To my Riemann zeros:
Odile, Julie and Michaël, my muse and offspring

Michel L. Lapidus

To Jena, David and Samuel

Machiel van Frankenhuijsen
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Fractal Geometry, Complex Dimensions and Zeta Functions
Geometry and Spectra of Fractal Strings

With 53 Illustrations
The front cover shows a tubular neighborhood of the Devil’s staircase (Figure 12.2, page 335) and the quasiperiodic pattern of the complex dimensions of a nonlattice self-similar string (Figure 3.7, page 86).

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This book is a self-contained unit which encompasses a broad range of topics that connect many areas of mathematics, including fractal geometry, number theory, spectral geometry, dynamical systems, complex analysis, distribution theory and mathematical physics.

The material in our earlier book, *Fractal Geometry and Number Theory* ([Lap-vF5], Birkhäuser, January 2000), has been recast and greatly expanded, taking into account the latest research developments in the field. Compared with that foundational monograph, the present book is longer by almost 200 pages and has twice as many illustrations. Further, it contains two new chapters, Chapter 3 about the complex dimensions of nonlattice self-similar strings, and Chapter 7 about flows, and a new appendix, on an application of Nevanlinna theory, as well as many new sections within the original chapters of [Lap-vF5]. It also provides a number of new examples, comments and theorems, many of which have not previously been published in the mathematical literature.

Chapter 1 provides a gentle introduction to some of the main topics. We extend the definition of self-similar string in Chapter 2 to include self-similar strings with more than one gap, following [Fra1, 2]. These self-similar strings correspond to (standard) self-similar sets in $\mathbb{R}$. Moreover, their geometric zeta functions may have both poles (i.e., complex dimensions) and zeros, so that, as shown in the new Section 2.3.3, cancellations may occur and some of the potential complex dimensions may disappear as a result.

Since the publication of [Lap-vF5], the material on Diophantine approximation of complex dimensions of nonlattice strings, which was included
in [Lap-vF5, Sections 2.4–2.6], has grown into a new chapter, Chapter 3. In that chapter, we give a large amount of new information on the fine structure of the complex dimensions of nonlattice self-similar strings. In particular, both qualitatively and quantitatively, we obtain a much better understanding of the quasiperiodic patterns of the complex dimensions. Our new theoretical and numerical results are illustrated by a variety of diagrams throughout Chapter 3, and several conjectures and open problems are proposed. Further, dimension-free regions are obtained that are used to derive good (and sometimes sharp) error estimates in much of the remainder of the book. In addition, we introduce Dirichlet polynomials and we slightly change the definition of generic nonlattice. We use a new approach to computing the density of the complex dimensions of a nonlattice self-similar string, improving upon the density results of Chapter 2. We have kept the argument of [Lap-vF5] in Chapter 2, and the relevant parts of Nevanlinna theory are presented in the new Appendix C. These results were first partly presented in the paper [Lap-vF7] (see also [Lap-vF9]).

We have added a new and previously unpublished discussion of the Euler product of the spectral operator in Section 6.3.2. The interest of this construction is that it provides an operator-valued Euler factorization of the spectral operator (associated with the spectral counting function) that is valid in the critical strip $0 < \Re s < 1$.

Chapter 7 has been largely expanded from Section 2.1.1 of [Lap-vF5]. It contains material about the periodic orbits of self-similar flows, leading to an Euler product representation of the geometric zeta function of a self-similar fractal string. We obtain an explicit formula (expressed in terms of the underlying dynamical complex dimensions) for the prime orbit counting function of a suspended flow, and thereby deduce a Prime Orbit Theorem in this context. Our results are most precise in the special case of self-similar flows for which we determine (via Diophantine approximation) dimension-free regions, from which we deduce a Prime Orbit Theorem with (often sharp) error term. This chapter is partly based on the paper [Lap-vF6].

We found a more efficient way to study the tubular neighborhoods of fractal strings (Chapter 6 of [Lap-vF5]), which led us to establish a pointwise tube formula in Section 8.1.1, Theorem 8.7. Under suitable somewhat stronger hypotheses, this theorem complements and improves upon the conclusion of Theorem 8.1 (the distributional tube formula). The latter theorem is central to our book (and played a key role in [Lap-vF5] as well). In fact, upon the request of some of our readers, we have provided more details for the proof of Theorem 8.1 as well as of several other results in Chapter 8. Thanks to Erin Pearse, we were also able to include Figure 8.1, illustrating the structure of the proof of Theorem 8.15 and the interdependence of many of the explicit formulas and other results in Chapters 5 and 8. In Sections 8.4.2 and 8.4.4, which discuss the important class of self-similar strings, we have provided a significantly more detailed discussion of the lattice case and of the nonlattice case. Furthermore, our earlier
statements are extended to general self-similar strings (i.e., lattice and non-lattice strings with multiple gaps). In Section 8.4.3, we define and compute the average Minkowski content of an arbitrary lattice string.

The geometry and the spectrum of Cantor strings and truncated Cantor strings is presented in Chapter 10 (Chapter 8 of [Lap-vF5]). The material on truncated Cantor strings is new, and is applied in Section 11.1.1.

We include in Chapter 11 (Chapter 9 of [Lap-vF5]) an exposition of the work of Mark Watkins on shifted arithmetic progressions [Watk,vFWatk]. We thank him for allowing us to include this result. We also include in Section 11.1.1 an exposition of [vF3], on finite arithmetic progressions of zeros of the Riemann zeta function. These works build upon and further develop the earlier work in [Lap-vF5, Chapter 9] on infinite arithmetic progressions of zeros. They also provide additional tools to attempt to solve some of the problems and conjectures proposed in [Lap-vF5, Section 10.1] (see Section 12.1).

Chapter 12 contains a summary of the recent results of Erin Pearse and the first author [LapPe1] on the complex dimensions and the volume of the tubular neighborhoods of the von Koch snowflake curve (see Section 12.3.1), which is pursued in a somewhat different direction in [LapPe2–4] (see Section 12.3.2). The latter work can be viewed as a step towards the long-term goal of developing a higher-dimensional theory of complex dimensions of fractals. We also discuss (in Section 12.4.1) recent results of Ben Hambly and the first author [HamLap] on the complex dimensions of random fractal strings, including random self-similar strings and the zero set of Brownian motion. Furthermore, in Section 12.4.2, we give a short introduction to the theory of fractal membranes (quantized fractal strings), proposed by the first author in the forthcoming book [Lap10] (see also [Lap9]) and further developed by Ryszard Nest and the first author in the papers in preparation [LapNes1–3]. The last section of the book, Section 12.7, corresponding to [Lap-vF5, Chapter 10.5], has been expanded, elaborating our proposed theory of complex cohomology.

In Appendix A, we have added a brief introduction to the two-variable zeta functions of Pellikaan [Pel], Schoof and van der Geer [SchoG], and Lagarias and Rains [LagR].

Several mistakes and misprints were pointed out to us by a number of people and have been corrected. We want to thank those people for their helpful comments. Without a doubt, new mistakes have been added, for which we take full responsibility. As was the case for the first book [Lap-vF5], we welcome comments from our readers.

Some, but by no means all of the main results of this book appeared in [Lap-vF1–7, 9]. Our earlier book [Lap-vF5] combined and superseded our two IHES preprints [Lap-vF1–2] and was announced in part in the paper [Lap-vF4], which was a slightly expanded version of the IHES preprint M/97/85. Most of the material in [Lap-vF5] was entirely new. The interested reader may wish to consult [Lap-vF4], as well as the research expos-
itary article [Lap-vF9]—in conjunction with the introduction and Chapter 1—to have an accessible overview of some of the main aspects of this work.

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May 2006
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Overview

In this book, we develop a theory of complex dimensions of fractal strings (i.e., one-dimensional drums with fractal boundary). These complex dimensions are defined as the poles of the corresponding (geometric or spectral) zeta function. They describe the oscillations in the geometry or the frequency spectrum of a fractal string by means of an explicit formula. Such oscillations are not observed in smooth geometries.

A long-term objective of this work is to merge aspects of fractal, spectral, and arithmetic geometries. From this perspective, the theory presented in this book enables us to put the theory of Dirichlet series (and of other zeta functions) in the geometric setting of fractal strings. It also allows us to view certain fractal geometries as arithmetic objects by applying number-theoretic methods to the study of the geometry and the spectrum of fractal strings.

In Chapter 1, we first give an introduction to fractal strings and their spectrum, and we precisely define the notion of complex dimension. We then make in Chapter 2 an extensive study of the complex dimensions of self-similar fractal strings. This study provides a large class of examples to which our theory can be applied fruitfully. In particular, we show in Chapter 3 that self-similar strings always have infinitely many complex dimensions with positive real part, and that their complex dimensions are quasiperiodically distributed. This is established by proving that the lattice strings—the complex dimensions of which are shown to be periodically distributed along finitely many vertical lines—are dense (in a suitable sense) in the set of all self-similar strings. We present the theory of Chapter 3—in which we analyze in detail the quasiperiodic pattern of complex dimen-
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sions, via Diophantine approximation—by using the more general notion of Dirichlet polynomial.

In Chapter 4, we extend the notion of fractal string to include (possibly virtual) geometries that are needed later on in our work. Then, in Chapter 5, we establish pointwise and distributional explicit formulas (explicit in the sense of Riemann’s original formula [Rie1], but more general), which should be considered as the basic tools of our theory. In Chapter 6, we apply our explicit formulas to construct the spectral operator, which expresses the spectrum in terms of the geometry of a fractal string. This operator has an Euler product that is convergent, in a suitable sense to be explained in Section 6.3.2, in the critical strip $0 < \Re s < 1$ of the Riemann zeta function. We also illustrate our formulas by studying a number of geometric and direct spectral problems associated with fractal strings.

In Chapter 7, we use the theory of Chapters 3 and 5 to study a class of suspended flows and define the associated dynamical complex dimensions. In particular, we establish an explicit formula for the periodic orbit counting function of such flows and deduce from it a prime orbit theorem with sharp error term for self-similar flows, thereby extending in this context the work of [PaPol1, 2]. We also obtain an Euler product for the zeta function of a self-similar fractal string (or flow).

In Chapter 8, we derive an explicit formula for the volume of the tubular neighborhoods of the boundary of a fractal string. We deduce a new criterion for the Minkowski measurability of a fractal string, in terms of its complex dimensions, extending the earlier criterion obtained by the first author and C. Pomerance (see [LapPo2]). This formula suggests analogies with aspects of Riemannian geometry, thereby giving substance to a geometric interpretation of the complex dimensions.

In the later chapters of this book, Chapters 9–11, we analyze the connections between oscillations in the geometry and the spectrum of fractal strings. Thus we place the spectral reformulation of the Riemann hypothesis, obtained by the first author and H. Maier [LapMa2], in a broader and more conceptual framework, which applies to a large class of zeta functions, including all those for which one expects the generalized Riemann hypothesis to hold. We also reprove—and extend to a large subclass of the aforementioned class—Putnam’s theorem according to which the Riemann zeta function does not have an infinite sequence of critical zeros in arithmetic progression. This work is supplemented in Section 11.1.1 with an upper bound for the possible length of an arithmetic progression of zeros, and in Section 11.4.1, where we present Mark Watkins’ work on the finiteness of shifted arithmetic progressions of zeros of $L$-series.

In the final Chapter 12, we propose as a new definition of fractality the presence of nonreal complex dimensions with positive real part. We also make several suggestions for future research in this area. In particular, we summarize the recent results of [LapPe1] on the complex dimensions and the volume of the tubular neighborhoods of the von Koch snowflake curve, which provides a first example of a higher-dimensional theory of complex dimensions of fractals.
Le plus court chemin entre deux vérités dans le domaine réel passe par le domaine complexe.

[The shortest path between two truths in the real domain passes through the complex domain.]

Jacques Hadamard
Introduction

A fractal drum is a bounded open subset of $\mathbb{R}^m$ with a fractal boundary. A difficult problem is to describe the relationship between the shape (geometry) of the drum and its sound (its spectrum). In this book, we restrict ourselves to the one-dimensional case of fractal strings, and their higher-dimensional analogues, fractal sprays. We develop a theory of complex dimensions of fractal strings, and we study how these complex dimensions relate the geometry and the spectrum of fractal strings. See the notes to Chapter 1 in Section 1.5 for references to the literature.

In Chapter 1, we define the basic object of our research, fractal strings. A standard fractal string is a bounded open subset of the real line. Such a set is a disjoint union of open intervals, the lengths of which form a sequence

$$\mathcal{L} = l_1, l_2, l_3, \ldots,$$

which we typically assume to be infinite. Important information about the geometry of $\mathcal{L}$ is contained in its geometric zeta function

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s,$$

We assume throughout that this function has a suitable meromorphic extension. The central notion of this book, the complex dimensions of a fractal string $\mathcal{L}$, is defined as the poles of the meromorphic extension of $\zeta_{\mathcal{L}}$. These can also be referred to as the complex fractal dimensions of $\mathcal{L}$. 
The spectrum of a fractal string consists of the sequence of frequencies

\[ f = k \cdot l_j^{-1} \quad (k, j = 1, 2, 3, \ldots). \]

The spectral zeta function of \( L \) is defined as

\[ \zeta_\nu(s) = \sum_{f} f^{-s}. \]

The geometry and the spectrum of \( L \) are connected by the following formula [Lap2]:

\[ \zeta_\nu(s) = \zeta_L(s)\zeta(s), \]

where \( \zeta(s) = 1 + 2^{-s} + 3^{-s} + \ldots \) is the classical Riemann zeta function, which in this context can be viewed as the spectral zeta function of the unit interval.

We also define a natural higher-dimensional analogue of fractal strings, fractal sprays, the spectra of which are described by more general zeta functions than \( \zeta(s) \) [LapPo3]. The counterpart of (*) still holds for fractal sprays and can be used to study their spectrum. We refer the interested reader to Appendix B for a brief review of aspects of spectral geometry—including spectral zeta functions and spectral asymptotics—in the classical case of smooth manifolds.

We illustrate these notions throughout Chapter 1 by working out the example of the Cantor string. In this example, we see that the various notions that we have introduced are described by the complex dimensions of the Cantor string. In higher dimensions, a similar example is provided by the Cantor sprays.

This theory of complex dimensions sheds new light on, and is partly motivated by, the earlier work of the first author in collaboration with C. Pomerance and H. Maier (see [LapPo2] and [LapMa2]). In particular, the heuristic notion of complex dimension suggested by the methods and results of [Lap1–3, LapPo1–3, LapMa1–2, HeLap1–2] is now precisely defined and turned into a useful tool.

In Chapters 2 and 3, we make an extensive study of the complex dimensions of self-similar strings, which form an important subclass of fractal strings. This amounts to studying the zeros of the function

\[ f(s) = 1 - r_1^s - r_2^s - \cdots - r_N^s \quad (s \in \mathbb{C}), \]

for a given set of real numbers \( r_j \in (0, 1), j = 1, \ldots, N, N \geq 2 \). We introduce the subclass of lattice self-similar strings, and find a remarkable difference between the complex dimensions of lattice and nonlattice self-similar

---

1The eigenvalues of the Dirichlet Laplacian \(-d^2/dx^2\) on this set are the numbers \( \lambda = \pi^2k^2l_j^{-2} \) \((k, j \in \mathbb{N}^*)\). The (normalized) frequencies of \( L \) are the numbers \( \sqrt{\lambda}/\pi \).
strings. In the lattice case, each number $r_j$ is a positive integral power of one fixed real number $r \in (0, 1)$. Then $f$ is a polynomial in $r^s$, and its zeros lie periodically on finitely many vertical lines. The Cantor string is the simplest example of a lattice self-similar string, and we refer to Section 2.3 for additional examples. In contrast, the complex dimensions of a nonlattice string are apparently randomly distributed in a vertical strip. In Chapter 3, however, we show that these complex dimensions are approximated by those of a sequence of lattice strings. Hence, they exhibit a quasiperiodic behavior (see Theorems 2.17 and 3.6, along with Section 3.4). On page 52, Figure 2.12, the reader finds a diagram of the complex dimensions of the golden string, one of the simplest nonlattice self-similar strings. This and other examples are discussed in Section 2.3, and many other such examples are discussed in more detail in Chapter 3. In fact, much of Chapter 3 is devoted to the careful study (via Diophantine approximation techniques) of the beautiful and intriguing quasiperiodic patterns of the complex dimensions of nonlattice strings, both rigorously and computationally. We also obtain in that chapter specific dimension-free regions of the complex plane for nonlattice self-similar strings (see Section 3.6). When combined with our explicit formulas from later chapters, this will enable us, in particular, to give good estimates for the error term in the resulting asymptotic formulas, depending on the Diophantine properties of the scaling ratios.

Chapters 4, 5 and 6 are devoted to the development of the technical tools needed to extract geometric and spectral information from the complex dimensions of a fractal string. In Chapter 4, we introduce the framework in which we will formulate our results, that of generalized fractal strings. These do not in general correspond to a geometric object. Nevertheless, they are not just a gratuitous generalization since they enable us, in particular, to deal with virtual geometries and their associated spectra—suitably defined by means of their zeta functions—as though they arose from actual fractal geometries. In Chapters 9, 10 and 11, the extra flexibility of this framework allows us to study the zeros of several classes of zeta functions.

The original explicit formula was given by Riemann [Rie1] in 1858 as an analytical tool to understand the distribution of primes. It was later extended by von Mangoldt [vM1–2] and led in 1896 to the first rigorous proof of the Prime Number Theorem, independently by Hadamard [Had2] and de la Vallée Poussin [dV1] (as described in [Edw]). Writing $f(x)$ for the function $= \sum_{p^n \leq x} \frac{1}{n}$ that counts prime powers $p^n \leq x$ with a weight $1/n$, the explicit formula of Riemann is (see [Edw, p. 304 and Section 1.16])

$$f(x) = \text{Li}(x) - \sum_{\rho} \text{Li}(x^\rho) + \int_x^\infty \frac{1}{t^2 - 1} \frac{dt}{t \log t} - \log 2,$$

where the sum is over all zeros $\rho$ of the Riemann zeta function, taken in order of increasing absolute value, and $\text{Li}(x)$ is the logarithmic integral $\int_0^x dt/(\log t)$ (see (5.78)). For numerical purposes, the left-hand side of
Riemann’s explicit formula is easy to compute. For theoretical purposes, however, the right-hand side is more useful. For example, the Prime Number Theorem $f(x) = \text{Li}(x)(1 + o(1))$ as $x \to \infty$,\(^2\) which was the primary motivation for Riemann’s investigations, follows if all zeros satisfy $\Re \rho < 1$, that is, the function $\log \zeta(s)$ has a singularity at $s = 1$ but no other singularities on the line $\Re s = 1$.

An example of an explicit formula in our theory is formula (***) below, which expresses the volume of the tubular neighborhoods $V(\varepsilon)$ of a fractal string as an infinite sum of oscillatory terms $\varepsilon^{1-\omega}$, where $\omega$ runs over the complex dimensions of the fractal string (and hence $1 - \omega$ runs over the complex codimensions). Much like Riemann, we use that formula to show that $V(\varepsilon) = \varepsilon^{1-D}(v + o(1))$ as $\varepsilon \to 0^+$, for some constant $v$ which is related to the $D$-dimensional volume of the fractal, if and only if there are no nonreal complex dimensions on the line $\Re \omega = D$. Here, $D$ denotes the Minkowski dimension of the fractal boundary of the string.

In Chapter 5, we state and prove our explicit formulas, which are the basic tools for obtaining asymptotic expansions of geometric, spectral or dynamical quantities associated with fractals. Our first explicit formula, which expresses the counting function of the lengths as a sum of oscillatory terms and an error term of smaller order, is only applicable under fairly restrictive assumptions. To obtain a more widely applicable theory, we show in Section 5.4 that this same function, interpreted as a distribution, is given by the same formula, now interpreted distributionally. The resulting distributional formula with error term is applicable under mild assumptions on the analytic continuation of the geometric zeta function. We also obtain a pointwise and a distributional formula without error term, which exists only for the geometry of a smaller class of fractal strings, including the self-similar strings and the so-called prime string. In Section 5.5, we use this analysis of the prime string to give a proof of the Prime Number Theorem.

We note that our explicit formulas are close relatives of—but are also significantly more general than—the usual explicit formulas encountered in number theory [Edw, In, Pat]. (See the end of Section 5.1 and the notes in Section 5.6 for further discussion and additional references.)

In the subsequent chapters, we investigate the geometric, dynamical, and spectral information contained in the complex dimensions. The main theme of these chapters is that the oscillations in the geometry or in the spectrum of a fractal string are reflected in the presence of oscillatory terms in the explicit formulas associated with the fractal string.

In Chapter 6, we work out the necessary computations to find the oscillatory terms in the explicit formulas of a fractal string. We define the spectral operator, which relates the spectrum of a fractal string with its

\(^2\)We have $\text{Li}(x) = \frac{x}{\log x}(1 + o(1))$ as $x \to \infty$, but the approximation $\text{Li}(x)$ for $f(x)$ is better than $\frac{x}{\log x}$.
geometry and obtain an Euler product representation for it, which provides a counterpart in this context to the usual Euler product expansion for the Riemann zeta function, but is convergent in the critical strip $0 < \text{Re } s < 1$. We also illustrate our results by considering a variety of examples of geometric and direct spectral problems. We study the geometric and spectral counting functions as well as the geometric and spectral partition functions of fractal strings. In particular, we analyze in detail the geometry and the spectrum of self-similar strings, both in the lattice and the nonlattice case.

We show in Chapter 7 that the geometric zeta function of a self-similar string coincides with a suitably defined dynamical (or Ruelle) zeta function, and hence that it admits an appropriate Euler product; see Section 7.2. For example, the zeta function of the self-similar flow (or dynamical system) with weights $w = w_1, w_2, \ldots, w_N$ is given by

$$\zeta_w(s) = \frac{1}{1 - \sum_{j=1}^{N} e^{-w_j s}},$$

which is the geometric zeta function of a self-similar fractal string with scaling ratios $r_j = e^{-w_j}$ $(j = 1, \ldots, N)$ and a single gap (see Chapter 2). The connection in this dynamical context with Marc Frantz’s more general self-similar strings with multiple gaps still remains to be clarified.\(^3\) We apply the theory of Chapters 5 and 6 to obtain a suitable explicit formula and an associated Prime Orbit Theorem for self-similar flows: the function which counts primitive periodic orbits with their weight has the asymptotic expansion

$$\psi_w(x) = G(x) \frac{x^D}{D} + \sum_{\text{Re } \omega < D} \frac{x^\omega}{\omega} + R(x),$$

where the sum is over all the (dynamical) complex dimensions of the flow, repeated according to their multiplicity, and $R(x) = O(1)$ as $x \to \infty$ (see Section 7.4). Here, $G(x) = 1$ in the nonlattice case, when $D$ is the only complex dimension with real part $D$, and $G(x)$ is multiplicatively periodic in the lattice case. As was alluded to earlier, using the results of Section 3.6 on dimension-free regions, we deduce from this formula a prime orbit theorem \textit{with error term}. We may analyze this error term (beyond the leading term) in terms of the Diophantine properties of the weights $w_j$. For example, self-similar flows with weights that are badly approximable by rationals have a larger dimension-free region and hence a better (i.e., smaller) error term in the above asymptotic formula. (See Section 7.5.)

\(^3\)Such self-similar strings, the boundary of which corresponds to self-similar sets in $\mathbb{R}$, were introduced in [Fra1, 2] after the publication of [Lap-vF5], where the prototypical case of a single gap was studied. They are discussed in Chapter 2. With the exception of Chapter 7, our theory can be applied immediately to this more general setting.
Analogous comments hold with regard to all our explicit formulas when they are applied in the self-similar case.

In Chapter 8, we derive an explicit formula for the volume \( V(\varepsilon) \) of the inner\(^4\) \( \varepsilon \)-neighborhood of the boundary of a fractal string. For example, when the complex dimensions of \( \mathcal{L} \) are simple, we obtain the following key formula:

\[
V(\varepsilon) = \sum_{\omega} c_{\omega} \frac{(2\varepsilon)^{1-\omega}}{\omega(1-\omega)} + R(\varepsilon),
\]

where \( \omega \) runs over the complex dimensions of the fractal string \( \mathcal{L} \), \( c_{\omega} \) denotes the residue of \( \zeta_{\mathcal{L}}(s) \) at \( s = \omega \), and \( R(\varepsilon) \) is an error term of lower order. Formula (**) yields a new criterion for the Minkowski measurability of a fractal string in terms of the absence of nonreal complex dimensions with real part \( D \), the dimension of the string. (See Section 8.3.) This extends the joint work of the first author with C. Pomerance [LapPo1; LapPo2, Theorem 2.2], in which a characterization of Minkowski measurability was obtained in terms of the absence of geometric oscillations in the string. A comparison, in Section 8.2, of our formula with Hermann Weyl’s formula for tubes in Riemannian geometry [BergGo, p. 235] suggests what kind of geometric information may be contained in the complex dimensions of a fractal string.

The last part of Chapter 8 (Section 8.4) is devoted to a detailed discussion of the tube formulas for the class of self-similar strings, both in the lattice and nonlattice case. The tube formula (**) takes a particularly concrete form for lattice strings, while the error term \( R(\varepsilon) \) in (**) can be estimated by using some of the results of Chapter 3. In particular, it follows that a self-similar string is Minkowski measurable if and only if it is nonlattice.

In Chapters 9, 10, and 11, we shift the emphasis from the geometry of fractal strings to the relationship between the geometry and the spectrum of fractal strings.

In Chapter 9, we study the inverse spectral problem, the problem of deducing geometric information from the spectrum of a fractal string: Does the absence of oscillations in the spectrum of a fractal string imply the absence of oscillations in its geometry? In other words, we consider the question (à la Mark Kac [Kac]) “Can one hear the shape of a fractal string?” This inverse spectral problem has been considered before by the first author jointly with H. Maier in [LapMa1–2], where it was shown that the

\[^4\]We use the inner tubular neighborhood (see Equation (1.3) in Chapter 1) so that \( V(\varepsilon) \) does not depend on the placement of the lengths \( l_j \) in space (the geometric realization of the fractal string). Likewise, the Minkowski dimension and Minkowski content are independent of the geometric realization of the fractal string, as opposed to the Hausdorff dimension and measure. This is the main reason why in [Lap1, LapPo2], the Minkowski dimension is used as well.
audibility of oscillations in the geometry of a fractal string of (Minkowski) dimension $D \in (0, 1)$ is equivalent to the absence of zeros of the Riemann zeta function $\zeta(s)$ on the line $\text{Re } s = D$. In our framework, this becomes the question of inverting the spectral operator. We deduce, in particular, that the spectral operator is invertible for all fractal strings of dimension $D \neq \frac{1}{2}$ if and only if the Riemann hypothesis holds, i.e., if and only if the Riemann zeta function $\zeta(s)$ does not vanish if $\text{Re } s \neq \frac{1}{2}$, $\text{Re } s > 0$.

By considering (generalized) fractal sprays, instead of fractal strings, we extend the above criterion for zeros of $\zeta(s)$ in the critical strip to a large class of zeta functions, including all those for which the analogue of the generalized Riemann hypothesis is expected to hold. We thus characterize the generalized Riemann hypothesis as a natural inverse spectral problem for fractal sprays. In addition to the Epstein zeta functions, this class includes all Dedekind zeta functions and Dirichlet $L$-series, and more generally, all Hecke $L$-series associated with an algebraic number field. It also includes all zeta functions associated with algebraic varieties over a finite field. We refer the interested reader to Appendix A for a brief review of such number-theoretic zeta functions.

In Chapter 10, we make an extensive study of the geometry and the spectrum of generalized Cantor strings. The complex dimensions of such strings form an infinite sequence in vertical arithmetic progression, with real part the Minkowski dimension $D$ of the string. We show that these strings always have oscillations of order $D$ in both their geometry and their spectrum. In Chapter 11, we deduce from this result that the explicit formulas for the geometry and the spectrum of Cantor strings always contain oscillatory terms of order $D$. On the other hand, if $\zeta(s)$ had a vertical arithmetic progression of zeros coinciding with the arithmetic progression of complex dimensions of $L$, then, by formula $(\ast)$, the explicit formula for the frequencies would only contain the term corresponding to $D$, and not any oscillatory term. Thus we prove that $\zeta(s)$ does not have such a sequence of zeros. This theorem was first obtained by Putnam [Pu1, 2] in 1954. However, his methods do not apply to prove the extension to more general zeta functions. We also apply this idea in Section 11.1.1 to the geometry and the spectrum of the truncated generalized Cantor strings, to deduce an explicit upper bound on the maximal possible length of a vertical arithmetic sequence of zeros.

By considering (generalized) Cantor sprays, we extend this result to a large subclass of the aforementioned class of zeta functions. This class includes all the Dedekind and Epstein zeta functions, as well as many Dirichlet series not satisfying a functional equation. It does not, however, include the zeta functions associated with varieties over a finite field, for which this result does not hold (this is explained in Section 11.5). Indeed, we show that every Dirichlet series with positive coefficients and with only finitely many poles has no infinite sequence of zeros forming a vertical arithmetic
progression. In Section 11.4.1, we present Mark Watkins’ extension to finite shifted arithmetic progressions.

We conclude this book with a chapter of a more speculative nature, Chapter 12, in which we make several suggestions for the direction of future research in this area. Our results suggest that important information about the fractality of a string is contained in its complex dimensions. In Section 12.2, we propose as a new definition of fractality the presence of at least one nonreal complex dimension with positive real part. In this new sense, every self-similar set in the real line is fractal. On the other hand, in agreement with geometric intuition, certain compact subsets of $\mathbb{R}$, associated with the so-called $a$-string, are shown here to be nonfractal, whereas they are fractal according to the definition of fractality based on the notion of Minkowski dimension. We suggest one possible way of defining the complex dimensions of higher-dimensional fractals, and we discuss the examples of the Devil’s staircase and of the snowflake drum (see Figures 12.1 and 12.6). In particular, the Devil’s staircase is not fractal according to the traditional definition based on the Hausdorff dimension. However, there is general agreement among fractal geometers that it should be called fractal. (See [Man1, p. 84].) We show that our new definition of fractality does indeed resolve this problem satisfactorily. In spite of this positive outcome, we stress that the theory of the complex dimensions of higher-dimensional fractals still needs to be further developed.

The first steps towards such a theory are discussed in Sections 12.3.1 and 12.3.2. In Section 12.3.1 (based on [LapPe1]), a tube formula for the Koch snowflake curve is given and the corresponding complex dimensions are inferred, by analogy with formula (**). In Section 12.3.2, we briefly discuss aspects of a work in progress ([Pe], [LapPe2–4]) where a theory of complex dimensions of self-similar fractals (and tilings) is developed.

In Section 12.4, we discuss two types of extensions of the main framework of this book, that of fractal strings. Namely, in Section 12.4.1—based on the paper [HamLap]—we give an overview of some of the main results on random fractal strings, such as random self-similar strings and stable random strings (which, in a special case, have for boundary the zero set of Brownian motion), and their associated complex dimensions. Moreover, in Section 12.4.2, we discuss the notion of fractal membrane (or quantized fractal string) introduced in [Lap10] and further developed in [LapNes1, 2].

In Sections 12.2 through 12.7, we discuss several conjectures and open problems regarding possible extensions and geometric, spectral, or dynamical interpretations of the present theory of complex dimensions, both for fractal strings and their higher-dimensional analogue, fractal drums, for which much research remains to be carried out in this context. We explain the connection with geometries over finite fields in Section 12.7.1. We also propose in Section 12.7.2 to develop a suitable fractal cohomology theory in the context of the theory of self-similar strings and their associated complex dimensions.
1
Complex Dimensions of Ordinary Fractal Strings

In this chapter, we recall some basic definitions pertaining to the notion of (ordinary) fractal string and introduce several new ones, the most important of which is the notion of complex dimension. We also give a brief overview of some of our results in this context by discussing the simple but illustrative example of the Cantor string. In the last section, we discuss fractal sprays, which are a higher-dimensional analogue of fractal strings.

1.1 The Geometry of a Fractal String

A (standard or ordinary) fractal string $\mathcal{L}$ is a bounded open subset $\Omega$ of $\mathbb{R}$. It is well known that such a set consists of countably many open intervals, the lengths of which will be denoted by $l_1, l_2, l_3, \ldots$, called the lengths of the string. Note that $\sum_{j=1}^{\infty} l_j$ is finite and equal to the Lebesgue measure of $\Omega$. From the point of view of this work, we can and will assume without loss of generality that

$$l_1 \geq l_2 \geq \cdots > 0,$$  \hspace{1cm} (1.1)

where each length is counted according to its multiplicity. We allow for $\Omega$ to be a finite union of open intervals, in which case the sequence of lengths is finite.

An ordinary fractal string can be thought of as a one-dimensional drum with fractal boundary. Actually, we have given here the usual terminology that is found in the literature. A different terminology may be more sug-