and only if its derivative, as a generalized function, is nonnegative.

Similarly, a function that is locally integrable on the interval (a,b) is convex if and only if its second derivative is a nonnegative generalized function.

Let us also indicate that the condition: the function $f : (a,b) \to R$ is absolutely continuous, is equivalent to the condition: the function f is locally integrable and its derivative, as a generalized function, is a locally integrable function.

S. L. Sobolev also constructed the theory of classes of functions with generalized derivatives, the so-called spaces $W_p^l(\Omega)$. In the literature these spaces are called Sobolev spaces. For applications of functional analysis to mathematical physics, besides the general principles, it is necessary to have large sets of Banach spaces that can be used in problems of mathematical physics. The spaces $W_p^l(\Omega)$ provide such sets.

Let Ω be a domain in R^n, let $l \geq 1$ and $p \geq 1$ be real numbers such that l is the integer, and let f be a generalized function defined in Ω. We say that f belongs to the class $W_p^l(\Omega)$, if all its derivatives D^α, $|\alpha| \leq l$, belong to the class $L_p(\Omega)$. Naturally, these derivatives are understood in the sense of the definition given above.

S. L. Sobolev built a theory of the classes $W_p^l(\Omega)$. These functional classes have become the object of careful attention of many researches. At the same time, the techniques of studying such functions and methods proposed by S. L. Sobolev were universally recognized; they continue to be applied in many various studies.

S. L. Sobolev had constructed integral representations of the functions from the classes $W_p^l(\Omega)$ and studied different norms of the classes $W_p^l(\Omega)$. He showed that these classes form Banach spaces. Here, the main result of S. L. Sobolev is embedding theorems establishing connections between these spaces.

Let us make some statements.

Theorem 1. *Let Ω be a bounded domain in the space R^n with a boundary satisfying certain conditions of geometrical nature. If $lp > n$, then any function $f \in W_p^l(\Omega)$ is continuous. Moreover, the following inequality holds:*

$$\|f\|_{C(\Omega)} \leq M\|f\|_{W_p^l(\Omega)},$$

where $M = M(l, p, n, \Omega)$ is a positive constant.

Theorem 2. *Let Ω be a bounded domain in the space R^n with a boundary satisfying certain conditions of geometrical nature. If $lp \leq n$, then any function $f \in W_p^l(\Omega)$ for any q such that $1 \leq q < \frac{np}{n-lp}$ belongs to the class $L_q(\Omega)$. Moreover, the following inequality holds:*

$$\|f\|_{L_q(\Omega)} \leq M\|f\|_{W_p^l(\Omega)},$$

where $M = M(l, p, q, n, \Omega)$ is a positive constant.

The conditions on the boundary of the domain Ω, indicated by S. L. Sobolev in these theorems, have quite general character. For example, they are satisfied for any domain with a smooth boundary.

S. L. Sobolev was one of the founders of Novosibirsk State University in Akademgorodok. He gave the first lecture during the opening of Novosibirsk State University. Working in the Siberian Division of the USSR Academy of Sciences for 25 years, he was the head of the Chair of Differential Equations in the Department of Mechanics and Mathematics, lectured the classical course on equations of mathematical physics and a special course on cubature formulas, the theory which he had developed. The result of this research is his book “Introduction to the Theory of Cubature Formulas” (Nauka, Moscow (1974)). The scientific school in the field of the theory of cubature formulas was formed under the lead of S. L. Sobolev.

In the 1960s, S. L. Sobolev was also engaged in the problem of construction of electronic computers with processing power at least 1 billion operations per second (in the terminology used now, supercomputers). In this connection a group was formed in the Institute of Mathematics of the Siberian Division of the USSR Academy of Sciences. Such a supercomputer had to be a cluster of separate computers (processors) performing in parallel different steps of the work. The main technical principle was micro computerization. The time allocated for the completion of this project was said to be 20–25 years. The journal titled “Computational Systems” was published in the institute. It published papers devoted to electronic computers of high productivity. An interinstitutional seminar was organized, where everybody who studied this subject could present. Unfortunately, for objective (and possibly, subjective) reasons this work was not finished, since it did not find proper understanding and support. We can now say that, during the work conducted in the Institute of Mathematics, there was given a prognosis of ways of development of computer science. This prognosis turned out to be precise. The ideas formulated in the process of that work were implemented in real devices later on. For example, the proposal to use for connecting processors the network of the faces of the n-dimensional cube first was formulated in one of the papers published in the journal“Computational Systems” in 1962. This idea was realized in many parallel supercomputers working now.

The fact that such a remarkable mathematician as Academician S. L. Sobolev arrived in Novosibirsk in 1957 had great significance for the Siberian Division and development of mathematics in Siberia. The Sobolev Institute of Mathematics has been one of the world centers of mathematical research already for more than 40 years.

Part I

Equations of Mathematical Physics

1. Application of the Theory of Plane Waves to the Lamb Problem*

S. L. Sobolev

Chapter 1

1. Professor H. Lamb in his article [1] considered the problem on propagation of disturbances in an infinite half-space.

At a point on a boundary of the half-space there is a force normal to the surface of boundary between the medium and the vacuum. The problem is to compute components of displacements at some other point of the surface (the observation point). The results obtained by H. Lamb allow us to compute these displacements in the form of definite integrals.

Our problem is to find analogous integral expressions for displacements at an arbitrary point inside the medium. Our method, in spite of a certain formal dissimilarity, is actually close to H. Lamb's method.

The essence of our method is the consideration of a disturbance propagating in the half-space as a sum of disturbances of a certain special type: the complex plane waves. We obtain these complex waves directly from the equations of elasticity; however, they can be obtained by summing in a certain order the multiple Fourier integrals used by H. Lamb. We are not going to prove the existence and uniqueness theorems for our integral representation, since they are the formal corollary of the corresponding theorems for the Fourier integrals. Moreover, the obtained result does not need a strict proof, since the final formulas allow us to verify all initial and boundary conditions.

Let us briefly outline the statement of the problem.

First, we investigate the two-dimensional Lamb problem, and then move on to the three-dimensional problem.

As a starting point, we take H. Lamb's expressions of the displacements on the boundary, which we use as the boundary conditions.

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* Tr. Seism. Inst., **18** (1932), 41 p.

Tr. Seism. Inst. is Transactions of the Seismological Institute of the USSR Academy of Sciences. – *Ed.*

As in any method of representing a solution as a definite integral, for solving a problem we need to define so-called density of spectrum in the representation. For this purpose, we identify integrals obtained by H. Lamb with those obtained by us.

After defining in such way the spectral function, we transform the obtained results to a form more convenient for calculation.

Now we move on to the presentation of our method.

2. In our note [2] we had already studied the reflection of longitudinal and transverse elastic plane waves falling at different angles on the plane. However, because of the great importance of the plane waves for the problem in question, and also since the presentation of this question can be significantly simplified by using the theory of functions of a complex variable, we review this question again.

Consider an infinite elastic half-space and direct the y-axis along the normal to the boundary plane inward to the elastic medium, and the x- and z-axis along its surface. Suppose that we deal with a plane problem, and that the disturbance picture does not depend on the coordinate z. In this case, as is known, the components of the displacements u and v have the form

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}, \tag{1}$$

where φ and ψ are scalar and vector potentials satisfying the equations

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = a^2 \frac{\partial^2 \varphi}{\partial t^2}, \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = b^2 \frac{\partial^2 \psi}{\partial t^2}. \tag{2}$$

Here

$$a = \sqrt{\frac{\rho}{\lambda + 2\mu}}, \qquad b = \sqrt{\frac{\rho}{\mu}}$$

are the values reciprocal to the velocity of propagation of the longitudinal and transverse waves.

Let us consider the coordinate system moving along the x-axis with the velocity $\frac{1}{\theta}$, and assume that in this moving system of coordinates the disturbance picture, i.e., both the displacements and potentials, remain constant. In what follows, this quantity $\frac{1}{\theta}$ is called *apparent velocity*, and the described motion is called the *plane wave*. The meaning of this name will be explained later.

If we denote $\xi = t - \theta x$, then the system of ξ and y coordinates is our moving system of coordinates with the rescaled abscissa axis.

Our assumption is equivalent to the fact that both φ and ψ depend only on ξ and y.

Substituting these expressions into equations (2), we obtain

$$\theta^2 \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial y^2} = a^2 \frac{\partial^2 \varphi}{\partial \xi^2} \quad \text{and} \quad \theta^2 \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial y^2} = b^2 \frac{\partial^2 \psi}{\partial \xi^2}$$