Exercises in Environmental Physics

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To Louine and Donovan

Contents

Preface

The study of environmental physics requires understanding topics from many different areas of physics as well as comprehension of physical aspects of the world around us. Several excellent textbooks are available covering most aspects of environmental physics and of applications of physics to the natural environment from various points of view. However, while teaching environmental physics to university students, I sorely missed a book specifically devoted to exercises for the environmental science student. Thus, the motivation for this book came about as in physics, as well as in many other disciplines, satisfactory knowledge of a subject cannot be acquired without practice. Usually students are not familiar with the various areas of physics that are required to describe both the environment and the human impact upon it. At the same time, students need to develop skills in the manipulation of the ideas and concepts learned in class. Therefore, this exercise book is addressed to all levels of university students in environmental sciences.

Because of the wide range of potential users this book contains both calculus-based and algebra-based problems ranging from very simple to advanced ones. Multiple solutions at different levels are presented for certain problems—the student who is just beginning to learn calculus will benefit from the comparison of the different methods of solution. The material is also useful for courses in atmospheric physics, environmental aspects of energy generation and transport, groundwater hydrology, soil physics, and ocean physics, and selected parts may even be used for basic undergraduate physics courses. This collection of exercises is based on courses taught at the University of Northern British Columbia and at the University of Victoria, Canada.

Each problem and its solution are self-contained so that they can be attempted or assigned independently. For students willing to deepen their knowledge of the subject, references to the literature are sometimes given in the text or the solution of the problems.

The problems are arranged by topic, although problems usually overlap two or more different categories. This should make the students aware of the fact that problems of the environment—even relatively simple ones—often involve different areas and require various techniques in an interdisciplinary approach. This is even more true for the complex problems that the environmental scientist encounters daily. To put it in John Muir's words, "When we try to pick out anything by itself, we find it hitched to everything else in the universe" [53].

This book is not comprehensive: covering the complete spectrum of topics in environmental physics would require a monumental work and most readers would have little appreciation for the more specialized topics. Many books or review papers on specific topics exist and they sometimes include exercises, but they are often too detailed for the purpose of a general course in environmental physics. The selection of topics contained in this book is to a certain extent arbitrary, as is the choice of subjects presented in most courses in environmental physics currently taught in university. However, I do believe that the essential topics common to any general environmental physics course are covered here. Rather than presenting exercises on the plethora of empirical formulas appearing in the literature on the various areas of environmental physics, the focus is on the unifying physical principles that can be applied to many different subjects.

How to Use This Book

The International System of units (*SI* system) is used in this book. Exercises are labeled with the letters **A**, **B**, or **C**. **A** denotes lower mathematical level (algebra-based) problems that can be solved without knowledge of calculus; whereas **B** indicates higher mathematical level problems usually requiring calculus for their solution. The letter **C** denotes conceptual questions that do not require calculations—these are inserted at the beginning of each chapter in lieu of lengthy review sections. Problems labeled **A** or **C** are not necessarily the easiest just because no calculus is required: they test the student's understanding and knowledge of the physical concepts and normally require more than just common sense for their solution.

The student should not browse through the solution before a problem has been attempted and a honest effort has been made to solve it. If a problem cannot be solved in spite of serious and repeated effort, the student should not be frustrated but should read and understand the solution and then review and correct his or her knowledge of the subject.

This is what exercises are for, after all, and the student will certainly learn from this process. Many exercises in the book require a sound mathematical background, and Chapter 1 reviews basic mathematical techniques. The section on vector calculus is particularly important to solve exercises that require the use of the transport equations.

First-year students may benefit from reading a general qualitative book such as Refs. [64, 60, 46] before delving into the details of specific areas of environmental physics. References [27, 28] contain entertaining and instructive solutions to selected problems using simplified quantitative models—for an advanced reference on environmental modeling, see Ref. [72]. Suggested readings are given at the beginning of each chapter or section. A recommendation for students just beginning in science and to whom many of these exercises are addressed: the problems should be solved using symbols for the physical quantities considered and the numerical values should only be inserted at the end of the mathematical calculations. It is strongly recommended to insert the corresponding units together with the numerical values of the various quantities, and to pay attention to the number of significant digits.

I have tried as much as possible to eliminate errors from the book, but I shall be grateful to readers informing of any errors that they may notice.

Lennoxville, Québec

March 2006

VALERIO FARAONI

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Chapter 1

MATHEMATICAL METHODS

A part of the secret of analysis is the art of using notations well.
—Gottfried Wilhelm Leibnitz

Environmental problems are often posed in the context of data collection and statistics, and extensive discussion of the social, economic, and legal aspects of environmental science is certainly required. However, it is not sufficient to talk about environmental problems or to collect data and make statistics. To begin analyzing and finding solutions to problems in environmental physics requires a precise formulation in mathematical terms, and the methods of mathematical physics are widely used. Many problems—even if well-posed mathematically—are too difficult to solve because of the complexity resulting from interdisciplinarity and because of their intrinsic nonlinearity. As a result, simplified models are often employed.

Mathematical modeling is an art in which one needs to capture the essential features of the phenomenon under study, yet keep the model sufficiently simple so that it is useful. Complications and details can be added later by modifying a model that has provided physical insight, and observational data and statistics are required in order to formulate the necessary boundary and initial conditions. One ends up using approximations, which are usually found on the basis of physical intuition rather than mathematical convenience, although sometimes the temptation to kill complicated terms in the equations has led to meaningful approximations. The assumptions of the model, however, should not oversimplify—the old adagio applies: no model is better than its assumptions.

Environmental science takes advantage of virtually every mathematical tool developed—here we review the basic mathematical concepts used in the solution of the exercises of this book.

1.1 Complex numbers

Complex numbers are used to describe physical systems ruled by linear differential equations, to represent physical quantities with Fourier series and Fourier integrals, to compute definite integrals of functions of a single variable, in quantum mechanics, fluid dynamics, and in many other applications.

1 **(A)** Solve the complex algebraic equation

$$x + iy + 2 + 3i = 1 - 2i.$$

Solution
This equation can be rewritten as

$$x + iy = -1 - 5i$$

and, by equating the real (respectively, imaginary) part of the left-hand side to the real (respectively, imaginary) part of the right-hand side, we obtain the complex solution $z = -1 - 5i$.

2 **(A)** Solve the complex algebraic equation

$$z^2 - i = 0.$$

Solution
One can rewrite this equation using the polar form of $i = \cos(\pi/2) + i\sin(\pi/2) = \mathrm{e}^{i\pi/2}$ as

$$z^2 = i = \mathrm{e}^{i(\pi/2+2n\pi)} \qquad (n = 0, 1, 2, 3, \ldots),$$

which has the two distinct solutions obtained for $n = 0, 1$

$$\begin{aligned} z_{1,2} &= \mathrm{e}^{i(\pi/4+n\pi)} = \left[\cos\left(\frac{\pi}{4} + n\pi\right) + i\sin\left(\frac{\pi}{4} + n\pi\right)\right] \\ &= \pm\frac{\sqrt{2}}{2}(1 + i). \end{aligned}$$

3 **(A)** What regions of the complex plane correspond to the following?
a) $|z| < 1$

b) $\text{Re}(z) > 3$
c) $\text{Im}(z) > 2$
d) $|z + 5| \leq 1$
e) $-1 \leq \text{Im}(z) \leq 1$
f) $2 < |z| < 3$

Solution
Let $z = x + iy$, where $x = \text{Re}(z)$ and $y = \text{Im}(z)$ are real. Then:
a) represents a circle of unit radius centered on the origin $z = 0$ and excluding the circumference of radius $r \equiv \sqrt{x^2 + y^2} = 1$
b) represents the half-plane $x > 3$ with arbitrary y
c) represents the half-plane $y > 2$ with x arbitrary
d) represents the circle of unit radius centered on $z = -5$ [or $(x, y) = (-5, 0)$] and including the circumference of unit radius
e) represents the horizontal strip $-1 \leq y \leq 1$ with arbitrary x
f) represents the annulus comprised between the circles of radii 2 and 3 and centered on the origin $z = 0$.

4 **(A)** Express the complex number $z = 1 + i\sqrt{3}$ in polar form.

Solution
The polar form is

$$z = \rho\, \mathrm{e}^{i(\theta + 2n\pi)} \qquad (n = 0, 1, 2, 3, \ldots),$$

where $\rho = |z| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ and $\theta = \text{tg}^{-1}(\sqrt{3}/1) = \pi/3$; hence

$$z = 2\, \mathrm{e}^{i(\pi/3 + 2n\pi)} \qquad (n = 0, 1, 2, 3, \ldots).$$

The argument of z obtained for $n = 0$ is called the *principal argument* of z.

5 **(A)** When studying oscillations of a physical system described by ordinary differential equations, is it always legitimate to represent an oscillating quantity A using a complex exponential as $A = A_0 \exp(i\omega t)$, and to take the real part of A at the end of the calculations as the physical result? If $x(t)$ and $y(t)$ are oscillating quantities represented by complex exponentials, is $\text{Re}(xy) = \text{Re}(x) \cdot \text{Re}(y)$?

Solution
No: the above representation is legitimate only when the oscillating quantity A obeys *linear* differential equations. Often a system described by a set of nonlinear equations may be described by the

linearized version of the full equations under the assumption of small motions or small oscillations (e.g., a simple pendulum), which may constitute a physically meaningful approximation.

If $x(t) = x_0\,\mathrm{e}^{i\omega_1 t}$ and $y(t) = y_0\,\mathrm{e}^{i\omega_2 t}$, then

$$x(t)\,y(t) = x_0 y_0\,\mathrm{e}^{i(\omega_1+\omega_2)t};$$

however,

$$\begin{aligned}\mathrm{Re}\,(xy) &= x_0 y_0 \cos\left[(\omega_1+\omega_2)\,t\right] \\ &= x_0 y_0 \left[\cos(\omega_1 t)\cos(\omega_2 t) - \sin(\omega_1 t)\sin(\omega_2 t)\right] \\ &\neq \mathrm{Re}(x)\cdot\mathrm{Re}(y) = x_0 y_0 \left[\cos(\omega_1 t)\cos(\omega_2 t)\right].\end{aligned}$$

6 **(A)** Prove that a phase factor $\mathrm{e}^{i\theta}$, where θ is real, has unit modulus.

Solution

We have

$$|z| \equiv \mathrm{e}^{i\theta} = |\cos\theta + i\sin\theta| = \left(\cos^2\theta + \sin^2\theta\right)^{1/2} = 1.$$

7 **(A)** Prove that

a) $\mathrm{Re}(z) = \frac{z+z^*}{2}$

b) $\mathrm{Im}(z) = \frac{z-z^*}{2i}$

c) $z^2 = (z^*)^2$ only if z is purely real or purely imaginary.

Solution

Let $z = x + iy$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ are real. Then we have

$$\frac{z+z^*}{2} = \frac{(x+iy)+(x-iy)}{2} = x,$$

$$\frac{z-z^*}{2i} = \frac{(x+iy)-(x-iy)}{2i} = y;$$

the equation $z^2 = (z^*)^2$ is equivalent to

$$(x+iy)^2 = (x-iy)^2,$$

or

$$x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy.$$

Equating the real part of the left-hand side to the real part of the right-hand side and doing the same for the imaginary parts yields $xy = 0$, with solutions $x = 0$, or $y = 0$, or both x and y vanishing.

8 **(A)** Show that there are exactly n distinct roots of a complex number $z \neq 0$.

Solution
Write z in its polar form

$$z = \rho\, \mathrm{e}^{i(\theta + 2k\pi)},$$

where $k = 0, 1, 2, 3, \ldots$ Then

$$z^{1/n} = \rho^{1/n}\, \mathrm{e}^{i\left(\frac{\theta}{n} + \frac{2k}{n}\pi\right)}.$$

The n distinct roots of z are obtained from this formula by letting k assume the n values

$$k = 0, 1, 2, \ldots, (n-1)\,.$$

9 **(B)** Use complex exponentials to derive the trigonometric identities

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\,\cos\theta, \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta. \end{aligned}$$

Solution
The de Moivre formula

$$\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$$

squared yields

$$\mathrm{e}^{2i\theta} = \left(\mathrm{e}^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.$$

On the other hand,

$$\mathrm{e}^{2i\theta} = \cos(2\theta) + i\sin(2\theta)\,;$$

by comparing the two expressions of $\mathrm{e}^{2i\theta}$ one deduces that

$$\cos(2\theta) + i\sin(2\theta) = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta;$$

and by equating the real and the imaginary parts of the two sides of this equation, we obtain

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\,\cos\theta, \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta. \end{aligned}$$

1.2 Differentiation and integration of functions of a single variable

Basic calculus begins by studying functions of a single variable: the most basic operations are taking limits, differentiation, and integration. In the mathematical modeling of a physical process or system one begins by choosing an independent variable and by letting other variables be functions of it—for example, in the problem of motion of a point particle the independent variable can be time and the particle coordinates are dependent variables.

1.2.1 Differentiation

1 **(B)** Compute the first and second derivatives of the function $f(x) = x\,\mathrm{e}^x \ln x + x^3 \sin x$.

Solution
The function $f(x)$ is defined on $(0, +\infty)$ and has derivatives of all orders on this interval. The first derivative is

$$f'(x) = \mathrm{e}^x \ln x + x\,\mathrm{e}^x \ln x + \mathrm{e}^x + 3x^2 \sin x + x^3 \cos x,$$

while the second derivative is

$$\begin{aligned} f''(x) &= \mathrm{e}^x \ln x + \frac{\mathrm{e}^x}{x} + \mathrm{e}^x \ln x + x\,\mathrm{e}^x \ln x + x\,\frac{\mathrm{e}^x}{x} + \mathrm{e}^x + 6x \sin x \\ &\quad +3x^2 \cos x + 3x^2 \cos x - x^3 \sin x \\ &= (x+2)\,\mathrm{e}^x \ln x + \left(\frac{1}{x} + 2\right)\mathrm{e}^x + x\left(6 - x^2\right)\sin x + 6x^2 \cos x. \end{aligned}$$

2 **(B)** Compute the derivative df/dx, where

$$f(x) = \sqrt{\cos\left(\sin^2 x\right)}.$$

Solution
We have

$$\begin{aligned} \frac{df}{dx} &= \frac{d\left(\cos\left(\sin^2 x\right)\right)/dx}{2\sqrt{\cos\left(\sin^2 x\right)}} = \frac{-\sin\left(\sin^2 x\right) d\left(\sin^2 x\right)/dx}{2\sqrt{\cos\left(\sin^2 x\right)}} \\ &= \frac{-\sin x \cos x \sin\left(\sin^2 x\right)}{\sqrt{\cos\left(\sin^2 x\right)}}. \end{aligned}$$

3 **(B)** Compute the derivative df/dx, where

$$f(x) = x^2\,\mathrm{e}^{-2x}\,\frac{3\,\mathrm{e}^{-2x}-1}{\left(3\,\mathrm{e}^{-2x}+1\right)^2}.$$

Solution
We have

$$\begin{aligned}\frac{df}{dx} &= 2x\,\mathrm{e}^{-2x}\,\frac{3\,\mathrm{e}^{-2x}-1}{\left(3\,\mathrm{e}^{-2x}+1\right)^2} - 2x^2\,\mathrm{e}^{-2x}\,\frac{3\,\mathrm{e}^{-2x}-1}{\left(3\,\mathrm{e}^{-2x}+1\right)^2} \\ &\quad +6x^2\,\mathrm{e}^{-4x}\left[\frac{-\left(3\,\mathrm{e}^{-2x}+1\right)+2\left(3\,\mathrm{e}^{-2x}-1\right)}{\left(3\,\mathrm{e}^{-2x}+1\right)^3}\right] \\ &= 2x\,\mathrm{e}^{-2x}\left[\frac{\left(3\,\mathrm{e}^{-2x}-1\right)}{\left(3\,\mathrm{e}^{-2x}+1\right)^2}\,(1-x)+9x\,\mathrm{e}^{-2x}\,\frac{\left(\mathrm{e}^{-2x}-1\right)}{\left(3\,\mathrm{e}^{-2x}+1\right)^3}\right] \\ &= \frac{2x\,\mathrm{e}^{-2x}}{\left(3\,\mathrm{e}^{-2x}+1\right)^3}\left(9\,\mathrm{e}^{-4x}-9x\,\mathrm{e}^{-2x}+x-1\right).\end{aligned}$$

4 **(B)** Compute the derivative of the function

$$f(x) = x\ln\left(3x^4+2x^2+|x|+1\right).$$

Solution
We apply the Leibnitz rule $(fg)' = f'g + fg'$ and the chain rule $\frac{d(f(g(x)))}{dx}dx = \frac{df}{dg}\frac{dg}{dx}$ obtaining, for $x \neq 0$,

$$\begin{aligned}\frac{df}{dx} &= \ln\left(3x^4+2x^2+|x|+1\right)+\frac{x\left(12x^3+4x+\frac{|x|}{x}\right)}{\left(3x^4+2x^2+|x|+1\right)} \\ &= \ln\left(3x^4+2x^2+|x|+1\right)+\frac{12x^4+4x^2+|x|}{3x^4+2x^2+|x|+1}.\end{aligned}$$

This result is obtained for $x \neq 0$, but the function $f(x)$ is defined at $x=0$ and since the two limits

$$\lim_{x\to 0^-}\frac{df}{dx} = \lim_{x\to 0^-}\left[\ln\left(3x^4+2x^2+|x|+1\right)+\frac{12x^4+4x^2+|x|}{3x^4+2x^2+|x|+1}\right]$$

$$= 0,$$

$$\lim_{x \to 0^+} \frac{df}{dx} = \lim_{x \to 0^+} \left[\ln \left(3x^4 + 2x^2 + |x| + 1 \right) + \frac{12x^4 + 4x^2 + |x|}{3x^4 + 2x^2 + |x| + 1} \right]$$

$$= 0,$$

exist and are equal, we conclude that the derivative of $f(x)$ at $x = 0$ exists and is zero.

5 **(B)** Prove that

$$\arcsin x + \arccos x = \frac{\pi}{2}, \qquad -1 \le x \le 1,$$

$$\operatorname{arctg} x + \operatorname{arccotg} x = \frac{\pi}{2}, \qquad -\infty < x < +\infty.$$

Solution
Differentiate $\arcsin x + \arccos x$ in the interval $(-1, 1)$:

$$\frac{d}{dx} (\arcsin x + \arccos x) = \frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - x^2}} = 0,$$

hence $\arcsin x + \arccos x = \text{const.}$ in $(-1, 1)$ and, by continuity, also in $[-1, 1]$. At $x = 1$ we have $\arcsin 1 + \arccos 1 = \pi/2$, which fixes the value of the constant. Hence $\arcsin x + \arccos x = \pi/2$ in $[-1, 1]$.

Let us consider the function $\operatorname{arctg} x + \operatorname{arccotg} x$ on the real axis: differentiating in this interval we obtain

$$\frac{d}{dx} (\operatorname{arctg} x + \operatorname{arccotg} x) = \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0$$

and hence $\operatorname{arctg} x + \operatorname{arccotg} x$ is constant. Since at $x = 1$

$$\operatorname{arctg} 1 + \operatorname{arccotg} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2},$$

the value of the constant is fixed and $\operatorname{arctg} x + \operatorname{arccotg} x = \pi/2$ over the entire real axis.

6 **(B)** Prove the identities

$$\operatorname{arctg} x + \operatorname{arctg} \left(\frac{1}{x} \right) = \frac{\pi}{2} \qquad (x > 0),$$

$$\operatorname{arctg} x + \operatorname{arctg} \left(\frac{1}{x} \right) = -\frac{\pi}{2} \qquad (x < 0).$$

Solution
The function $\operatorname{arctg} x + \operatorname{arctg}\left(\frac{1}{x}\right)$ is singular at $x = 0$ and therefore we have to consider separately the two semi-infinite intervals $x < 0$ and $x > 0$. Differentiation yields

$$\frac{d}{dx}\left[\operatorname{arctg} x + \operatorname{arctg}\left(\frac{1}{x}\right)\right] = \frac{1}{1+x^2} + \frac{1}{1+1/x^2}\left(\frac{-1}{x^2}\right) = 0$$

for any $x \neq 0$. Hence the function $\operatorname{arctg} x + \operatorname{arctg}\left(\frac{1}{x}\right)$ is constant, but the value of the constant is different in the two disconnected intervals $x < 0$ and $x > 0$. In fact, for $x = -1$ it is

$$\operatorname{arctg}(-1) + \operatorname{arctg}(-1) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2},$$

while for $x = 1$ it is

$$\operatorname{arctg} 1 + \operatorname{arctg} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2},$$

which fixes the values of the constants.

7 **(B)** Determine whether there exist values of α and β such that the curves representing the two functions

$$\begin{aligned} f(x) &= \frac{3\alpha}{4}x^4 + \beta x^2, \\ g(x) &= x^2 + 3, \end{aligned}$$

have parallel tangents at some point, and find the values of x for which this happens.

Solution
The functions f and g are continuous with all their derivatives of any order on $(-\infty, +\infty)$. The points with the desired property are those where the first derivatives of f and g are equal, i.e., where

$$3\alpha x^3 + 2\beta x = 2x$$

or

$$x\left[3\alpha x^2 + 2(\beta - 1)\right] = 0.$$

The point $x = 0$ has the desired property for any value of α and β: the tangent to both curves representing $f(x)$ and $g(x)$ is horizontal here.

If $\alpha = 0$, one finds immediately that setting $\beta = 1$ all points x have the desired property.

If $\alpha \neq 0$, then the points x with the desired property satisfy the equation

$$x^2 = -\frac{2}{3\alpha}(\beta - 1);$$

this equation has solutions for $\alpha < 0$ and $\beta \geq 1$, or for $\alpha > 0$ and $\beta \leq 1$; the points x with the desired property are

$$x = \pm\sqrt{\frac{2}{3}\left|\frac{\beta - 1}{\alpha}\right|}.$$

To summarize, the values of α and β that allow for the desired property are
any (α, β) and $x = 0$,
$(\alpha, \beta) = (0, 1)$ and any x,
(α, β) with $\alpha < 0$ and $\beta \geq 1$, and $x = \pm\sqrt{\frac{2}{3}\left|\frac{\beta-1}{\alpha}\right|}$,
(α, β) with $\alpha > 0$ and $\beta \leq 1$, and $x = \pm\sqrt{\frac{2}{3}\left|\frac{\beta-1}{\alpha}\right|}$.

1.2.2 Integration

1 **(B)** Compute the indefinite integral

$$\int dx\, \left(x\,\mathrm{e}^x + 3x^2\right).$$

Solution
Because of the linearity of the integral, we have

$$\int dx\, \left(x\,\mathrm{e}^x + 3x^2\right) = \int dx\, x\,\mathrm{e}^x + 3\int dx\, x^2.$$

The first integral on the right-hand side is evaluated by parts, obtaining

$$\int dx\, x\,\mathrm{e}^x = x\,\mathrm{e}^x - \int dx\,\mathrm{e}^x = (x - 1)\,\mathrm{e}^x.$$

As a check, one can take the derivative of this last term,

$$\frac{d}{dx}\left[(x - 1)\,\mathrm{e}^x\right] = \mathrm{e}^x + (x - 1)\,\mathrm{e}^x = x\,\mathrm{e}^x,$$

which assures us of the correctness of this first integral. The second integral is elementary,

$$\int dx\, x^2 = \frac{x^3}{3},$$

and therefore we have

$$\int dx\, \left(x\, e^x + 3x^2\right) = x\, e^x + x^3 + \text{constant}.$$

2 **(B)** Compute the definite integrals

$$I_1 = \int_{-\infty}^{+\infty} dx\, f(x)\, x,$$

$$I_2 = \int_{-\infty}^{+\infty} dx\, f(x)\, x^3,$$

$$I_3 = \int_{-\infty}^{+\infty} dx\, g(x)\, x^2,$$

$$I_4 = \int_{-\infty}^{+\infty} dx\, g(x)\, x^8,$$

where the functions $f(x)$ and $g(x)$ are defined and regular over the entire real axis and are, respectively, even and odd, i.e., $f(-x) = f(x)$ and $g(-x) = -g(x)$ for any real value of x.

Solution
We have

$$I_1 = I_2 = I_3 = I_4 = 0$$

because in all these cases the integrand is an odd function of x and the integrals are computed over an interval symmetric with respect to $x = 0$ (the entire real axis). The contribution to the integral coming from regions with $x < 0$ cancels the corresponding contribution, with opposite sign, from symmetric regions with $x > 0$.

3 **(B)** Compute the integral

$$\int_{1}^{+\infty} dx\, \frac{1}{x\,(x+1)}.$$

Solution
We decompose the fraction in the integrand as follows:

$$\frac{1}{x\,(x+1)} = \frac{A}{x} + \frac{B}{x+1},$$

where the constants A and B are determined by writing the two terms on the right-hand side with common denominator

$$\frac{A}{x} + \frac{B}{x+1} = \frac{(A+B)\,x + A}{x\,(x+1)}$$

and setting this equal to $1/x(x+1)$, which yields

$$A + B = 0,$$

$$A = 1,$$

or $(A, B) = (1, -1)$. Therefore,

$$\begin{aligned}
\int_1^{+\infty} dx\, \frac{1}{x\,(x+1)} &= \int_1^{+\infty} \frac{dx}{x} - \int_1^{+\infty} \frac{dx}{x+1} \\
&= [\ln x - \ln (x+1)]_1^{+\infty} \\
&= \lim_{M\to+\infty} \left[\ln \left(\frac{M}{M+1} \right) - \ln \frac{1}{2} \right] \\
&= \left[\lim_{M\to+\infty} \left(\frac{M}{M+1} \right) \right] + \ln 2 = \ln 2.
\end{aligned}$$

4 **(B)** Consider a river modeled as a straight channel of width a with irregular depth. Using horizontal x- and y- axes pointing in the direction of the flow and in the transversal direction, respectively, the depth profile across the river is given by the function

$$h(y) = \begin{cases} -h_0 \sin\left[\pi\left(1 - \frac{y}{a}\right)\right] & \text{if } 0 \le y < a, \\ 0 & \text{if } y < 0 \text{ or } y \ge a, \end{cases}$$

where h_0 is a constant with the dimensions of a length. Compute the cross-sectional area of the river.

Solution

The area of a cross section of the river is

$$\begin{aligned}
A &= \int_0^a dy\, |h(y)| = -h_0 \int_0^a dy \sin\left[\pi\left(\frac{y}{a} - 1\right)\right] \\
&= h_0\, \frac{a}{\pi} \cos\left[\pi\left(\frac{y}{a} - 1\right)\right]\Big|_0^a = \frac{2h_0 a}{\pi} \simeq 0.6367 h_0 a.
\end{aligned}$$

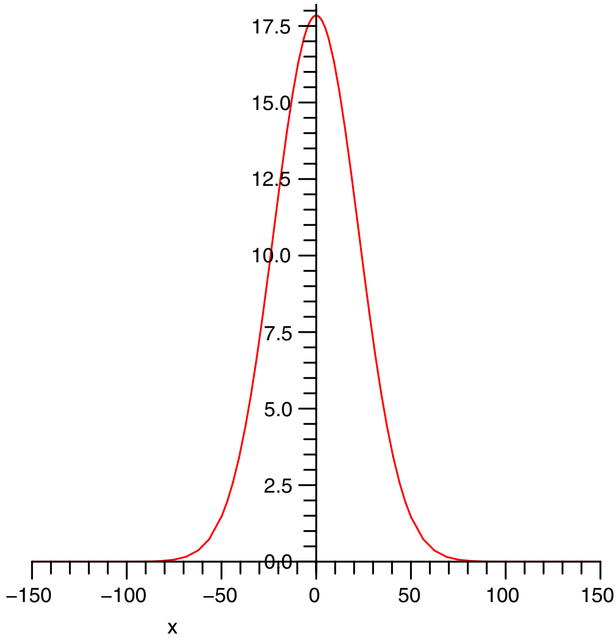

Figure 1.1. The normalized Gaussian (1.2).

5 **(B)** Compute the integral

$$I = \int_{-\infty}^{+\infty} dx\, \mathrm{e}^{-\alpha x^2}, \tag{1.1}$$

which represents the area of the region of plane delimited by the x-axis and by the graph of a Gaussian[1] (Fig. 1.1). Normalize the Gaussian in such a way that

$$f(x) \equiv N\, \mathrm{e}^{-\alpha\, x^2}, \tag{1.2}$$

where N is a constant, satisfies

$$\int_{-\infty}^{+\infty} dx\, f(x) = 1.$$

[1]The Gaussian function is widely used in statistics and in many models (*Gaussian plume models*) describing the spreading of pollutants in water or in the atmosphere.

Solution
Consider the quantity

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{+\infty} dx\, \mathrm{e}^{-\alpha x^2}\right)^2 = \left(\int_{-\infty}^{+\infty} dx\, \mathrm{e}^{-\alpha x^2}\right) \cdot \left(\int_{-\infty}^{+\infty} dy\, \mathrm{e}^{-\alpha y^2}\right) \\ &= \int\int_{R^2} dx\, dy\, \mathrm{e}^{-\alpha\left(x^2+y^2\right)}. \end{aligned}$$

By using polar coordinates (r, φ), where

$$x = r\cos\varphi,$$
$$y = r\sin\varphi,$$

and inserting the Jacobian factor r corresponding to the transformation from Cartesian to polar coordinates $(x, y) \to (r, \varphi)$, we obtain

$$\begin{aligned} I^2 &= \int_0^{+\infty} dr \int_0^{\pi} d\varphi\, r\, \mathrm{e}^{-\alpha r^2} = 2\pi \int_0^{+\infty} dr \left(-\frac{1}{2\alpha}\right) \frac{d}{dr}\left(\mathrm{e}^{-\alpha r^2}\right) \\ &= -\frac{\pi}{\alpha}\left[\mathrm{e}^{-\alpha r^2}\right]_0^{+\infty} = \frac{\pi}{\alpha}, \end{aligned}$$

and therefore

$$I = \sqrt{\frac{\pi}{\alpha}}.$$

In order to find a normalization factor N such that $\int_{-\infty}^{+\infty} dx\, f(x) = 1$, one needs to impose the condition

$$\int_{-\infty}^{+\infty} dx\, f(x) = 1,$$

and hence $N = 1/I$. With the choice $N = \sqrt{\alpha/\pi}$, the normalized Gaussian

$$f(x) = \sqrt{\frac{\alpha}{\pi}}\, \mathrm{e}^{-\alpha x^2}$$

satisfies

$$\int_{-\infty}^{+\infty} dx\, N\mathrm{e}^{-\alpha x^2} = 1.$$

1.2.3 Maxima, minima, and graphs

Calculus allows one to compute maxima, minima, and inflection points, to study the behavior of functions of a single variable and to construct their graphs. In physics, these tools are used to find states of equilibrium, study stability, or optimize choices.

1 **(B)** Study the graph of the function $f(x) = x^2 \ln |x|$.

Solution
The function is defined on $(-\infty, 0) \cup (0, +\infty)$ and is continuous with all its derivatives of any order there. The function is even, i.e., $f(x) = f(-x)$ for all values of x in the intervals on which f is defined. We also notice that $f(x) > 0$ for $|x| > 1$, that $f(x) < 0$ in the intervals $-1 < x < 0$ and $0 < x < 1$, and $f(\pm 1) = 0$. The points $x = \pm 1$ are the only zeros of f.

Let us compute the limits of $f(x)$; as $x \to 0$ we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\ln|x|}{1/x^2} = \lim_{x\to 0} \frac{\frac{1}{|x|}\,\frac{|x|}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0} \frac{-x^2}{2} = 0,$$

by using de l'Hôpital rule. The function $f(x)$ can be redefined so that

$$\tilde{f}(x) \equiv \begin{cases} f(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$. The other limits of $f(x)$ are

$$\lim_{x\to\pm\infty} x^2 \ln|x| = +\infty.$$

The first derivative of the function f is

$$f'(x) = x\left(1 + 2\ln|x|\right),$$

and its sign is determined by studying the sign of $1 + 2\ln|x|$, which is positive for $|x| > \mathrm{e}^{-1/2}$, negative for $-\mathrm{e}^{-1/2} < x < \mathrm{e}^{-1/2}$, and zero at $\pm\mathrm{e}^{-1/2}$. Therefore:
$f'(x) < 0$ and f is strictly decreasing if $x < -1/\sqrt{\mathrm{e}}$ and $0 < x < 1/\sqrt{\mathrm{e}}$;
$f'\left(\pm 1/\sqrt{\mathrm{e}}\right) = 0$ and f has horizontal tangent there;
$f'(x) > 0$ and f is strictly increasing if $-1/\sqrt{\mathrm{e}} < x < 0$ and $x > 1/\sqrt{\mathrm{e}}$.
This information, plus what we know about the continuity of f, is sufficient to establish that $f(x)$ has local and absolute minima at $x = \pm 1/\sqrt{\mathrm{e}}$, and the minimum is $f\left(\pm 1/\sqrt{\mathrm{e}}\right) = -1/2\mathrm{e}$. The graph of the function is reported in Fig. 1.2.

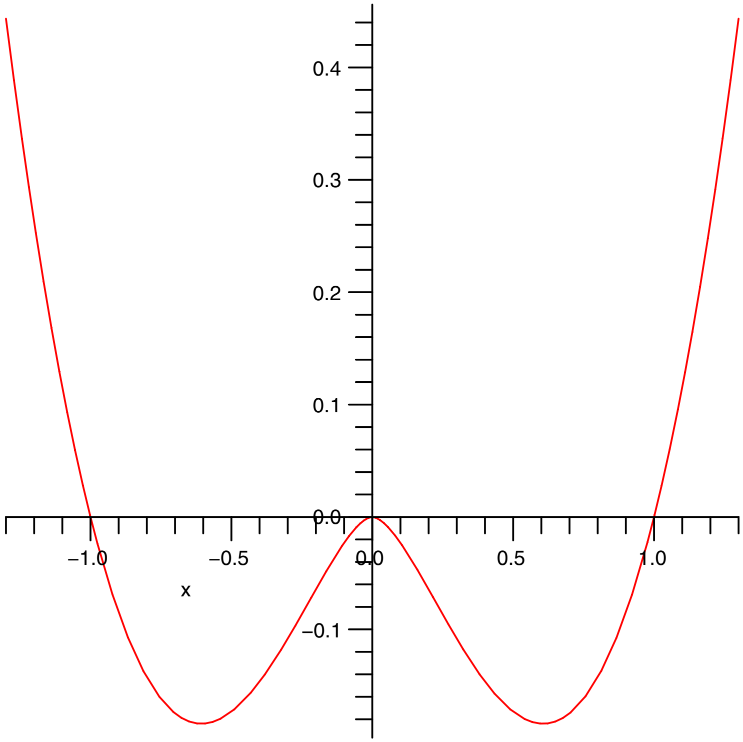

Figure 1.2. The graph of $f(x) = x^2 \ln |x|$.

2 **(B)** It can be shown [12, 27, 4] that the work delivered by a heat engine is

$$f(x) = xa\left(T_H - \frac{T_C}{1-x}\right),$$

where x is the efficiency of the engine and T_H and T_C (with $0 < T_C < T_H$) are the absolute temperatures of the hot and cold reservoir, respectively, while a is a positive constant. Find the efficiency that maximizes the work delivered in the interval[2] $[0, 1 - T_C/T_H]$.

Solution

We look for a maximum of the function $f(x)$ in the efficiency interval $[0, 1 - T_C/T_H]$. The function f is continuous with all its derivatives

[2]Thermodynamics imposes the fundamental upper limit on the efficiency $0 \leq x \leq x_c$, where $x_c \equiv 1 - T_C/T_H < 1$ is the *Carnot factor*.

in this interval and its first derivative is

$$\frac{df}{dx} = a\left(T_H - \frac{T_c}{1-x}\right) - \frac{aT_C x}{(1-x)^2} = \frac{aT_H}{(1-x)^2}\left[(1-x)^2 - \frac{T_C}{T_H}\right].$$

One has $df/dx > 0$ and f strictly increasing if $(1-x)^2 > T_C/T_H$, $df/dx = 0$ (horizontal tangent) if $(1-x)^2 = T_C/T_H$, while $df/dx < 0$ and f is strictly decreasing if $(1-x)^2 < T_C/T_H$. The inequality $(1-x)^2 > T_C/T_H$ corresponds to

$$x < 1 - \sqrt{\frac{T_C}{T_H}} < 1 - \frac{T_C}{T_H},$$

and the above results are sufficient to conclude that $f(x)$ has a local maximum at $x_* = 1 - \sqrt{T_C/T_H}$, which has the value

$$\begin{aligned} f\text{max} &= f\left(1 - \sqrt{\frac{T_C}{T_H}}\right) = x_* a T_H \left(1 - \frac{1}{1-x_*}\frac{T_C}{T_H}\right) \\ &= a\left(1 - \sqrt{\frac{T_C}{T_H}}\right)^2. \end{aligned}$$

Since $f\text{max} > f(0) = f(1 - T_C/T_H) = 0$ and f is continuous on $[0, 1 - T_C/T_H]$, the local maximum is also an absolute maximum.

3 **(B)** Study the graph of the function $f(x) = x\,|x|\,\mathrm{e}^x$.

Solution

The function is defined on $(-\infty, +\infty)$ and is continuous in this interval. All its derivatives exist and are continuous on $(-\infty, 0) \cup (0, +\infty)$. The limits of the function at the boundaries of this interval are

$$\lim_{x\to+\infty} f(x) = +\infty,$$

$$\lim_{x\to-\infty} f(x) = 0.$$

The function is negative for $x < 0$, vanishes only at $x = 0$, and is positive for $x > 0$. The first derivative of $f(x)$ for $x \neq 0$ is

$$f'(x) = |x|\,\mathrm{e}^x\,(x+2)\,.$$

Since both limits

$$\lim_{x\to 0^-} f'(x) = 0,$$

$$\lim_{x\to 0^+} f'(x) = 0$$

exist and are finite and equal, the first derivative of $f(x)$ exists also at $x = 0$ and has zero value.

The study of the sign of $f'(x)$ allows one to conclude that
$f'(x) < 0$ for $x < -2$, where f is strictly decreasing;
$f'(-2) = 0$, where the graph of f has horizontal tangent;
$f'(x) > 0$ for $x > -2$, where $f(x)$ is strictly increasing.
The function f has a local minimum at $x = -2$, which is $f(-2) = -4/\mathrm{e}^2$. This minimum is also an absolute minimum.

The second derivative $f''(x)$ is defined on the set $(-\infty, 0) \cup (0, +\infty)$ and has the value

$$f''(x) = |x|\,\mathrm{e}^x \left(\frac{2}{x} + x + 4\right) :$$

it is not defined at $x = 0$. By studying the sign of $f''(x)$ one concludes that
$f''(x) < 0$ for $x < -\left(2 + \sqrt{2}\right)$ and for $-2 + \sqrt{2} < x < 0$; the graph of the function has concavity facing downward in these intervals.
$f''\left(-2 \pm \sqrt{2}\right) = 0$ and the graph of $f(x)$ changes concavity at $x = -2 \pm \sqrt{2}$.
$f''(x) > 0$ for $-\left(2 + \sqrt{2}\right) < x < -2 + \sqrt{2}$ and for $x > 0$, where the curve representing $f(x)$ has upward-facing concavity. Therefore, the graph of $f(x)$ is as follows: the x-axis is a horizontal asymptote as $x \to -\infty$. Beginning from $x \to -\infty$, the function is negative with downward-facing concavity, decreases until it reaches its absolute minimum at $x = -2$ (changing concavity at $x = -2-\sqrt{2}$ before it reaches its minimum), then it starts increasing and is always strictly increasing for $x > -2$ (it changes concavity again at $x = -2 + \sqrt{2}$ past its minimum point). It reaches its zero at $x = 0$, where the second derivative has a jump discontinuity (from -2 as $x \to 0^-$ to $+2$ as $x \to 0^+$), and diverges as $x^2\,\mathrm{e}^x$ as $x \to +\infty$. The graph is reported in Fig. 1.3.

4 **(B)** Study the function $f(x) = x\,\mathrm{e}^{\lambda x}$ as the real parameter λ varies, and sketch its graph.

Solution
The function is continuous with all its derivatives of any order on $(-\infty, +\infty)$. The sign of $f(x)$ is easy to study—we have, for any real value of the parameter λ:
$f(x) > 0$ if $x > 0$;
$f(x) = 0$ only at $x = 0$;
$f(x) < 0$ if $x < 0$.

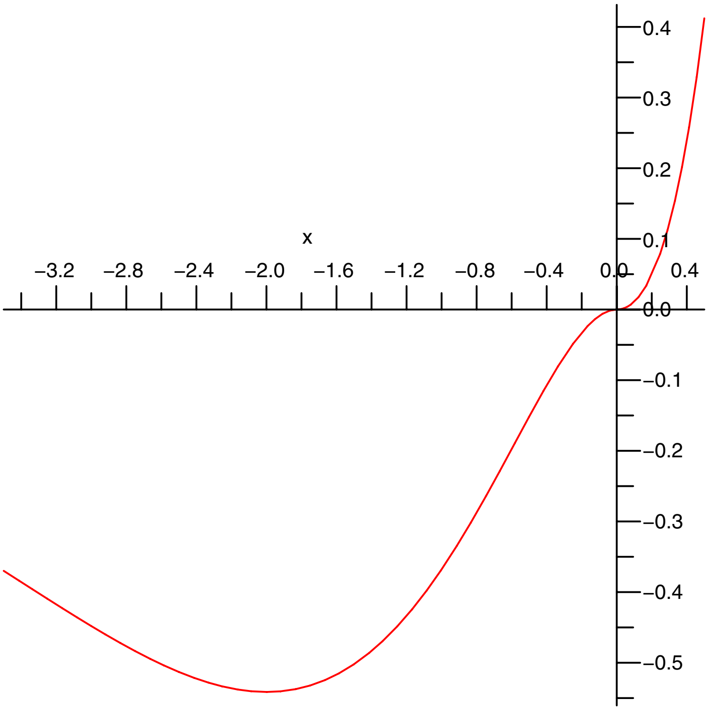

Figure 1.3. The graph of $f(x) = x\,|x|\,\mathrm{e}^{x}$.

The first and second derivatives of f are

$$\begin{aligned} f'(x) &= (\lambda x + 1)\,\mathrm{e}^{\lambda x}, \\ f''(x) &= \lambda\,\mathrm{e}^{\lambda x}\,(\lambda x + 2)\,, \end{aligned}$$

respectively. We now consider the possible values of λ separately. If $\lambda > 0$, the limits of $f(x)$ are

$$\lim_{x\to+\infty} x\,\mathrm{e}^{\lambda x} = +\infty,$$

$$\lim_{x\to-\infty} x\,\mathrm{e}^{\lambda x} = 0.$$

The sign of the first derivative of f is as follows:
$f'(x) < 0$ for $x < -1/\lambda$, where f is strictly decreasing;
$f'\,(-1/\lambda) = 0$ (f has horizontal tangent);
$f'(x) > 0$ for $x > -1/\lambda$, where f is strictly increasing.