Tales of Mathematicians and Physicists

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Translated from the Russian by Alan Shuchat

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Preface to the English Edition

Tales of Physicists and Mathematicians is a translation of a book that was published in Russia in 2001 and is based on articles that appeared from 1960–1980. The first edition of the book, less than half the size of the current one, was published in the Soviet Union in 1981 and in English in 1988. Thus the book has its own history, and I would like to share some of the circumstances under which it appeared to the western reader.

This was a time not only of a surprising flourishing of mathematics in the Soviet Union but also of its surprisingly great prestige in society, perhaps not seen since the time of Plato's Academy in Athens. Mathematics attracted talented youth not only as an area where they could stretch themselves intellectually but also as one that minimized the influence of the official Marxist ideology that deeply penetrated into the lives of the "Soviet people." The profession of scientist, and in particular of mathematician, carried great authority. Here is an interesting observation in this regard. Children of the top Communist elite, including some "members of the Politburo," sometimes chose mathematics or another science as their professions, just as future kings often studied with Plato. Mathematics was lucky: it was never a personal "concern" of Stalin, as were biology, linguistics, and economics, which inevitably led to annihilating, punitive operations against them. In a fantasy of Solzhenitsyn, Stalin looked through a high school mathematics text, choosing the next science to be the subject of his concern. It is hard to imagine what would happen next. The opinion "upstairs" that a high level in the exact sciences was important for the military industry no doubt helped. Gradually, it became the fashion to have mathematicians in any serious organization. Often they enjoyed some freedom, but this is reminiscent of the freedom of the court jester. The comparative idyll between mathematicians and those in power ended in the late 1960s when many mathematicians signed a letter to the government defending their colleague Alexander Esenin-Volpin, who had been sent to a mental hospital for political reasons.

Mathematical life itself was not without clouds. The most violent anti-Semitism was supported not only by bureaucrats who carried out ideological surveillance and did not take part in real scientific work themselves, but also by some leading mathematicians. A distorted system of entrance exams closed off the way to mathematics for many talented people.

In the 1930s, the work of attracting young people to mathematics began to flourish. Mathematics is the unique area of science where children can begin serious work and obtain outstanding results very early. I recall A. N. Kolmogorov's story of how he became interested in mathematics. He said that one should not seriously study mathematics "too early," "not before the age of 12": at an earlier age there are many competing things to do that are less intellectual. Mathematics competitions (olympiads) and clubs (circles) were organized and many interesting books were written. This mainly took place around the universities in Moscow and Leningrad, and both well-known mathematicians and brilliant young university students played the leading role. Some real changes took place in the 1960s. Olympiads began to be held for students from the whole country and mathematics circles were replaced by mathematics high schools, bringing together many children devoted to mathematics who could be taught with an intensiveness and with results not previously seen. In Moscow and Leningrad, boarding schools opened where children from far away could be taught. A. N. Kolmogorov, I. M. Gel'fand, E. B. Dynkin, and other leading mathematicians gave regular lessons in such classes. Not infrequently, students obtained their first serious results before they finished high school.

The physics-mathematics journal *Kvant* (*Quantum*) began to come out and most of the activities described here were concentrated around it. The articles I wrote that make up this book appeared in *Kvant*. I began with the story of the first two discoveries of the 19-year old Gauss, with complete proofs. It seemed to me that this possibility of following the first steps of a genius was invaluable for young people who were starting along their paths in science. Gradually, I told not only more about mathematics but also about the people who created it. I thought that it was always important to understand the people of science better, but this was especially urgent given the conditions in which we lived.

It was rather unusual for a professional mathematician to write about its history. There were some highly qualified historians of mathematics in the country, but mathematicians were basically suspicious about historical studies, seeing in them a direction in which the official ideology could influence mathematics. There was no shortage of examples of this. The influential "Communist commissar" at Moscow University was an expert on the mathematical writings of Karl Marx.

I wanted to show the great mathematicians as living human beings. Maybe it sounds strange today that this was in contradiction with the official tradition. It would not be a gross exaggeration to say that a black-and-white picture of the world was created in which scientists were divided into progressive materialists (with no shortcomings) and reactionaries and idealists (with no merit), and whether you belonged to one category or the other was decided at a very high level. Pasternak wrote,[1]

Komu byt' zhivym i hvalimym,	Who is to be honoured and living
Kto dolzhen byt' mërtv i hulim,	And who without honour and dead
Izvestno u nas podhalimam	Nobody knows in our country
Vliyatel'nym tol'ko odnim.	Till Establishment yes-men have said.

Such a world without shades of gray probably made it easier for those at the top to keep an eye on everyone. Russian scientists had a special advantage. Their primacy (real or imagined) was carefully cultivated (disrespect to them could easily be interpreted as slander), and western scientists were rarely "fully" progressive. Today it would be funny to see biographical movies of those years. I remember Euler in a film about Lomonosov,[2] reading with great surprise and delight Lomonosov's text on the conservation of energy and verifying the law by shoving one chair towards another, which began to move on impact. In view of Euler's foreign origins the level of his progressivity was not clear, not withstanding his long work in Russia.

It seemed to me that information about the fact that mathematicians like Euler or Gauss were basically ordinary people who spent a lot of energy solving ordinary problems of life in no way disparaged them. I saw no reason to cover up the history of how the aging great Euler wanted to become a (civil) general on returning to Russia from Prussia but that Catherine the Great explained (through an intermediary!) that he could be given a rank no higher than colonel. A comparison with influential Soviet mathematicians who dreamed of becoming Heroes of Socialist Labor

[1] From "The Wind (Four Fragments about Blok)" in Boris Pasternak, *Selected Poems*, translated by Jon Stallworthy and Peter France, W. W. Norton, New York, 1983, p. 147.

[2] Mikhail Lomonosov (1711–1765) is traditionally thought of as the first Russian scientist and was influential in founding the university that carries his name today, Lomonosov Moscow University.—*Transl.*

(and twice was better!) lay on the surface. "Double heroes" were eligible to have monuments erected to them during their lives (while Euler did not achieve this honor even after his death; see the story on p. 212). At times I succeeded in some counterestablishment action. The story about Chernyshevsky writing complete nonsense about Lobachevsky and his geometry was against the rules, since Chernyshevsky was officially classified as a "revolutionary democrat," which was only one step below a "Marxist revolutionary." More often, I was put in my place: a large part of the article on Pascal, devoted to his *Pensées*, was deleted. A progressive scientist could not be a religious writer (so they bashfully tried to overlook Gogol's religious searching during the last years of his life). An article on von Neumann was rejected, since I refused to say that he was a "servant of American militarism." The last trick for getting the book published was to switch "mathematicians" and "physicists" in the title and declare it to be a book about physics: there was no chance of getting it past the publishing committee on mathematics. Life taught us to fight for survival.

It is always instructive to compare similar events separated in time. Mandel'shtam wrote,[3]

Vsë bylo vstar',	Everything's been told before,
vsë povtoritsya snova,	everything will happen again,
I sladok nam lish' uznavan'ya mig.	and all that's sweet is the instant of recognition.

But in that life such comparisons could be risky. It was hard not to compare the story of the Göttingen professors' letter to the king about violating the constitution (which interrupted the collaboration of Gauss and Weber) with the letter that mentioned Esenin-Volpin. The limits within which Cardinal Bellarmino proposed to place Galileo turned out to be fantastically gentle compared to what the Soviet ideological machine required of scientists. Pascal's tragic thoughts about the sinfulness of science acquired new nuances in the 20th century. The fate of the French scientists who were happy to have a chance to participate in governing France at the time of the Revolution had direct associations with Soviet reality.

While recently rereading what I wrote in preparing this edition, I felt that after such a long time it seems to be the writing of another person. I think that it would have been wrong to change anything. Of course texts exist independently of the context in which they were written, but all the same I decided to use the occasion to recall in this very important stage of my life when this book was written.

[3]From Tristia in Osip Mandelstam, *Complete Poetry of Osip Emilevich Mandelstam*, translated by Burton Raffel and Alla Burago, State University of New York Press, Albany, NY, 1973, p. 103.

I want to thank Alan Shuchat for his enormous selfless work in translating both editions of this book. I am also grateful to my son Daniel, who read the translation and made several suggestions. Ann Kostant's support was very important for the publication of the translation.

Preface to the Third Russian Edition

The first edition of this book appeared in 1981 in the *Kvant* (*Quantum*) collection. It was reprinted several times in large print runs until 1985, more than half a million copies were sold, and it was translated into English, French, and Japanese. The book was based on articles that were published earlier in *Kvant* magazine. In this edition, some material is added that existed in 1981 but was not included then because of strict limitations on size. Some additional chapters were written later. More than twenty years have passed since a significant part of this book was written and today I would have written much of it differently, but I preferred to limit myself only to correcting errors that have been pointed out and to inaccuracies.

Among the additional subjects we note the history of the cycloid, a curve of unusual destiny, which seemed to 17th century mathematicians to be a curve of paramount importance and figured in the research of the strongest mathematicians but turned out in the end to be a curiosity in the history of mathematics. The story of the 17th century, the heroic century of mathematical analysis, is completed by the chapter on Leibniz, one of the most surprising figures in the history of science.

The 18th century that followed is represented by a trio of the most important mathematicians of the century: Euler, Lagrange, and Laplace (the last two worked into the 19th century). By the usual logic of the history of science, this should have been a relatively quiet century of putting in order the unpolished facts accumulated during the preceding revolutionary century of differential and integral calculus. However the great genius Euler, who felt confined by the mathematics of his day, broke all the rules and made surprising discoveries that were extraordinarily ahead of their time. At the end of the century, scientists turned out to be the objects of a critical historical experiment: the French Revolution tempted some of them with the possibility of taking a direct role in government, and this temptation cost many of them their lives. The fates of Laplace and Lagrange are two examples of the behavior of scientists under these conditions.

The 19th and 20th centuries are represented, apart from Gauss, by stories about Klein, Poincaré, and Ramanujan. Of course, this choice is random enough but their histories are instructive from our view. Finally, we brought to completion two articles about the history of projective geometry and its connections with one of the most modern theories of mathematical physics—Penrose's twistor theory. The mathematical part of this dramatic history assumes a greater degree of preparation than the rest of the book.

I want to remind the reader again that this is not a systematically written book but a collection of articles that were first of all intended for students interested in mathematics, and so wherever possible I tried to include detailed mathematical fragments in the historical tales. Since then, it has turned out that the circle of readers of this book was significantly wider. I discovered, not without surprise, that even some professional mathematicians and physicists found something in it for themselves. On the other hand, there were readers who skipped all the mathematics and found something instructive in the remainder. I would also like to warn against treating this book as a serious work on the history of mathematics: I did not work with original sources, did not thoroughly verify details, and did not furnish the text with citations and references. I only wanted to share with the reader who, like myself, loves mathematics and physics a picture that appeared to me after I became familiar with considerable historico-scientific material in connection with my professional mathematical studies. It would have been ideal for me to present this history not in serious history books (which are doubtless important) but rather in the novels of Dumas.

Although this book does not give a systematic picture of the development of mathematics, it contains significant material for reflecting on some astonishing paths in this development. I have already pointed out certain recurring subjects in the preface to the first edition. The additional chapters touch on some new and important examples (we recall the apocalyptic ideas of Leibniz and Lagrange on the coming end of mathematics). Unknowable laws govern mathematical fashion! How can we understand why Fermat, well-respected by his contemporaries, could not interest any serious 17th century mathematician in his work in arithmetic? Only as the result of a fortunate coincidence was his work continued in the next century by Euler, who passed the baton to Lagrange and Gauss, thus guaranteeing the continued development of number theory. By contrast, projective geometry, one of the greatest achievements of human thought and discovered in the same 17th century by Desargues and Pascal, was quickly forgotten and rediscovered only in the 19th century.

I do not try to explain in this book the laws of the development of mathematics: I do not know them. I only observe this process with interest, trying to draw the reader into searching for the logic hidden within it. Does there exist a natural time for the creation of a mathematical theory? One can bring many arguments in favor of this proposition. The construction of differential and integral calculus was begun by several 17th century mathematicians at once and in the end was completed independently by Newton and Leibniz; analytic geometry was independently constructed by Descartes and Fermat. Some problems that remained unsolved for many years were suddenly solved in a short interval of time by several mathematicians at once (strangely, often by three). Non-Euclidean geometry was discovered independently by Gauss, Lobachevsky, and Bolyai; the theory of elliptic functions was constructed independently by Gauss, Abel, and Jacobi. On the other hand, there have been great scholars who were very much ahead of their time and made discoveries that did not lead naturally to advancing science. Sometimes these discoveries were welcomed by their contemporaries (in the case of Archimedes or Euler) and sometimes they were forgotten (as in the case of Nicole Oresme in the 14th century, who used coordinates and considered uniformly accelerated motion 250 years before Galileo; see also the above examples about arithmetic and projective geometry). We find the richest information about the laws of mathematical creativity in the history of Ramanujan's surprising life.

What role do personalities play in the history of mathematics? For example, how decisive for the fate of mathematics was Plato's uncompromising position on the question of the subject of mathematics, given his unlimited influence on the science of his day? Was the development of geometry as an axiomatic science predetermined, or under different circumstances could it have evolved as more of an experimental science? Did Plato's almost extreme requirement of using only a straightedge and compass in geometric constructions help or hurt? Without it, what would have been discovered about unsolvable geometric problems, algebraic equations not solvable in radicals, and transcendental numbers?

I belong to the generation of Russian mathematicians who sometimes experience an ambivalent nostalgia for the time when mathematics flourished against the background of all the horrors of Soviet reality (the word "despite" would have been out of place in this context). Mathematics was a prestigious profession that attracted many talented young people who aspired to an intellectual activity that was relatively free from the influence of the prevailing Marxist ideology. This phenomenon has been talked about a lot during the last ten years, and we will not try to continue this important discussion here.

Today the position of mathematics has changed in an important way. I am able to observe a significant decrease in the standing of mathematics and of science in general in American life. I do not see it as a tragedy that most talented youngsters prefer professions with incomparably better prospects for financial success than a scientific career, but I am frightened by the needlessly utilitarian view of the role of mathematics in education, a view that absolutely misunderstands the unique place of mathematics in the general intellectual development of the individual. Recall that in the past all future rulers, rather than future scholars, studied geometry in Plato's Academy (the Spartans did not share this piety towards mathematics and the Romans did not include it among the values they inherited from Greek civilization). Graduates of mathematics schools in the former Soviet Union were successful far beyond mathematics. Today, many young professional mathematicians have decided to leave mathematics for careers in business. They are often successful, thanks not to some particular bit of mathematical knowledge but rather to the intellectual training they received while preparing for the mathematics profession.

In today's Russia the conditions of life have changed, and mathematics is going through difficult times. Mathematicians run into everyday problems that are unknown to their western colleagues. Glancing at some Russian newspapers one day, I thought that perhaps it was in vain that in the 18th century mathematicians had happily eliminated constructing horoscopes from their professional obligations; today it might have turned out to be a useful occupational addition.

It will soon be 50 years that I have been engaged in mathematics, and I never cease to be enraptured by this amazing science. I am used to expecting that many other people, including the young, share my love for it. This book is above all addressed to them.

I warmly thank the editor of this book, S. M. L'vovskiy, for his invaluable help in preparing this edition.

February 11, 2001
Princeton, NJ

Preface to the First Russian Edition

This book is based on articles published in *Kvant* (*Quantum*) over the course of several years. This explains a certain element of randomness in the choice of the people and events to which the stories collected in the book are devoted. However, it seems to us that the book discusses the principal events in the history of science that deserve the attention of devotees of mathematics and physics.

We cover a time span of four centuries, beginning with the sixteenth. The 16th century was a very important one for European mathematics, when its rebirth began a thousand years after the decline of ancient mathematics. Our story begins at the very moment when, after a 300-year-long apprenticeship, European mathematicians were able to obtain results unknown to the mathematicians of either ancient Greece or the East: they found a formula for the solution of the third-order polynomial equation. The events of the next series of tales begin at the dawn of the 17th century when Galileo, investigating free fall, laid the foundation for the development of both the new mechanics and the analysis of infinitely small quantities. The parallel formulation of these two theories was one of the most notable scientific events of the 17th century (from Galileo to Newton and Leibniz). We also tell of Galileo's remarkable astronomical discoveries, which interrupted his study of mechanics, and of his dramatic struggle on behalf of the claims of Copernicus. Our next hero, Huygens, was Galileo's immediate scientific successor. The subject we take is his work over the course of forty years to create and perfect the pendulum clock. A significant part of Huygens' achievements in both physics and mathematics was directly stimulated by this activity. The 17th century is also represented here by Pascal, one of the most surprising personalities in human history. Pascal began as a geometer, and his youthful work signified that European mathematics was already capable of competing with the great Greek mathematicians in their own territory—geometry. A hundred years had passed since the first successes of European mathematics in algebra.

Towards the end of the 18th century, mathematics unexpectedly found itself with no fundamental problems on which the leading scholars would otherwise have concentrated their efforts. Some approximation of mathematical analysis had been constructed; neither algebra nor geometry had brought forth suitable problems up to that time. Celestial mechanics "saved the day." The greatest efforts of the best mathematicians, beginning with Newton, were needed to construct the theory of motion of heavenly bodies, based on the law of universal gravitation. For a long time, almost all good mathematicians had considered it a matter of honor to demonstrate their prowess on some problem of celestial mechanics. Even Gauss, to whom the last part of this book is devoted, was no exception. But Gauss came to these problems as a mature scholar, and instead made his debut in an unprecedented way. He solved a problem that had been outstanding for 2000 years: He proved it was possible to construct a regular 17-gon with a straightedge and compass. The ancient Greeks had known how to construct regular n-gons for $n = 2^k, 3 \cdot 2^k, 5 \cdot 2^k$, and $15 \cdot 2^k$, and had spent much energy on unsuccessful attempts to devise a construction for other values of n. From a technical point of view, Gauss' discovery was based on arithmetical considerations. His work summed up a century and a half of converting arithmetic from a collection of surprising facts about specific numbers, accumulated from the deep past, into a science. This process began with the work of Fermat and was continued by Euler, Lagrange, and Legendre. It was startling that the young Gauss, with no access to the mathematical literature, independently reproduced most of the results of his great predecessors.

Observing the history of science from points chosen more or less at random turns out to be instructive in many ways. For example, numerous connections revealing the unity of science in space and time come into view. Connections of a different kind are revealed in the material considered in this book: the immediate succession from Galileo to Huygens, Tartaglia's ideas on the trajectory of a projectile carried by Galileo to a precise result, Galileo's profiting from Cardano's proposal for using the human pulse to measure time, Pascal's problems on cycloids being opportune for Huygens' work on the isochronous pendulum, the theory of motion of Jupiter's moons, which were discovered by Galileo, to which scholars of several generations tried to make some small contribution, and so on.

One can note many situations in the history of science that repeat, often with small variations (in the words of the French historian de Tocqueville, "history is an art gallery with few originals and many copies"). Consider, for example, how the evaluation of a scientist changes over the centuries. Cardano had no doubt that his primary merit lay in medicine and not in mathematics. Similarly, Kepler considered his main achievement to be the "discovery" of a mythical connection between the planetary orbits and the regular polyhedra. Galileo valued none of his discoveries more than the erroneous assertion that the tides prove the true motion of the earth (to a significant extent, he sacrificed his material well-being for the sake of its publication). Huygens considered his most important result to be the application of the cycloid pendulum to clocks, which turned out to be completely useless in practice, and Huygens could have considered himself generally unsuccessful since he could not solve his greatest problem—to construct a naval chronometer (much of what is considered today to be his fundamental contribution was only a means for constructing naval chronometers).

The greatest people are defenseless against errors of prognosis. In fact, a scientist sometimes makes the critical decision to interrupt one line of research in favor of another. Thus, Galileo refused to carry through to publication the results of his twenty-year-long work in mechanics, first being diverted for a year to make astronomical observations and then essentially ceasing scientific research, in the true sense of the word, for twenty years in order to popularize the heliocentric system. A century and a half later, Gauss' work on elliptic functions remained unpublished, again for the sake of astronomy. Probably neither foresaw how long the interruption would be, and neither saw around him anyone who could have threatened his priority. Galileo succeeded in publishing his work in mechanics after 30 years(!), when the verdict of the Inquisition closed off for him the possibility of other endeavors. Only a communication by Cavalieri about the trajectory of a projectile being parabolic forced Galileo to worry a bit, although it did not encroach on his priority. Gauss did not find time to complete his results, also for thirty years, and they were rediscovered by Abel and Jacobi.

The selection of material and the nature of its presentation were dictated by the fact that the book and the articles on which it is based are addressed to lovers of mathematics and physics and, most of all, to students. We have always given priority to a precise account of specific scientific achievements (Galileo's work in mechanics, Huygens' mathematical and mechanical research in connection with pendulum clocks, and Gauss' first two mathematical works). Unfortunately, this is not always possible, even with ancient works. There is no greater satisfaction than following the flight of fancy of a genius, no matter how long ago he lived. It is not only a matter of this being beyond the reach of the amateur in the case of contemporary works. To be able to feel the revolutionary character of an achievement of the past is an important part of culture.

We wish to stress that the tales collected in this book do not have the nature of texts in the history of science. This is revealed in the extensive adaptation of the historical realities. We freely modernize the reasoning of our scientists: we use algebraic symbols in Cardano's proofs, we introduce free-fall acceleration in Galileo's and Huygens' calculations (in order not to bother the reader with endless ratios), we work with natural logarithms instead of Naperian ones in the story of Napier's discovery, and we use Galileo's latest statements in order to reconstruct the logic of his early studies in mechanics. Throughout, we consciously disregard details that are appropriate for a work in the history of science in order to present vividly a small number of fundamental ideas.

Translator's Note

Wherever possible, citations have been made to English versions of the works discussed in the book. In addition, since many of the quotations that appear were taken from various European languages (including English), I have tried to use existing translations or work directly from the original. It has been difficult to locate the sources of some quotations and these have thus been translated twice, first into Russian and then into English, and inaccuracies may have crept in. There is an apocryphal story about a computer that translated "the spirit is willing but the flesh is weak" into Russian and back again, ending up with "the wine is strong but the meat is rancid." I trust these results are more palatable! *A. S.*

Ars Magna (The Great Art)

In 1545 a book by Gerolamo Cardano appeared whose title began with the Latin words *Ars Magna*. It was essentially devoted to solving third- and fourth-order equations, but its value for the history of mathematics far surpassed the limits of this specific problem. Even in the 20th century, Felix Klein, evaluating this book, wrote, "This most valuable work contains the germ of modern algebra, surpassing the bounds of ancient mathematics."

The 16th century was the century in which European mathematics was reborn after the hibernation of the Middle Ages. For a thousand years the work of the great Greek geometers was forgotten, and in part irrevocably lost. From Arab texts, the Europeans learned not only about the mathematics of the East but also about the ancient mathematics of the West. It is characteristic that in the spread of mathematics across Europe a major role was played by traders, for whom journeys were a means of both obtaining information and spreading it. The figure of Leonardo of Pisa (1180–1240), better known as Fibonacci (son of Bonacci), especially stands out. His name is immortalized by a remarkable numerical sequence (the Fibonacci numbers). Science can lose its royal status very quickly and centuries may be needed to reestablish it. For three centuries European mathematicians remained as apprentices, although Fibonacci undoubtedly did some interesting work. Only in 16th century Europe did significant mathematical results appear that neither the ancient nor the Eastern mathematicians knew. We are talking about the solution of third- and fourth-degree equations.

Typically, the achievements of the new European mathematics were in algebra, a new field of mathematics that arose in the East and was essentially taking only its first steps. For at least a hundred years, it would be beyond the power of the European mathematicians not only to achieve something in geometry comparable to the great geometers Euclid, Archimedes, and Apollonius, but even to master their results fully.

Legend ascribes to Pythagoras the phrase "all is number." But after Pythagoras, geometry gradually came to dominate all of Greek mathematics. Euclid even put the elements of algebra into geometric form. For example, a square was divided, by lines parallel to its sides, into two smaller squares and two equal rectangles. The formula $(a+b)^2 = a^2 + b^2 + 2ab$ was obtained by comparing areas. But to be sure, there was no algebraic notation at the time, and expressing the result in terms of areas was definitive. Mathematical statements were very awkward. In essence, construction problems with straightedge and compass led to solving quadratic equations and to considering expressions that contained square roots (quadratic irrationals). For example, Euclid considered expressions of the form

$$\sqrt{(a+\sqrt{b})}$$

in detail (in different language). To a certain extent, the Greek geometers understood the link between the classical unsolved construction problems (duplicating a cube and trisecting an angle) and cubic equations.

With the Arab mathematicians, algebra gradually became distinct from geometry. However, as we will see below, the solution of the cubic equation was obtained by geometric means (the debut of algebraic formulas for solving even the quadratic equation came only with Bombelli in 1572). The algebraic assertions of the Arab mathematicians are stated as recipes for the solution of one-of-a-kind arithmetic problems, usually of an "everyday" sort (for example, dividing an inheritance). Rules are formulated for specific examples but so that similar problems can be solved. Until recently rules for solving arithmetic problems (the rule of three,[1] and so on) were sometimes stated this way. Stating rules in general form almost inevitably requires a developed symbolism, which was still far off. The Arab mathematicians did not go further than solving quadratic equations and some specially chosen cubics.

The problem of solving cubic equations bothered both the Arab mathematicians and their European apprentices. A surprising result in this direction belongs to Leonardo of Pisa. He showed that the roots of the equation $x^3 + 2x^2 + 10x = 20$ cannot be expressed in terms of Euclidean irrationals of the form

[1] A mechanical way of solving proportion problems.—*Transl.*

$$\sqrt{(a+\sqrt{b})}.$$

This statement is startling for the beginning of the 13th century and foreshadows the problem of solving equations in radicals, which was thought of significantly later. Mathematicians did not see the path that led to solving the general cubic equation.

The state of mathematics at the turn of the 16th century was summed up by Fra Luca Pacioli (1445–1514) in his book, *Summa de Arithmetica* (1494), one of the first printed mathematics books and written in Italian rather than Latin.[2] At the end of the book he states that "the means [for solving cubic equations] by the art of algebra are not yet given, just as the means for squaring the circle are not given." The comparison sounds impressive, and Pacioli's authority was so great that most mathematicians (even including our heroes at first, as we shall see) believed that the cubic equation could not be solved in general.

Scipione dal Ferro

There was a man who was not deterred by Pacioli's opinion. He was a professor of mathematics in Bologna named Scipione dal Ferro (1465–1526), who found a way to solve the equation

$$x^3 + ax = b. \quad (1)$$

Negative numbers were not yet in use and, for example,

$$x^3 = ax + b \quad (2)$$

was thought of as a completely different equation! We have only indirect information about this solution. Dal Ferro told it to his son-in-law and successor on the faculty, Annibale della Nave, and to his student Antonio Maria Fior. The latter decided, after his teacher's death, to use the secret confided to him to become invincible in the problem-solving "duels" that were then quite widespread. On February 12, 1535, Niccolò Tartaglia, one of the major heroes of our story, nearly became his victim.

Niccolò Tartaglia

Tartaglia was born around 1500 in Brescia into the family of a poor mounted postman named Fontana. During his childhood, when his native city was captured by the French, he was wounded in the larynx and thereafter spoke with difficulty. Because of this he was given the nickname "Tartaglia"

[2] Despite its title.—*Transl.*

The only known portrait of Niccolò Tartaglia.

(stutterer). Early on he came under the influence of his mother, who tried to enroll him in school. But the money ran out when the class reached the letter "k," and Tartaglia left school without having learned to write his name. He continued to study on his own and became an "abacus master" (something like an arithmetic teacher) in a private school for commerce. He traveled a lot throughout Italy until landing in Venice in 1534. Here his scientific studies were stimulated by contact with engineers and artillerymen of the famed Venetian arsenal. They asked Tartaglia, for example, at what angle to aim a gun so that it shoots the farthest. His answer, a 45° angle, surprised his questioners. They did not believe that they had to raise the barrel so high, but "several private experiments" proved he was right. Although Tartaglia said he had "mathematical reasons" for this assertion, it was more of an empirical observation (Galileo gave the first proof).

Tartaglia published two books, one a sequel of the other: *La Nuova Scientia* (*The New Science* [*of Artillery*], 1537) and *Quesiti et Inventioni Diverse* (*Problems and Various Inventions*, 1546), where the reader is promised ". . .new inventions, not stolen from Plato, from Plotinus, or from any other Greek or Roman, but obtained only by art, measurement, and reasoning." The books were written in Italian in the form of a dialogue, which was later adopted by Galileo. In several respects, Tartaglia was Galileo's predecessor. Although in the first of these books he followed Aristotle in saying that a projectile launched at an angle first flies along an inclined straight line, then along a circular arc, and finally falls vertically, in the second book he wrote that the trajectory "does not have a single part that is perfectly straight." Tartaglia was interested in the equilibrium of bodies on an inclined plane and in free fall (his student Giovanni Benedetti (1530–1590) convincingly showed that the behavior of a falling body does not depend on its weight). Tartaglia's translations of Archimedes' and Euclid's work into Italian and his detailed commentaries played an important role (he called Italian the "national" language, as opposed to Latin). In his personal qualities Tartaglia was far from irreproachable and was very difficult with interpersonal relations. Bombelli (who was admittedly not impartial; more on him later) wrote that "this man was by nature so inclined to speak badly, that he took any sort of abuse as a compliment." According to other information (Pedro Nuñes) "he was at times so excited that he seemed mad."

Let us return to the duel before us. Tartaglia was an experienced combatant and hoped to win an easy victory over Fior. He was not frightened even when he discovered that all thirty of Fior's problems contained equation (1), for various values of a and b. Tartaglia thought that Fior himself could not solve these problems, and hoped to unmask him: "I thought that not a single one could be solved, because Fra Luca [Pacioli] assures us of their difficulty that such an equation cannot be solved by a general formula." After fifty days, Tartaglia was supposed to submit the solution to a notary. When the time limit had almost elapsed, he heard a rumor that Fior had a secret method for solving equation (1). He was not pleased by the prospect of hosting a victory meal for Fior's friends, one friend for each problem the victor solved (those were the rules!). Tartaglia put forth a titanic effort, and fortune smiled on him eight days before the deadline of February 12, 1535: He found the method he had hoped for! He solved all the problems in two hours. His opponent did not solve a single one of the problems Tartaglia had given him. Strangely enough, Fior could not handle one problem that could be solved by dal Ferro's formula (Tartaglia had posed it with a certain trick in mind for solving it), but we will see that the formula is not easy to use. Within a day Tartaglia found a method for solving equation (2).

Many people knew about the Tartaglia–Fior duel. In this situation a secret weapon could not help but could rather hurt Tartaglia in further duels. Who would agree to compete with him if the outcome were predetermined? All the same, Tartaglia turned down several requests to reveal his method for solving cubic equations. But one who made the request achieved his goal. This was Gerolamo Cardano, the second hero of our tale.

Gerolamo Cardano

He was born in Pavia on September 24, 1501. His father, Fazio Cardano, an educated lawyer with broad interests, was mentioned by Leonardo da Vinci. Fazio was his son's first teacher. After graduating from the University of Padua, Gerolamo decided to devote himself to medicine. But he was an illegitimate child and so was denied admission to the College of Physicians in Milan. Cardano practiced in the provinces for a long time until August, 1539 when the college admitted him anyway, specially changing the rules to do so. Cardano was one of the most famous doctors of his time, probably only second to his friend, Andreas Vesalius. In his declining years, Cardano wrote his autobiography, *De Vita Propria Liber* (*The Book of My Life*). It contains recollections of his mathematical work, as well as detailed descriptions of his medical research. He claimed that he prescribed cures for up to 5000 difficult diseases and solved some 40,000 problems and questions, as well as up to 200,000 smaller ones. Of course these figures should be taken with a large dose of skepticism, but Cardano was undoubtedly a famous physician. He described cases from his medical practice where he focused on curing noted personalities (Archbishop James Hamilton of Scotland, Cardinal Morone, etc.), claiming that he had only three failures. In modern terms he was evidently an outstanding diagnostician, but he did not pay great attention to anatomical information, unlike Leonardo da Vinci and Vesalius. In his autobiography Cardano places himself alongside Hippocrates, Galen, and Avicenna (the latter's ideas were especially close to his own).

However, medical studies did not fill up Cardano's time. In his free moments he studied everything under the sun. For example, he constructed horoscopes for persons both living and dead (Christ, King Edward VI of England, Petrarch, Dürer, Vesalius, and Luther). These studies harmed his reputation among his successors (according to one unkind legend, Cardano committed suicide in order to confirm his own horoscope). But we must remember that at that time astrology was completely respectable (astronomy was a part of astrology—natural astrology as opposed to the astrology of predictions). The pope himself utilized the work of Cardano the astrologer.

In his scientific activities Cardano was an encyclopedist, but a lone encyclopedist, which was typical for the time of the Renaissance. Only after a century and a half did the first academies appear, in which scholars specialized in more or less narrow fields. Real encyclopedias could only be created with such collaborative efforts. The lone encyclopedist was in no position to verify much of the information he was given. In Cardano's case a large role was played by the peculiarities of his personality and psycho-

Gerolamo Cardano.

logical bent. He believed in magic, premonitions, demons, and in his own supernatural ability. He described in detail the events that convinced him of this (there was no bleeding in any collision he saw, from neither people nor animals, not even in hunting; he learned in advance, from signs, about the events leading up to his son's death, etc.). Cardano believed he possessed a gift of vision (he called it a "harpocratic" feeling) that allowed him to divine both an inflamed organ in an ill patient and the fall of the dice in a game of chance, and to see the mark of death on an interlocutor's face. Dreams, which he remembered in the finest detail and described carefully, played a great role in his life. Contemporary psychiatrists have used these descriptions to try to determine his disease. Cardano writes that constantly recurring dreams, together with the desire to immortalize his name, were his main reasons for writing books. In his encyclopedias *De Subtilitate Rerum* (*On Subtlety*) and *De Rerum Varietate* (*On a Variety of Matters*), he again gave a lot of space to descriptions of the author and his father.

But these books also contain many personal observations and carefully digested communications from others. His readiness to discuss fantastic theories and his peculiar credulity did not only play a negative role. Thanks to them, he discussed things that his more careful colleagues decided to speak of only many years later (see below about complex numbers). It does not always pay to follow authority. It is not clear how familiar Cardano was with the works of Leonardo da Vinci (this also applies to other 16th century Italian authors; Leonardo became widely known only at the very end of the 18th century). *De Subtilitate Rerum* was brought to France and served as a popular textbook on statics and hydrostatics throughout the 17th century. Galileo employed Cardano's instructions for using the human pulse to measure time (in particular, for observing the oscillations of the cathedral chandelier). Cardano asserted that perpetual motion is impossible, some of his remarks can be interpreted as the principle of virtual displacements (according to Pierre Duhem (1861–1916), the well-known historian of physics), and he studied the expansion of steam. Cardano adhered to the theory, first conceived of in the 3rd century B.C., that explained the tides by the motion of the moon and sun. He was the first to clearly explain the difference between magnetic and electrical attraction (we have in mind the type of phenomenon observed as early as Thales (c.640–c.546 B.C.), such as the attraction of straw to polished amber).

Cardano was no stranger to experimental research either, or to the construction of practical devices. In his declining years he established experimentally that the ratio of the density of air to water is 1/50. In 1541, when King Charles V of Spain conquered Milan and entered the city in triumph, Cardano, as Rector of the College of Physicians, walked alongside him near the baldachin (canopy). In response to the honor shown to him, he offered to supply the royal team with a suspension from two shafts, which would keep the coach horizontal when it rocked (the roads in Charles' empire were long and bad). Such a system is now called a Cardan suspension (Cardan shaft, Cardan joint) and is used in automobiles. The truth requires us to note that the idea of such a system arose in antiquity and that, at the very least, there is a drawing of a ship's compass with a Cardan suspension in Leonardo da Vinci's *Codice Atlantico*. Such compasses became common during the first half of the 16th century, obviously without Cardano's influence.

Cardano wrote a great many books, of which some were published, some remained as manuscripts, and some were destroyed by him in Rome in anticipation of arrest. His voluminous book, *De Libris Propriis* (*On My Own Books*), contained only a description of the books he had written. His books on philosophy and ethics were popular for many years, and *On Consolation* was translated into English and influenced Shakespeare. Some Shakespeare-philes even claim that Hamlet speaks his monologue "To be or not to be..." while holding this book in his hands.

Much can be said about Cardano's personality. He was passionate, quick-tempered, and often played games of chance. Cardano gambled at chess for forty years ("I could never express in a few words how much damage this caused my home life, without any compensation") and at dice for twenty-five ("but dice harmed me even more than chess"). From time to time he threw away his studies for gambling, and fell into unpleasant situations. A collateral product of Cardano's passion was *Liber de Ludo Aleae* (*The Book on Games of Chance*), written in 1526 but published only in 1663. This book contains the beginnings of probability theory, including a preliminary statement of the law of large numbers, some combinatoric questions, and observations on the psychology of gamblers.

Here are a few words about Cardano's nature. He himself writes, "This I recognize as unique and outstanding among my faults—the habit, which I persist in, of preferring to say above all things what I know to be displeasing to the ears of my hearers. I am aware of this, yet I keep it up willfully.... And I have made many, nay, numberless blunders, wherever I wished to mingle with my fellows.... I blundered, almost unavoidably, not solely because of lack of deliberations, and an ignorance of... manners and customs, but because I did not duly regard certain of those conventions which I learned about long afterwards, and with which cultivated men, for the most part, are acquainted."[3] For friends and students he could be yet another person. Bombelli wrote that Cardano had "a more godlike than human appearance."

Cardano and Tartaglia

Towards 1539, Cardano was completing his first mathematical book, *Practica Arithmeticae Generalis*, envisioned to replace Pacioli's book. Cardano burned with desire to adorn his book with Tartaglia's secret. At his request, the bookseller Zuan Antonio da Bassano met with Tartaglia in Venice on January 2, 1539. He asked Tartaglia, in the name of "a worthy man, physician of Milan, named Messer Gerolamo Cardano," to give him the rule for solving equation (1), either to publish in the book or under promise to keep it secret. The response was negative: "Tell his Excellency that he must pardon me, that when I publish my invention it will be in my own work and not in that of others...."[4] Tartaglia also refused to communicate the solutions to Fior's thirty problems and only stated the questions (which could have been obtained from the notary), and refused to solve seven problems sent by Cardano. Tartaglia suspected that Cardano was a straw man for the mathematician Zuanne de Tonini da Coi, who had long been trying

[3]From Jerome Cardan (Gerolamo Cardano), *The Book of My Life*, translated by Jean Stoner, E. P. Dutton, New York, 1930. Reprinted with the permission of E. P. Dutton, a division of NAL Penguin, Inc.

[4]Øystein Ore, *Cardano, the Gambling Scholar*, Princeton University Press, Princeton, NJ, 1953 (© renewed 1981). Reprinted with permission.

unsuccessfully to learn the secret.

On February 12th, Cardano sent Tartaglia comments about his book, *La Nuova Scientia*, and repeated his requests. Tartaglia was implacable, agreeing to solve only two of Cardano's problems. On March 13th Cardano invited Tartaglia to visit him, expressed interest in his artillery instruments, and promised to present him to the Marchese del Vasto, the Spanish governor of Lombardy. Evidently, this perspective enticed Tartaglia, since he accepted the invitation and the critical meeting took place on March 25th at Cardano's home.

Here is an excerpt from the notes of this meeting (one must keep in mind that the record was made by Tartaglia; Ferrari, Cardano's student, claimed that it does not completely correspond to the facts):

"**Niccolò:** I say to you: I refused you not just because of this one chapter and the discoveries made in it, since this is the key that unlocks the way to the study of countless other areas. I would have long ago found a general rule for many other problems, if I had not at present been occupied with translating Euclid into the national language (I have now brought the translation up to Book XIII). But when this task, which I have already begun, is done, I plan to publish the work for practical application together with a new algebra. . . . If I give it to some theorist (such as your Excellency), then he could easily find other chapters with the help of this explanation (for it is easy to apply this explanation to other questions) and publish the fruit of my discovery under his own name. All my plans would be ruined by this.

Messer Gerolamo: I swear to you by the Sacred Gospel, and on my faith as a gentleman, not only never to publish your discoveries, if you tell them to me, but I also promise and pledge my faith as a true Christian to put them down in cipher so that after my death no one shall be able to understand them. If I, in your opinion, am trustworthy then do it, and if not then let us end this conversation.

Niccolò: If I did not believe an oath such as yours then, of course, I myself would deserve to be considered a nonbeliever."

Thus, Tartaglia convinced himself. He communicated his solution in the form of a Latin poem. Is it not true that it is hard to understand from these notes *what* induced Tartaglia to change his decision? Was he really shaken by Cardano's vow? What happened later is not well understood. Having revealed his secret, the uneasy Tartaglia left immediately, refusing to meet the marchese for whom he had undertaken the journey. Could Cardano have hypnotized him? In all likelihood, Tartaglia's account is inaccurate.

Tartaglia was somewhat reassured when on May 12th he received the *Practica Arithmeticae Generalis*, freshly printed, without his recipe. In an accompanying letter, Cardano wrote, "I have verified the formula and believe it has broad significance."

Cardano received from Tartaglia a ready-to-use method for solving equation (1), without any hint of proof. He spent a great deal of effort on carefully verifying and substantiating the rule. From our standpoint it is not easy to understand the difficulty: Just substitute into the equation and verify it! But the absence of a well-developed algebraic notation made what any schoolchild today can do automatically, accessible to only a select few. Without knowing the original texts from that time we cannot appreciate how much the algebraic apparatus "economizes" thought. The reader must always keep this in mind, so as not to be deluded by the "triviality" of the problems over which passions seethed in the 16th century.

Cardano put in years of intense work trying to understand the solution of cubic equations thoroughly. He obtained a recipe (after all, they did not know how to write formulas!) for solving equations (1) and (2), as well as

$$x^3 + b = ax \tag{3}$$

and equations containing x^2. He certainly "outstripped" Tartaglia. All this happened against the background of a consolidation of Cardano's position: in 1543 he became professor at Pavia. "My knowledge of astrology," wrote Cardano, "led me to the conclusion that I would not live more than forty years and, in any case, would not reach the age of forty-five.... The year arrived that was supposed to be the last one of my life and that, on the contrary, turned out to be its beginning—namely, the forty-fourth."

Luigi Ferrari

For some time Cardano had been assisted in his mathematical work by Luigi Ferrari (1522–1565). In a list Cardano made of his fourteen students, Ferrari appears as the second chronologically and one of the three most outstanding. Cardano, believing in signs, wrote that on November 14, 1536, when the fourteen-year old Luigi and his brother arrived in Bologna, "a magpie in the courtyard chirred for such an unusually long time that we all expected someone to arrive." Ferrari was a man of phenomenal ability. He had such a stormy temper that even Cardano was sometimes afraid to speak with him. We know that at seventeen, Ferrari returned from a brawl without a single finger on his right hand. He was unreservedly devoted to his teacher and for a long time was his secretary and confidant. Ferrari's contribution to Cardano's mathematical work was quite substantial.

In 1543 Cardano traveled with Ferrari to Bologna, where della Nave allowed him to examine the papers of the late dal Ferro. They became convinced that dal Ferro had known Tartaglia's rule. It is interesting that they evidently knew almost nothing about dal Ferro's formula. Cardano would hardly have pursued Tartaglia so energetically had he known that the same information could have been obtained from della Nave (he had not consulted him before 1543). Almost everyone now agrees that dal Ferro had the formula, that Fior knew it, and that Tartaglia rediscovered it knowing that Fior had it. However, not one of the steps in this chain has been strictly proven! Cardano spoke of it, but Tartaglia wrote at the end of his life, "...I can testify that the theorem described was not proved before by Euclid or by anyone else but only by one Gerolamo Cardano, to whom we showed it.... In 1534 [elsewhere February 4, 1535–S.G.] in Venice, I found a general formula for the equation...." It is hard to untangle this confused story.

Ars Magna

Familiarity with dal Ferro's papers, strong pressure from Ferrari, or, most likely, an unwillingness to bury the results of many years' work led Cardano to include everything he knew about cubic equations in this book, *Artis Magnae Sive de Regulis Algebraicis* (*The Great Art, or the Rules of Algebra*), which appeared in 1545. It has come to be called simply *Ars Magna* (*The Great Art*).

At the beginning, Cardano lays out the history of the problem: "...In our own days Scipione del Ferro of Bologna has solved the case of the cube and first power equal to a constant, a very elegant and admirable accomplishment. Since this art surpasses all human subtlety and the perspicuity of mortal talent and is a truly celestial gift and a very clear test of the capacity of men's minds, whoever applies himself to it will believe that there is nothing that he cannot understand. In emulation of him, my friend Niccolò Tartaglia of Brescia, wanting not to be outdone, solved the same case when he got into a contest with his [Scipione's] pupil. Antonio Maria Fior, moved by my many entreaties, gave it to me. For I had been deceived by the words of Luca Paccioli, who denied that any general rule could be rediscovered other than his own. Notwithstanding the many things which I had already discovered, as is well known, I had despaired and had not attempted to look any further. Then, however, having received Tartaglia's solution and seeking its proof, I came to understand that there were a great many other things that could also be had. Pursuing this thought and with increased confidence, I discovered these others, partly by myself and partly through Lodovico Ferrari, formerly my pupil."[5]

[5] Girolamo Cardano, *The Great Art or the Rules of Algebra*, translated by T. Richard Witmer,