Stochastic Ageing and Dependence for Reliability

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Foreword by Richard E. Barlow

Springer

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This book is dedicated to
Ai Ing, Joseph, Eugene, Serena,
my brother Chin-Yii and my parents
C. D. L

Wenhong, Jessica, Harry, William
M. X

Foreword

I first met Min Xie in the 1980s at Linkoping University in Sweden. He was working with Professor Bo Bergman, the Professor in Quality at the University. As I recall, Min Xie was a very serious student of reliability theory at the time. He was very familiar with the book *Mathematical Theory of Reliability* by myself and Frank Proschan.

My first meeting with C. D. Lai was in 1999 at Massey University in New Zealand. I was impressed then by his serious interest in research.

The subject of this monograph is ageing and dependence in the context of reliability. Both of these ideas are important and controversial. Ageing is a phenomenon experienced by both machines and people. There has been a great deal of progress in understanding ageing relative to people by molecular biologists such as Giusseppe Attardi at the California Institute of Technology. Other researchers have even tried to apply ideas in mathematical reliability theory to biological ageing. Unfortunately, it seems that this is not a useful activity. This is because biological organisms are capable of self-repair and reproduction while machines at this point in time are not.

Probabilistic dependence has also been discussed at length by many mathematicians and philosophers. One of the best classical mathematical discussions can be found in *Statistical Independence in Probability Analysis and Number Theory* by Mark Kac (1959). However, this work is solely applied mathematics and leaves the subject somewhat mysterious at the philosophical level which is also the level at which applications need to be made.

From another point of view, de Finetti, in 1937, for the first time presented a rigorous and systematic treatment of the concept of exchangeability together with the fundamental result which became known as "de Finetti's representation theorem." [See Kotz and Johnson (1992)]. De Finetti's paper illuminates the conditions under which frequencies may be related to subjective probabilities (that is, probabilities based on judgment) and also formalizes this connection. It replaces the classical notion of observations assumed to be "independent and identically distributed with unknown distribution" by the concept of exchangeable observations. This helps to resolve the mystery behind the ideas of independence and dependence. De Finetti also helped in the understanding of conditional probability. Conditional dependence is closely tied to finite populations (i.e., all populations in this world) while unconditional independence is relative to conceptually infinite populations.

To illustrate, consider n binary random quantities $(x_1, x_2, , x_n)$ judged *a priori* to be exchangeable, i.e., distributed with the hypergeometric distribution with parameters (N, S) where $S = \sum_{i=1}^{N} x_i$ is unknown since in this case observations $(x_{n+1}, x_{n+2}, , x_N)$ are not available. Although N is known, S is unknown. We are interested in inference concerning S. Now $(x_1, x_2, ...x_n)$ are *a priori* dependent, conditional on S. However, if S has a prior distribution which is judged binomial with parameters N (the known population size) and

n specified, then $(x_1, x_2, , x_n)$ are *a priori* unconditionally independent, with joint probability

$$\prod_{i=1}^{n} \rho^{x_i} (1-\rho)^{1-x_i}$$

Since the binomial distribution with parameters (N, ρ) is only suitable for conceptually infinite populations, we begin to see the connection between independence and infinite populations. (In the binomial case, N would be the sample size, not the population size.) This is presented as an exercise on page 52 of Barlow (1998). It was pointed out to me by a colleague, Max Mendel. Of course once $(x_1, x_2, , x_n)$ are observed they are no longer random quantities. Any judgment concerning S would require knowledge of the problem at hand and this judgment is only partly a mathematical problem.

The present monograph deals with life distributions belonging to various classes of failure (hazard) rate functions and mean residual life functions. The so-called 'bathtub' distributions are featured prominently and a brief introduction of the Bayesian approach on ageing concepts is given. The text provides a lot of material on test procedures and bivariate life distributions, with various concepts and measures of dependence. The material concerning reliability of coherent systems with positively dependent components is very important as component lifetimes are generally dependent in practice.

The book should be considered as a very useful reference. Results of the last three decades are brought together without delving into unnecessary detail. The reader is referred to papers, which are listed in the bibliography. It covers most of results in the literature pertaining to ageing classes and bivariate life distributions; so it can be regarded as a compendium of ageing concepts. It is encyclopedic in scope, contains much information, and will be useful to researchers in reliability engineering and other disciplines.

Berkeley, September 2005 Richard E. Barlow

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Preface

Reliability is an important and challenging subject, which involves the disciplines of science and engineering. Researchers in both these fields have been working on reliability problems for several decades. The aim of this book is to summarize various ageing and dependence concepts of the lifetimes that have been widely studied in the field of reliability.

Chapter 1 provides a summary of the book and notations and acronyms are also listed for easy reference later on. Chapter 2 deals with various concepts of stochastic ageing starting with the definition of the failure rate function (or hazard function). In this book, we will use the term failure rate instead of hazard rate, which is more common in survival analysis. Part of the reason is because most abbreviations such as IFR/DFR/IFRA/DFRA, etc., contain FR which stands for failure rate. We think that more confusion will be caused if the abbreviations are changed. Moreover, the acronym 'failure rate' is more commonly used in reliability engineering, especially for non-repairable systems.

Chapters 3–7 deal with some specific concepts of ageing and lifetime distributions. In particular, we consider bathtub shaped life distributions in Chapter 3. Existing models are grouped and summarized with their properties listed. Chapter 4 considers the mean residual lifetime function which is an important measure of ageing in reliability applications. Chapter 5 deals with the Weibull distribution and its generalizations that can be flexible in modeling lifetime data. Chapter 6 considers ageing concepts for discrete distributions. Chapter 7 summarizes statistical tests of ageing.

Chapter 8 extends the univariate ageing concepts to two or more variables. A brief introduction to the Bayesian approach to multivariate ageing in terms of majorization and Schur-concavity is given.

Dependence concepts, dependence orderings and measures of dependence are dealt with in Chapter 9. This is an extensive and important topic which caught the attention of many authors in recent years. We emphasize the positive (negative) quadrant dependence as this property is verifiable and realistic in many situations. All relevant results concerning dependence are summarized in this chapter. However, most of these results are related to statistical concepts and some are theoretical probability applications. We expect further research and applications in this area to be carried out by researchers. As a follow-up, Chapter 10 discusses the reliability of coherent systems with positively dependent components. We feel that this topic is a very important one in reliability applications.

Last but not least, in Chapter 11, we list 33 data sets of failure times or survival times. This could be useful for researchers and students in their future study in this field. The book ends with a large collection of references with nearly eight hundred entries.

It is our aim to provide a comprehensive treatment of both ageing and dependence concepts with emphasis on reliability and survival analysis. Proofs of many results are omitted, especially when they are either obvious or are long. The interested readers may refer to references listed in the bibliography section for detailed proofs. The readers should, however, have some basic knowledge in probability and statistics before reading this book.

Apart from the excellent classical text by Barlow and Proschan (1981), Gertsbach (1989) is another good book on statistical reliability. There is an excellent book on ageing, written from a Bayesian point of view, by Spizzichino (2001). Also on dependence concepts and stochastic ageing, there is an excellent book by Shaked and Shanthikumar (1994). On multivariate dependence concepts, Joe (1997) has provided us an excellent monograph.

We hope that both reliability researchers and practitioners find the book useful for reference and for some new ideas. This book will also be useful for graduate students in reliability or applied probability.

This book is a summary of the work carried out by many people. It would be too long a list if we acknowledge them one by one – most of the names can be found in the reference list at the end of the book. We wish to thank, in particular, Mr. John Kimmel of Springer who had guided us through the whole project with much encouragement and professionalism. We also wish to record our our sincere thanks to several anonymous reviewers for their constructive comments. We appreciate very much the help from all of them, and other colleagues and students of us.

C. D. Lai, Massey University, New Zealand
M. Xie, National University of Singapore, Singapore
September 2005

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Introduction

1.1 Aim and Scope of the Book

As the title suggests, the main aim of this book is to bring together different facets of ageing concepts of the lifetime of a device or a system. An ageing concept largely describes how a device ages with time. Though in most cases, ageing has an averse effect on a 'product', there are some other cases in which ageing is beneficial. These ageing concepts have a direct impact on the behaviors of two important reliability measures (i) the failure rate function and (ii) the mean residual life function. These reliability measures are important in maintenance planning, replacement planning, resource allocation and other reliability related decisions.

Looking from another angle, these ageing concepts were defined in the first place through the characteristics of these two named functions, especially their shapes. Another important aspect of a reliability study is that components of a system may not always be mutually independent so a description of how two or more component lifetimes depend on one another may be of interest. Further, it will be helpful to find some indices that quantify the degree or strength of relationships between them (according to a defined concept such as the linear dependence). Thus we will consider in this book some measures of dependence that are relevant in reliability or survival analysis.

The present book provides a comprehensive treatment of both ageing and dependence with the emphasis on reliability and survival analysis. Proofs of many results are not given, but extensive references are provided, so interested readers can refer to them. The book assumes a basic course in mathematical statistics and some familiarity of the classical reliability text by Barlow and Proschan (1981) "*Statistical Theory of Reliability and Life Testing: Probability Models*". It is intended that reliability researchers and practitioners may find the book useful for reference and new ideas. This book will also be useful for graduate students in reliability or applied probability.

1.2 Brief Overview

Chapter 1 – Introduction

This chapter provides a bird's eye view of the book which focuses on the usefulness of ageing and dependence concepts in real life, particularly in reliability engineering and survival analysis. We also give three lists of acronyms and nomenclatures that will appear frequently throughout the book.

Chapter 2 – Concepts and applications of stochastic ageing

This chapter begins with defining the failure rate function (hazard rate) which forms one of the pillars for reliability and survival analysis. We then introduce various ageing concepts based on the reliability characteristics such as the failure rate function, survival function or the mean residual life function. Their relative strengths are compared and a chain of relationships is given. Several examples of lifetime distributions together with their ageing properties are given. Special attention is given to discuss the ageing behavior of finite mixtures of life distributions. Partial orderings based on ageing concepts are also introduced. The chapter ends with a discussion on some existing or potential applications.

Chapter 3 – Bathtub shape life distributions

It is common for failure rate function (hazard rate) to have a bathtub shape. This chapter presents theoretical and practical discussion on this and presents several life distributions that can be used to model a bathtub curve. The change point (turning point) of the bathtub shaped failure rate function is also discussed as it plays an important role in establishing the optimal burn-in time of a product. Applications of bathtub models are also indicated.

Chapter 4 – Mean residual lifetime (MRL)

This chapter focuses on the use of MRL, which is an important measure in reliability applications. The traditional reliability analysis has been based on the failure rate function, but usually it is the residual life that is of great interest when one considers repair and replacement strategies. The former relates only to the risk of immediate failure whereas the mean residual life summarizes the entire remaining life distribution. We investigate how the shapes of MRL are related to the shapes of the failure rate functions; these relationships provide us a strategy to determine the optimal burn-in time and to solve other maintenance problems. In addition to burn-in time determination, the chapter also lists several other applications in diverse disciplines including demography and social studies.

Chapter 5 – Weibull and generalized distributions

The Weibull distribution is a generalization of the exponential distribution that has no ageing. However, the Weibull distribution can only be used to model increasing or decreasing failure rate distributions so it is not sufficiently flexible. Various extensions or generalizations have been added in the reliability literature to give rise to more flexible distributions. This chapter discusses various properties of the Weibull distribution and its generalizations. Reliability operations such as mixtures and formation of additive and multiplicative systems from the Weibull family are also discussed.

Chapter 6 – Ageing concepts for discrete distributions

When the failure time is in the form of discrete measurement such as on–off switching, a discrete distribution should be used. We review some common discrete failure time models together with their discrete ageing properties. As expected, most of these properties are analogous to their continuous counterparts. We also include an alternative definition of a failure rate which is closer to the continuous time failure rate than the traditional definition of a discrete failure rate.

Chapter 7 – Tests of ageing

An important problem in practice is to test the constant failure rate (hazard rate) versus other forms of ageing property. Many tests have been proposed in the past three decades and these tests will be discussed in this chapter.

Chapter 8 – Bivariate and multivariate ageing concepts

Univariate ageing concepts are generalized to bivariate and multivariate distributions. Various versions are available for the same marginal ageing concepts. Tests of bivariate ageing concepts are also briefly given. We also introduce the Bayesian approach to multivariate ageing through majorization and Schur-concavity of a joint survival function.

Chapter 9 – Dependence concepts and measures of dependence

Various types of dependence among two or more lifetime variables are considered. We pay a special attention to the so-called ‘positive dependence’ concept such as ‘association’, positively quadrant dependent, etc. Several bivariate distributions with positive dependence property are given to illustrate the theory. A chain of relations among positive ageing concepts is also presented.

Included also is a discussion of dependence orderings that give the relative strength of dependence between two pairs of lifetime random variables with respect to the same concept. For example, we say F is more positively quadrant dependent than G if the joint survival function of the former dominates the latter.

Various measures of dependence between two variables are available in the literature, e.g., Pearson's correlation, rank correlations, etc. We give an overview of these measures and show how these numerical indices vary with or are related to the dependence concepts.

Some local dependence measures, as opposed to the traditional global measures of dependence, are also introduced.

Chapter 10 – Reliability systems with dependent or independent components

Further analysis of the use of stochastic dependence in reliability studies will be presented. For example, dependence is common among the component lifetimes of a system, thus it plays an important role in redundancy improvement. In particular, we consider the reliability performance of parallel and series systems of two components with dependent component lifetimes. We discuss how the efficiency of redundancy is often determined by whether they are positively or negatively dependent.

For a system with independent components, we examine whether active spare allocation at the component level is superior (in some sense) to active spare at the system level. We also compare two k-out-of-n systems with different k or n using some partial ordering concepts.

Chapter 11 – Failure time data sets

We have collected 33 data sets of failure times or survival times which are now given in this chapter. These data sets are arranged according to the ageing classes they belong to. One of the primary aims of this chapter is to illustrate the existence of real data sets that have either bathtub or upside-down bathtub shaped failure rates. Also, the data may be a suitable testing ground for sophisticated techniques that the original author did not think of.

1.3 Acronyms and Nomenclatures

In this book, we follow a general convention regarding the shape of a function. We say that a function is increasing if it is nondecreasing. Similarly, we say a function is decreasing if it is nonincreasing.

We now provide three lists of acronyms and nomenclatures: (i) general, (ii) ageing concepts, and (iii) dependence concepts.

Table 1.1. General List

$B(a,b)$	The beta function of two parameters
cdf	Cumulative distribution function
D	A class of decreasing functions
E	Expectation
$E_1(t)$	Exponential integral function
$\mathcal{N}$	Set of all integers
$\mathcal{N}^+$	Set of all positive integers
$\Gamma(x)$	Gamma function
$F(x)$	(Cumulative) distribution function
$f(x) = F'(x)$	Density function if exists
$\in$	Belongs to a class or in a class
$H_0 :$	Null hypothesis
$H_1 :$	Alternative hypothesis
I	A class of increasing functions
$I(a) = \begin{cases} 0 \text{ if } a \leq 0 \\ 1, \text{ if } a > 0 \end{cases}$	Indicator function
i.i.d.	Independent and identically distributed
log	Natural log (based on e)
MTTF	Mean time to failure
μ	Mean of lifetime variable (Mean time to failure)
μ_X	Mean of the random variable X
μ'_k	kth moment about the origin (zero)
pdf	Probability density function
Pr(E)	Probability of event E to occur
R	Set of real numbers
R^+	Set of positive real numbers
T	Lifetime variable
$\tau_{k\|n}$	System lifetime of a k-out-of-n system
$X_1, X_2, ..., X_n$	Random sample from a population with distribution function F
$X_{(1)} < X_{(2)} < ... < X_{(n)}$	Order statistics from a sample of size n
$X_{i:n}$	ith order statistic of a k-out-of-n system
$[x]$	The largest integer that is less than or equal to x
$[x]^+$	The largest positive integer that is less than or equal to x
$\bar{X}$	Sample mean
U_n	U-statistic
var	Variance
$\leq_*$	Partial ordering with respect to an ageing characteristic *

Table 1.2. Ageing Concepts and Dependence List

$\bar{F}(t) = 1 - F(t)$	Survival function of a lifetime random varaible
$\bar{F}(x\|t) = \bar{F}(x+t)/\bar{F}(t)$	Conditional reliability of a unit of age t
$\bar{F}(x, y)$	Joint survival function of X and Y
$F_n(x)$	Empirical cdf
$F_X(x)$ $(F_Y(y))$	cdf of marginal random variable X (Y)
$\mu(t)$	Mean residual life function
MRL	Mean residual life
$\mu_{(1)}$	MTTF of the series system of two components
$\mu_{(2)}$	MTTF of the parallel system of two components
$R(\cdot)$	Reliability function or survival function
$r(t)$	Failure rate (hazard rate) function
T	Lifetime random varaible
T_1	Lifetime of a series system of two components
T_2	Lifetime of a parallel system of two components
τ	Change point
X	Lifetime random varaible
IFR (DFR)	Increasing (decreasing) failure rate
IFRA (DFRA)	Increasing (decreasing) failure rate average
MBT	Modified bathtub shaped
NBU (NWU)	New better (worse) than used
NBUE (NWUE)	New better (worse) than used in expectation
BT (UBT)	Bathtub shaped (Upside-down bathtub shape)
DIMRL (IDMRL)	Decreasing (increasing) then increasing (decreasing) mean residual life.
NWBUE (NBWUE)	New worse then better than used in expectation (New better then worse than used in expectation)
DMRLHA	Decreasing mean residual life in harmony average
DPRL-α(IPRL-α)	Decreasing (Increasing) α-percentile residual life

Table 1.3. Dependence Concepts List

PQD (NQD)	Positive (Negative) quadrant dependence
LTD (RTI)	Left-tail decreasing (Right-tail increasing)
SI (alias PRD)	Stochastically increasing (alias positively regression dependent)
RCSI (LCSD)	Right corner set increasing (Left corner set decreasing)
TP_2 (alias LRD)	Totally positive of order 2 (alias likelihood ratio dependent)
WPQD	Weakly positive quadrant dependent
PDO	Positive dependent ordering
RR_2	Reverse regular of order 2
$\bar{F} = 1 - F$	$\bar{F}$ survival function, F cumulative distribution function
ρ	Pearson product-moment correlation coefficient
τ	Kendall's tau
ρ_S	Spearson's rho

2

Concepts and Applications of Stochastic Ageing

2.1 Introduction

The concept of ageing is very important in reliability analysis. 'No ageing' means that the age of a component has no effect on the distribution of residual lifetime of the component. 'Positive ageing' (also known as 'averse ageing') describes the situation where residual lifetime tends to decrease, in some probabilistic sense, with increasing age of a component. This situation is common in reliability engineering as components tend to become worse with time due to increased wear and tear. On the other hand, 'negative ageing' has an opposite effect on the residual lifetime. 'Negative ageing' is also known as 'beneficial ageing'. Although this is less common, when a system undergoes regular testing and improvement, there are cases for which we have reliability growth phenomenon. Though we concentrate on positive ageing in this book, it is being understood that a parallel development of negative ageing can also be carried out.

Concepts of ageing describe how a component or system improves or deteriorates with age. Many classes of life distributions are categorized or defined in the literature according to their ageing properties. An important aspect of such classifications is that the exponential distribution is nearly always a member of each class. The notion of stochastic ageing plays an important role in any reliability analysis and many test statistics have been developed in the literature for testing exponentiality against different ageing alternatives. Our aim in this chapter is to provide an overview of these developments.

By 'life distributions' we mean those for which negative values do not occur, i.e., $F(x) = 0$ for $x < 0$. The nonnegative variate X is thought of as the time to failure (or death) of an electrical or mechanical component (or organism), but other interpretations may be possible – an inter-event time is normally necessarily positive.

In this chapter, we focus on classes of life distributions based on notions of ageing–IFR (increasing failure rate) is perhaps the best-known, but we shall meet several others also, and study their interrelationships whenever possible.

The chapter may serve as a continuation of the ageing concepts developed in the pioneering book Barlow and Proschan (1981), which was first printed by Holt, Reinhart and Winston in 1975.

The major parts of the current chapter are devoted to

- Introducing different ageing characteristics,
- Classifications of life classes based on various ageing characteristics and establishing their interrelationships,
- Failure rates of mixtures of distributions,
- Elementary properties of these life classes,
- Partial orderings of two life distributions based on comparison of their ageing properties.

From the definitions of the life distribution classes, results may be derived concerning such things as properties of systems (based upon properties of components), bounds for survival functions, moment inequalities, and algorithms for use in maintenance policies (Hollander and Proschan, 1984).

Most readers will know that statistical theory applied to distributions of lifetime lengths plays an important part in both the reliability engineering and the biometrics literature. We may also note a third applications area: Heckman and Singer (1986) review econometric work on duration variables (e.g., lengths of periods of unemployment, or time intervals between purchases of a certain good), much of which, they say, has borrowed freely and often uncritically from reliability theory and biostatistics.

Section 2.2 gives characterizations of lifetime distributions by their survival, failure rate or mean residual life functions. In Section 2.3, we list several commonly used life distributions together with their basic properties. In Section 2.4 we give formal definitions of ten basic ageing notions and their interrelationships together with a table of summary furnished with key references. Section 2.5 discusses the properties of some of these basic ageing classes and Section 2.6 is devoted to the non-monotonic failure rate classes such as the bathtub and upside-down bathtub life distributions, which are important in reliability applications. Section 2.7 briefly presents some additional but less known ageing classes. In Section 2.8, we consider failure rates of mixtures of life distributions. This has an important application in burn-in. Section 2.9 provides an introduction to partial ordering through which the strength of the ageing property of the two life distributions within the same class is compared. Section 2.10 considers briefly the matter of relative ageing of two life distributions. Relative ageing is really a form of partial ordering. We discuss in Section 2.11 how the relationship between the sth and the $(s+1)$th equilibrium distribution can be used to describe the relationship between the shape of the failure rate and the shape of mean residual life function of a distribution. Finally in Section 2.12, we tidy up the loose ends on stochastic ageing and the section ends with some remarks concerning future research directions that may bridge the theory and applications.

Abbreviations

The following table of acronyms and abbreviations will be a useful reference. Although this has largely been given in Chapter 1, the list here gives a more exhaustive coverage for ageing concepts.

Table 2.1. List of Ageing Class Abbreviations

Abbreviation	**Ageing Class**
BT (UBT)	Bathtub shape (Upside-down bathtub shape)
DMRL (IMRL)	Decreasing mean residual life (Increasing mean residual life)
HNBUE	Harmonically new better than used in expectation
(HNWUE)	(Harmonically new worse than used in expectation)
IFR (DFR)	Increasing failure rate (Decreasing failure rate)
IFRA (DFRA)	Increasing failure rate average (Decreasing failure rate average)
$\mathcal{L}$-class	Laplace class of distributions
NBU (NWU)	New better than used (New worse than used)
NBUE	New better than used in expectation
(NWUE)	(New worse than used in expectation)
NBUC	New better than used in convex ordering
(NWUC)	(New worse than used in convex ordering)
NBUFR	New better than used in failure rate
(NWUFR)	(New worse than used in failure rate)
NBUFRA	New better than used in failure rate average
(NWUFRA)	New worse than used in failure rate average
NBWUE	New better then worse than used in expectation
(NWBUE)	(New worse then better than used in expectation)

We note that NBUFRA is also known as NBAFR.

2.2 Characterizations of Lifetime Distributions

Rather than $F(t)$, we often think of $\bar{F}(t) = \Pr(X > t) = 1 - F(t)$, which is known as the survival function or reliability function. Here, X denotes the lifetime of a component, i.e., time to first failure. The expected value of X is denoted by μ. The function

$$\bar{F}(x \,|\, t) = \bar{F}(t + x)/\bar{F}(t), \quad x, t \geq 0, \tag{2.1}$$

represents the survival function of a unit of age t, i.e., the conditional probability that a unit of age t will survive for an additional x units of time. The expected value of the remaining (residual) life, at age t, is $\mu(t) = E(X - t \,|X > t)$ which may be shown to be $\int_0^\infty \bar{F}(x \,|\, t)\, dx$. It is obvious that $\mu(0) = \mu$.

When $F'(t) = f(t)$ exists, we can define the failure rate (hazard rate or force of mortality) of a component as

$$r(t) = f(t)/\bar{F}(t) \tag{2.2}$$

for t such that $\bar{F}(t) > 0$. This can also be written as

$$r(t) = \lim_{\Delta \to 0} \frac{\Pr(t \leq X < t + \Delta |\ t \leq X)}{\Delta}. \tag{2.3}$$

Thus for small Δ, $r(t)\Delta$ is approximately the probability of a failure occurring in $(t, t + \Delta]$ given no failure has occurred in $(0, t]$.

It follows that, if $r(t)$ exists, then

$$-\log \bar{F}(t) = \int_0^t r(x)\, dx \tag{2.4}$$

represents the cumulative failure (hazard) rate which may be designated by $H(t)$. Equivalently

$$\bar{F}(t) = \exp\left\{-\int_0^t r(x)\, dx\right\} = \exp\left\{-H(t)\right\}. \tag{2.5}$$

A lifetime distribution can also be characterized by its mean residual life (MRL) defined by

$$\mu(t) = E(X - t \,|\, X > t) \tag{2.6}$$

through

$$\bar{F}(t) = \frac{\mu}{\mu(t)} \exp\left\{-\int_0^t \mu(x)^{-1}\, dx\right\},\ t \geq 0. \tag{2.7}$$

We will discuss MRL more fully in Chapter 4.

In short, a lifetime distribution may be characterized by $\bar{F}(t)$, the conditional survival function $\bar{F}(x \,|\, t)$, $r(t)$ or $\mu(t)$. In addition, Galambos and Hagwood (1992) have shown that a life distribution may also be characterized by the second moment of the residual life $E[(X - t)^2 \,|\, X > t]$.

Remarks on terminology

Calling the function $r(t)$ the failure rate in (2.2) could cause some confusion if this terminology is not adequately explained. The confusion aries because another 'failure rate' is also used by some authors in the context of a point process of failures. We now follow the approach of Thompson (1981) to highlight this confusion and attempt to provide a distinction between the two concepts.

Let $N(t)$ denote the number of failures in the interval $(0, t]$. Set $M(t) = EN(t)$ and let $\xi(t) = M'(t)$ and so $\xi(t)$ is the instantaneous rate of change of the expected number of failure with respect to time; thus we may call $\xi(t)$ the failure rate of the process.

Another characteristic of interest in a failure process is

$$\lambda(t) = \lim_{\Delta \to 0} \frac{\Pr[N(t, t+\Delta) \geq 1]}{\Delta}. \tag{2.8}$$

If $\lambda(t)$ exists, then for small Δ, $\lambda(t)\Delta$ is approximately the probability of failure in the interval $(t, t+\Delta]$. Assuming the simultaneous failures do not occur (which is true for most applications), $\xi(t) = \lambda(t)$, if they exist.

Clearly, $r(t)$ is not the same as $\lambda(t)$ since the $r(t)\Delta$ as defined via (2.2) is (approximately) a conditional probability of a failure in $(t, t+\Delta]$ whereas $\lambda(t)\Delta$ is not conditional on the event prior to t.

Under the framework of a stochastic point process, Thompson (1981) discussed basic ways to characterize reliability. The distinction between the failure rate of a process, useful for repairable systems, and the failure rate of a distribution, useful for nonrepairable systems is drawn.

In the point process literature, the failure rate of the process $\xi(t)$ or $\lambda(t)$ is generally known as the intensity function. In reliability modeling, this is sometimes called the 'rate of occurrence of failure (ROCOF)' for repairable systems so that it is not to be confused with the traditional failure rate concept for the lifetime distribution. For further discussion, see Ascher and Feingold (1984).

Note that in the case of a homogeneous Poisson process, the failure rate of the process is λ which is also the failure rate of the the exponential distribution. We wish to emphasize here the 'failure rate' used in this book is the failure rate of a life distribution F defined in (2.2); it is *not* the failure rate of a point process of failures.

One of the reasons for our usage of the acronym 'failure rate' instead of 'hazard rate' in this book is that IHR (DHR) is rarely used in the literature on classification of life distributions. The 'near' universal use of the ageing notions such as IFR (DFR) is consistent with our choice in calling $r(t)$ the failure rate of a life distribution.

2.2.1 Shape of a Failure Rate Function

We assume that the failure rate function $r(t)$ is a real-valued differentiable function $r(t) : R^+ \to R^+$. As usual, by increasing we mean nondecreasing and by decreasing, we mean nonincreasing. $r(t)$ is said to be

(1) strictly increasing if $r'(t) > 0$ for all t and is denoted by I;

(2) strictly decreasing if $r'(t) < 0$ for all t and is denoted by D;

(3) bathtub shaped if $r'(t) < 0$ for $t \in (0, t_0)$, $r'(t_0) = 0$, $r'(t) > 0$ for $t > t_0$, and is denoted by BT;

(4) upside-down bathtub shaped if $r'(t) > 0$ for $t \in (0, t_0)$, $r'(t_0) = 0$, $r'(t) > 0$ for $t > t_0$, and is denoted by UBT;