Geometry of Quantum Theory

Second Edition

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TO MY PARENTS

PREFACE TO VOLUME I OF THE FIRST EDITION

The present work is the first volume of a substantially enlarged version of the mimeographed notes of a course of lectures first given by me in the Indian Statistical Institute, Calcutta, India, during 1964-65. When it was suggested that these lectures be developed into a book, I readily agreed and took the opportunity to extend the scope of the material covered.

No background in physics is in principle necessary for understanding the essential ideas in this work. However, a high degree of mathematical maturity is certainly indispensable. It is safe to say that I aim at an audience composed of professional mathematicians, advanced graduate students, and, hopefully, the rapidly increasing group of mathematical physicists who are attracted to fundamental mathematical questions.

Over the years, the mathematics of quantum theory has become more abstract and, consequently, simpler. Hilbert spaces have been used from the very beginning and, after Weyl and Wigner, group representations have come in conclusively. Recent discoveries seem to indicate that the role of group representations is destined for further expansion, not to speak of the impact of the theory of several complex variables and function-space analysis. But all of this pertains to the world of interacting subatomic particles; the more modest view of the microscopic world presented in this book requires somewhat less. The reader with a knowledge of abstract integration, Hilbert space theory, and topological groups will find the going easy.

Part of the work which went into the writing of this book was supported by the National Science Foundation Grant No. GP-5224. I have profited greatly from conversations with many friends and colleagues at various institutions. To all of them, especially to R. Arens, R. J. Blattner, R. Ranga Rao, K. R. Parthasarathy, and S. R. S. Varadhan, my sincere thanks. I want to record my deep thanks to my colleague Don Babbitt who read through the manuscript carefully, discovered many mistakes, and was responsible for significant improvement of the manuscript. My apologies are due to all those whose work has been ignored or, possibly, incorrectly (and/or insufficiently) discussed. Finally, I want to acknowledge that this book might never

have made its way into print but for my wife. She typed the entire manuscript, encouraged me when my enthusiasm went down, and made me understand some of the meaning of our ancient words,

कर्मण्येवाधिकारस्ते मा फलेषु कदाचन । .*

To her my deep gratitude.

Spring, 1968 V. S. VARADARAJAN

* *Bhagavadgita, 2:47a.*

PREFACE TO THE SECOND EDITION

कर्मण्येवाधिकारस्ते मा फलेषु कदाचन ।.*

It was about four years ago that Springer-Verlag suggested that a revised edition in a single volume of my two-volume work may be worthwhile. I agreed enthusiastically but the project was delayed for many reasons, one of the most important of which was that I did not have at that time any clear idea as to how the revision was to be carried out. Eventually I decided to leave intact most of the original material, but make the current edition a little more up-to-date by adding, in the form of notes to the individual chapters, some recent references and occasional brief discussions of topics not treated in the original text. The only substantive change from the earlier work is in the treatment of projective geometry; Chapters II through V of the original Volume I have been condensed and streamlined into a single Chapter II. I wish to express my deep gratitude to Donald Babbitt for his generous advice that helped me in organizing this revision, and to Springer-Verlag for their patience and understanding that went beyond what one has a right to expect from a publisher.

I suppose an author's feelings are always mixed when one of his books that is comparatively old is brought out once again. The progress of Science in our time is so explosive that a discovery is hardly made before it becomes obsolete; and yet, precisely because of this, it is essential to keep in sight the origins of things that are taken for granted, if only to lend some perspective to what we are trying to achieve. All I can say is that there are times when one should look back as well as forward, and that the ancient lines, part of which are quoted above still capture the spirit of my thoughts.

Pacific Palisades,
Dec. 22, 1984

V. S. VARADARAJAN

* *Bhagavadgita, 2:47a.*

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INTRODUCTION

As laid down by Dirac in his great classic [1], the principle of superposition of states is the fundamental concept on which the quantum theory of atomic systems is to be erected. Dirac's development of quantum mechanics on an axiomatic basis is undoubtedly in keeping with the greatest traditions of the physical sciences. The scope and power of this principle can be recognized at once if one recalls that it survived virtually unmodified throughout the subsequent transition to a relativistic view of the atomic world. It must be pointed out, however, that the precise mathematical nature of the superposition principle was only implicit in the discussions of Dirac; we are indebted to John von Neumann for explicit formulation. In his characteristic way, he discovered that the set of experimental statements of a quantum mechanical system formed a projective geometry—the projective geometry of subspaces of a complex, separable, infinite dimensional Hilbert space. With this as a point of departure, he carried out a mathematical analysis of the axiomatic foundations of quantum mechanics which must certainly rank among his greatest achievements [1] [3] [4] [5] [6].

Once the geometric point of view is accepted, impressive consequences follow. The automorphisms of the geometry describe the dynamical and kinematical structure of quantum mechanical systems, thus leading to the *linear* character of quantum mechanics. The covariance of the physical laws under appropriate space-time groups consequently expresses itself in the form of projective unitary representations of these groups. The economy of thought as well as the unification of method that this point of view brings forth is truly immense; the Schrödinger equation, for example, is obtained from a representation of the time-translation group, the Dirac equation from a representation of the inhomogeneous Lorentz group. This development is the work of many mathematicians and physicists. However, insofar as the mathematical theory is concerned, no contribution is more outstanding than that of Eugene P. Wigner. Beginning with his famous article on time inversion and throughout his great papers on relativistic invariance [1] [3] [4] [5] [6], we find a beautiful and coherent approach to the mathematical description of the quantum mechanical world which achieves nothing less than the fusion of group theory and quantum mechanics, and moreover does this without compromising in any manner the axiomatic principles formulated by Dirac and von Neumann.

My own interest in the mathematical foundations of quantum mechanics received a great stimulus from the inspiring lectures given by Professor George W. Mackey at the University of Washington in Seattle during the summer of 1961. The present volumes are in great part the result of my interest in a detailed elaboration of the main features of the theory sketched by Mackey in those lectures. In sum, my indebtedness to Professor Mackey's lectures and to the books and papers of von Neumann and Wigner is immense and carries through this entire work.

There exist today many expositions of the basic principles of quantum mechanics. At the most sophisticated mathematical level, there are the books of von Neumann [1], Hermann Weyl [1] and Mackey [1]. But, insofar as I am aware, there is no account of the technical features of the geometry and group theory of quantum mechanical systems that is both reasonably self-contained and comprehensive enough to be able to include Lorentz invariance. Moreover, recent re-examinations of the fundamental ideas by numerous mathematicians have produced insights that have substantially added to our understanding of quantum foundations. From among these I want to single out for special mention Gleason's proof that quantum mechanical states are represented by the so-called density matrices, Mackey's extensive work on systems of imprimitivity and group representations, and Bargmann's work on the cohomology of Lie groups, particularly of the physically interesting groups and their extensions. All of this has made possible a conceptually unified and technically cogent development of the theory of quantum mechanical systems from a completely geometric point of view. The present work is an attempt to present such an approach.

Our approach may be described by means of a brief outline of the contents of the three parts that make up this work. The first part begins by introducing the viewpoint of von Neumann according to which every physical system has in its background a certain orthocomplemented lattice whose elements may be identified with the experimentally verifiable propositions about the system. For classical systems this lattice (called the logic of the system) is a Boolean σ-algebra while for quantum systems it is highly nondistributive. This points to the relevance of the theory of complemented lattices to the axiomatic foundations of quantum mechanics. In the presence of modularity and finiteness of rank, these lattices decompose into a direct sum of irreducible ones, called geometries. A typical example of a geometry is the lattice of subspaces of a finite dimensional vector space over a division ring. The theory of these vector geometries is taken up in Chapter II. The isomorphisms of such a geometry are induced in a natural fashion by semilinear transformations. Orthocomplementations are induced by definite semi-bilinear forms which are symmetric with respect to suitable involutive anti-automorphisms of the basic division ring. If the division ring is the reals, complexes or quaternions, this leads to the Hilbert space structures. In this chapter, we also examine the relation between axiomatic geometry and analytic geometry along classical lines with suitable modifications in order to handle the infinite dimensional case also. The main result of this chapter is the theorem which asserts that an abstractly given generalized geometry (i.e., one whose dimension need not be finite) of rank ≥ 4 is isomorphic to the lattice of all finite dimensional subspaces of a vector space over a division ring. The division ring is an invariant of the lattice.

The second part analyzes the structure of the logics of quantum mechanical systems. In Chapter III, we introduce the notion of an abstract logic (= orthocomplemented weakly modular σ-lattice) and the observables and states associated with it. It is possible that certain observables need not be simultaneously observable. It is proved that for a given family of observables to be simultaneously measurable, it is necessary and sufficient that the observables of the family be classically related, i.e., that there exists a Boolean sub σ-algebra of the logic in question to which all the members of the given family are associated. Given an observable and a state, it is shown how to compute the probability distribution of the observable in that state. In Chapter IV, we take up the problem of singling out the logic of all subspaces of a Hilbert space by a set of neat axioms. Using the results of Chapter II, it is proved that the standard logics are precisely the projective ones. The analysis of the notions of an observable and a state carried out in Chapter III now leads to the correspondence between observables and self-adjoint operators, and between the pure states and the rays of the underlying Hilbert space. The automorphisms of the standard logics are shown to be induced by the unitary and antiunitary operators. With this the von Neumann program of a deductive description of the principles of quantum mechanics is completed. The remarkable fact that there is a Hilbert space whose self-adjoint operators represent the observables and whose rays describe the (pure) states is thus finally established to be a consequence of the projective nature of the underlying logic.

The third and final part of the work deals with specialized questions. The main problem is that of a covariant description of a quantum mechanical system, the covariance being with respect to suitable symmetry groups of the system. The theory of such systems leads to sophisticated problems of harmonic analysis on locally compact groups. Chapters V, VI, and VII are devoted to these purely mathematical questions. The results obtained are then applied to yield the basic physical results in Chapters VIII and IX. In Chapter VIII, the Schrödinger equation is obtained and the relations between the Heisenberg and Schrödinger formulations of quantum mechanics are analyzed. The usual expressions for the position, momentum, and energy observables of a quantum mechanical particle are shown to be inevitable consequences of the basic axioms and the requirement of covariance. In addition, a classification of single particle systems is obtained in terms of the spin of the particle. The spin of a particle, which is so characteristic of quantum mechanics, is a manifestation of the *geometry* of the configuration space of the particle.

The final chapter discusses the description of free particles from the relativistic viewpoint. The results of Chapters V, VI and VII are used to obtain a classification of these particles in terms of their mass and spin. With each particle it is possible to associate a vector bundle whose square integrable sections constitute the Hilbert space of the particle. These abstract results lead to the standard transformation formulae for the (one particle) states under the elements of the relativity group. By taking Fourier transforms, it is possible to associate with each particle a definite wave equation. In particular, the Dirac equation of the free electron is obtained in this manner. The same methods lead to the localization in space, for a given time instant, of the particles of nonzero rest mass. The chapter ends with an analysis of Galilean relativity. It is shown that the free particles which are governed by Galilei's principle of relativity are none other than the Schrödinger particles of *positive* mass and arbitrary spin.

With this the program of obtaining a geometric view of the quantum mechanical world is completed. It is my belief that no other approach leads so clearly and smoothly to the fundamental results. It may be hoped that such methods may also lead to a successful description of the world of interacting particles and their fields. The realization of such hopes seems to be a matter for the future.

V. S. VARADARAJAN

CHAPTER I

BOOLEAN ALGEBRAS ON A CLASSICAL PHASE SPACE

1. THE CLASSICAL PHASE SPACE

We begin with a brief account of the usual description of a classical mechanical system with a finite number of *degrees of freedom*. Associated with such a system there is an integer n, and an open set M of the n-dimensional space R^n of n-tuples $(x_1, x_2, \cdots, x_n)$ of real numbers. n is called the *number* of degrees of freedom of the system. The points of M represent the possible *configurations* of the system. A *state* of the system at any instant of time is specified completely by giving a $2n$-tuple $(x_1, x_2, \cdots, x_n, p_1, \cdots, p_n)$ such that $(x_1, \cdots, x_n)$ represents the configuration and $(p_1, \cdots, p_n)$ the momentum vector, of the system at that instant of time. The possible states of the system are thus represented by the points of the open set $M \times R^n$ of R^{2n}. The law of evolution of the system is specified by a smooth function H on $M \times R^n$, called the *Hamiltonian* of the system. If $(x_1(t), \cdots, x_n(t), p_1(t), \cdots, p_n(t))$ represents the state of the system at time t, then the functions $x_i(\cdot)$, $p_i(\cdot)$, $i = 1, 2, \cdots, n$, satisfy the well known differential equations:

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial H}{\partial p_i}, \qquad i = 1, 2, \cdots, n, \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial x_i}, \qquad i = 1, 2, \cdots, n. \end{aligned} \tag{1}$$

For most of the systems which arise in practice these equations have unique solutions for all t in the sense that given any real number t_0, and a point $(x_1{}^0, x_2{}^0, \cdots, x_n{}^0, p_1{}^0, \cdots, p_n{}^0)$ of $M \times R^n$, there exists a unique differentiable map $t \to (x_1(t), \cdots, x_n(t), p_1(t), \cdots, p_n(t))$ of R^1 into $M \times R^n$ such that $x_i(\cdot)$ and $p_i(\cdot)$ satisfy the equations (1) with the *initial conditions*

$$x_i(t_0) = x_i{}^0, \quad p_i(t_0) = p_i{}^0, \qquad i = 1, 2, \cdots, n. \tag{2}$$

If we denote by s an arbitrary point of $M \times R^n$, it then follows in the standard fashion that for any t there exists a mapping $D(t)(s \to D(t)s)$

of $M \times R^n$ into itself with the property that if s is the state of the system at time t_0, $D(t)s$ is the state of the system at time $t+t_0$. The transformations $D(t)$ are one-one, map $M \times R^n$ onto itself and satisfy the equations:

$$\begin{aligned} D(0) &= I \quad \text{(the identity mapping)}, \\ D(-t) &= D(t)^{-1}, \\ D(t_1+t_2) &= D(t_1)D(t_2). \end{aligned} \tag{3}$$

If, in addition, H is an indefinitely differentiable function, then the $D(t)$ are also indefinitely differentiable and the correspondence $t \to D(t)$ defines a one-parameter differentiable transformation group of $M \times R^n$ so that the map $t, s \to D(t)s$ of $R^1 \times M \times R^n$ into $M \times R^n$ is indefinitely differentiable. The set $M \times R^n$ of all the possible states of the system is called the *phase space* of the system.

In the formulation described above, the physical quantities or the *observables* of the system are described by real valued functions on $M \times R^n$. For example, if the system is that of a single particle of mass m which moves under some potential field, then $n=3$, $M=R^3$, and the Hamiltonian H is given by

$$H(x_1,x_2,x_3,p_1,p_2,p_3) = \frac{1}{2m}(p_1^2+p_2^2+p_3^2)+V(x_1,x_2,x_3). \tag{4}$$

The function $s \to (p_1^2+p_2^2+p_3^2)/2m$ is the kinetic energy of the particle and the function $s \to V(x_1,x_2,x_3)$ is the potential energy of the particle. The function $s \to p_i$ $(i=1, 2, 3)$ represents the x_i-component of the momentum of the particle. In the general case, if f is a function on $M \times R^n$ which describes an observable, then $f(s)$ gives the value of that observable when the system is in the state s.

This formulation of the basic ideas relating to the mechanics of a classical system can be generalized significantly (Mackey [1], Sternberg [1]). Briefly, this generalization consists in replacing the assumption that M is an open subset of R^n by the more general one that M is an abstract C^∞ manifold of dimension n. The set of all possible configurations of the system is now M, and for any $x \in M$, the momenta of the system at this configuration are the elements of the vector space M_x^*, which is the dual of the *tangent vector space* M_x of M at x. The phase space of the system is then the set of all possible pairs (x,p), where $x \in M$ and $p \in M_x^*$. This set, say S, comes equipped with a natural differentiable structure under which it is a C^∞ manifold of dimension $2n$, the so-called *cotangent bundle* of M. The manifold S admits further a canonical 2-form which is everywhere nonsingular and this gives rise to a natural isomorphism J of the module of all C^∞ vector fields on S onto the module of all 1-forms (both being considered as modules over the ring of C^∞ functions on S). The dynamical development of the system is then specified by a C^∞ function H on S, the *Hamiltonian* of the system. If $t \to s(t)$ is a curve representing a possible evolution of the system, then we have the differential equations:

$$\frac{ds(t)}{dt} = [J^{-1}(dH)](s(t)). \tag{5}$$

Here $ds(t)/dt$ is the tangent vector to S at the point $s(t)$ along the curve $t \to s(t)$ and $J^{-1}(dH)$ is the vector field on S corresponding to the 1-form dH; the right side of the equation (5) being the value of this vector field at the point $s(t)$ of S. In the special case when M is an open set in R^n and $x_1, x_2, \cdots, x_n$ are the global affine coordinates on M, S is canonically identified with $M \times R^n$ and, under this identification, J goes over into the map which transforms the vector field

$$\sum_{i=1}^{n} A_i(\partial/\partial x_i) + \sum_{i=1}^{n} B_i(\partial/\partial p_i)$$

into the 1-form

$$-\sum_{i=1}^{n} B_i \, dx_i + \sum_{i=1}^{n} A_i \, dp_i.$$

The equation (5) then goes over to (1) (cf. Chevalley [1], Helgason [1] for a discussion of the general theory of differentiable manifolds).

In this general setup, the dynamical development of the system is given by the integral curves of the vector field $J^{-1}(dH)$. It is necessary to assume that the integral curves are defined for *all* values of the time parameter t. One can then use the standard theory of vector fields to deduce the existence of a diffeomorphism $D(t)$ of S for each t such that the correspondence $t \to D(t)$ satisfies the conditions (3), and the map $t, s \to D(t)s$ of $R^1 \times S$ into S is C^∞. If the system is at the state s at time t_0, then its state at time $t+t_0$ is $D(t)s$. The physical observables of the system are then represented by real valued functions on S. A special class of Hamiltonian functions, analogous to (4), may be defined in this general framework. Let $x \to \langle .\,, . \rangle_x$ be a C^∞ Riemannian metric on M, $\langle .\,, . \rangle_x$ being a positive definite inner product on $M_x \times M_x$. For each $x \in M$, we then have a natural isomorphism $p \to p^*$ of $M_x{}^*$ onto M_x such that $p(u) = \langle u,p^* \rangle_x$ for all $p \in M_x{}^*$ and for all $u \in M_x$. The analogue of (4) is then the Hamiltonian H given by

$$H(x,p) = \langle p^*,p^* \rangle_x + V(x), \tag{6}$$

where V is a C^∞ function on M. The function $x, p \to \langle p^*,p^* \rangle_x$ then represents the kinetic energy of the system in question.

It may be pointed out that one can introduce the concept of the *momenta* of the system in this setup. Let

$$\gamma : t \to \gamma_t \tag{7}$$

be a one-parameter group of *symmetries* of the configuration space M, i.e., $\gamma(t \to \gamma_t)$ is a one-parameter group of C^∞ diffeomorphisms of M onto itself such that the map $t, x \to \gamma_t(x)$ of $R^1 \times M$ into M is C^∞. The infinitesimal generator of γ is a C^∞ vector field, say X_γ, on M; for any $x \in M$ and any real valued C^∞ function f defined around x,

$$(X_\gamma f)(x) = \left\{\frac{d}{dt} f(\gamma_{-t}(x))\right\}_{t=0}.$$

X_γ defines, in a natural fashion, a C^∞ function μ_γ on S. In fact, if $x \in M$ and $p \in M_x{}^*$,

$$\mu_\gamma(x,p) = p(X_\gamma(x))$$

(here $X_\gamma(x)$ denotes the tangent vector to M at x which is the value of X_γ at x). The observable corresponding to the function μ_γ is called the *momentum* of the system *corresponding to the one-parameter group of symmetries* γ. If $M = R^n$, if $x_1, \cdots, x_n$ are the global affine coordinates on M, and if

$$\gamma_t{}^{\mathbf{c}}(x_1, \cdots, x_n) = x_1 - tc_1, \cdots, x_n - tc_n,$$

then the observable corresponding to $\mu_{\gamma^{\mathbf{c}}}$ is called the *component of the linear momentum along* $(c_1, \cdots, c_n)$. In the same case, if

$$\gamma_t^{i,j}(x_1, \cdots, x_n) = (y_1, \cdots, y_n),$$

where

$$y_r = x_r, \quad r \neq i, j,$$

$$y_i = x_i \cos t + x_j \sin t, \quad y_j = -x_i \sin t + x_j \cos t,$$

then the observable corresponding to $\mu_{\gamma^{i,j}}$ is called the *angular momentum with respect to a rotation in the i-j plane.* A straightforward calculation shows that in the case when $M = R^n$, $S = R^n \times R^n$, and $x_1, \cdots, x_n, p_1, \cdots, p_n$ are global coordinates on S ($(x_1, \cdots, x_n, p_1, \cdots, p_n)$ depicts $\sum_{i=1}^n p_i(dx_i)_x$),

$$\mu_{\gamma^{\mathbf{c}}}(x,p) = c_1 p_1 + \cdots + c_n p_n,$$

and

$$\mu_{\gamma^{i,j}}(x,p) = x_i p_j - x_j p_i.$$

Suppose now that M is a general C^∞ manifold and S its cotangent bundle. If f and g are two C^∞ functions on S, then we can form $J^{-1}(df)$, which is a C^∞ vector field on S, and apply it to g to get another C^∞ function on S, denoted by $[f,g]$:

$$[f,g] = (J^{-1}(df))g. \tag{8}$$

$[f,g]$ is called the *Poisson Bracket* of f with g. If we use local coordinates $x_1, \cdots, x_n$ on M and the induced coordinates $x_1, \cdots, x_n, p_1, \cdots, p_n$ on S (so that $(x_1, \cdots, x_n, p_1, \cdots, p_n)$ represents $\sum_i p_i(dx_i)$), then J goes over into the map which (locally) sends $\sum_i A_i(\partial/\partial x_i) + \sum_i B_i(\partial/\partial p_i)$ into $-\sum_i B_i\, dx_i + \sum_i A_i\, dp_i$, and $[f,g]$ becomes

$$[f,g] = \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i}\right). \tag{9}$$

The map $f, g \to [f,g]$ is bilinear, skew symmetric, and satisfies the identity

$$[[f,g],h] + [[g,h],f] + [[h,f],g] = 0,$$

as is easily verified from (9). If X is any C^∞ vector field on M and μ_X is the C^∞ function on S defined by

$$\mu_X(x,p) = p(X(x)),$$

then one can verify, using (9), that

$$\mu_{aX+bY} = a\mu_X + b\mu_Y \qquad (a, b \text{ constants}),$$

$$\mu_{[X,Y]} = [\mu_X, \mu_Y],$$

where $[X,Y]$ is the *Lie bracket* of the vector fields X and Y. If f is any C^∞ function on M and f^0 is the lifted function on S, i.e.,

$$f^0(x,p) = f(x),$$

then we may use (9) once again to check that

$$[\mu_X, f^0] = (Xf)^0$$

for any C^∞ vector field X on M.

In many problems, there is a Lie group G which acts on M and provides the natural symmetries of the problem. For $g \in G$ we write $x \to g\cdot x$ for the symmetry associated with g and assume that $g, x \to g\cdot x$ is C^∞ from $G \times M$ into M. In such problems, one restricts oneself to the momenta specified by the one-parameter groups of M. If $\mathfrak{g}$ is the Lie algebra of G (cf. Chevalley [1]) and if we associate for $X \in \mathfrak{g}$, the vector field on M denoted by X also and defined by

$$(Xf)(x) = \left(\frac{d}{dt} f(\exp tX\cdot x)\right)_{t=0},$$

then we obtain the relations

$$\begin{aligned} \mu_{[X,Y]} &= [\mu_X, \mu_Y], \\ [\mu_X, f^0] &= (Xf)^0 \end{aligned} \tag{10}$$

between the configuration observables f^0 and the momentum observables μ_X. These relations are usually referred to as *commutation rules.*

2. THE LOGIC OF A CLASSICAL SYSTEM

We shall now examine the algebraic aspects of a general classical system. In view of the discussion carried out just now, it is clear that for any classical system $\mathfrak{S}$ there is associated a space S called the phase space of $\mathfrak{S}$. The states of the system are in one-one correspondence with the points of S. The notion of a state is so formulated that if one knows the state of the system at an instant of time t_0, and also the dynamical law of evolution of the system, then one can determine the state of the system at time $t+t_0$. The observables or physical quantities which are of interest to the observer are then represented by real valued functions on S. If f is the function corresponding to a particular observable, its value $f(s)$ at the point s of S is interpreted as the value of the physical quantity when the system is in the state s. If s is the state of the system at time t_0, we can write $D(t)s$ for the state of the system at time $t+t_0$. We thus have a transformation $D(t)$ of S into itself. For each t, $D(t)$ is invertible and maps S onto itself. The correspondence $t \to D(t)$ satisfies the equations (3). $t \to D(t)$ is then a one-parameter group of transformations of S. It is called the *dynamical group* of the system $\mathfrak{S}$.

These concepts make sense in every classical system. In the case of any such system the most general statement which can be made about it is one which asserts that the value of a certain observable lies in a real number set E. If the observable is represented by the function f on S, then such a statement is equivalent to the statement that the state of the system lies in the set $f^{-1}(E)$ of the space S. In other words, the physically meaningful statements that can be made about the system are in correspondence with certain subsets of S. The inclusion relations for subsets naturally correspond to implications of statements. In mathematical terms, this means that at the background of the classical system there is a Boolean algebra of subsets of the space S, the elements of which represent the statements about the physical system. It is natural to call this Boolean algebra the *logic* of the system.

Suppose now that $\mathfrak{S}$ is a system which does not follow the laws of classical mechanics. Then one cannot associate with it a phase space in general. It is nevertheless meaningful to consider the totality of experimentally verifiable statements which may be made about the system. This collection, which may be called the *logic* of $\mathfrak{S}$, comes equipped with the relations of implication and negation which convert it into a complemented partially ordered set. For a classical system this partially ordered set is a Boolean algebra. Clearly, it is possible to conceive of mechanical systems whose logics are not Boolean algebras. *We take the point of view that quantum mechanical systems are those whose logics form some sort of projective geometries and which are consequently nondistributive* *lattices*. With such a point of view it is possible to understand the role played by simultaneously observable quantities, the uncertainty relations, and the complementarity principles. These phenomena, which are so peculiar to quantum systems, will then be seen to be consequences of the nondistributive nature of the logic in the background of the system $\mathfrak{S}$.

It might seem a bit surprising that the basic assumption on a quantum system is that its logic is not a distributive lattice. It would be natural to argue that statements about a physical system should obey the same rules as the rules of ordinary set theory. The well known critiques of von Neumann and Heisenberg address this question (von Neumann [1], Birkhoff-von Neumann [1], Heisenberg [1]). The point is that only *experimentally verifiable statements* are to be regarded as members of the logic of the system. Consequently, as it happens in many questions in atomic physics, it may be impossible to verify *experimentally* statements which involve the values of two physical quantities of the system—for example, measurements of the position and momentum of an electron. One can verify statements about one of them but not, in general, those which involve both of them. What the basic assumptions imply is that the statements regarding position or momentum form two Boolean subalgebras of the logic but that there is in general no Boolean algebra which contains both of these Boolean subalgebras.

Before beginning an analysis of the logic of general quantum mechanical systems it would be helpful to recast at least some of the features of the formulation given in section 1 in terms of the logic of the classical system. In the first place it is natural to strengthen the hypothesis and assume that the logic of a given classical system $\mathfrak{S}$ is a Boolean σ-algebra, say $\mathscr{L}$, of subsets of S, the phase space of $\mathfrak{S}$. Suppose now, that an observable associated with the system is represented by the real valued function f on S. The statements concerning the observable are then those which assert that its value lies in an arbitrary Borel set E of the real line and these are represented by the subsets $f^{-1}(E)$ of S. The observable can thus be represented, without any loss of physical content, equally by the map $E \to f^{-1}(E)$ of the class of Borel subsets of the real line into $\mathscr{L}$. The range of this mapping is a sub-σ-algebra, say $\mathscr{L}_f$. Suppose g is a real valued Borel function on the real line. Then, the observable represented by the function $g \circ f$ $(s \to g(f(s)))$ can also be represented by the map $E \to f^{-1}(g^{-1}(E))$ from which we conclude that $\mathscr{L}_{g\circ f}$ is contained in $\mathscr{L}_f$.

In order to formulate the general features of a classical mechanical system in terms of its logic $\mathscr{L}$, it is therefore necessary to determine to what extent an abstract σ-algebra $\mathscr{L}$ can be regarded as a σ-algebra of subsets of some space S; further to determine the class of mappings from the σ-algebra of Borel sets of the real line into $\mathscr{L}$ which correspond to real valued functions on S; and to clarify the concept of functional dependence in this general context. We shall now proceed to a discussion of these questions.

3. BOOLEAN ALGEBRAS

Let $\mathscr{L}$ be a nonempty set. $\mathscr{L}$ is said to be *partially ordered* if there is a relation $<$ between some pairs of elements of $\mathscr{L}$ such that (i) $a<a$ for all a in $\mathscr{L}$; (ii) $a<b$ and $b<a$ imply $a=b$; (iii) $a<b$ and $b<c$ imply $a<c$. If $\mathscr{L}$ is partially ordered, there is at most one element called the *null* or *zero* element and denoted by 0, such that $0<a$ for all a in $\mathscr{L}$. Similarly there is at most one element called the *unit* element and denoted by 1, such that $a<1$ for all a in $\mathscr{L}$. More generally, for any nonempty subset F of $\mathscr{L}$ there exists at most one element c of $\mathscr{L}$ such that (i) $a<c$ for all $a \in F$; (ii) if d is any element of $\mathscr{L}$ such that $a<d$ for all $a \in F$, then $c<d$. We shall write $\bigvee_{a\in F} a$ for c whenever it exists. If F is a finite set, say $F=\{a_1,\cdots,a_n\}$, it is customary to write $\bigvee_{i=1}^n a_i$ or $a_1 \vee a_2 \vee \cdots \vee a_n$ instead of $\bigvee_{a\in F} a$. In an analogous fashion, for any subset F of $\mathscr{L}$ there exists at most one element c such that (i) $c<a$ for all $a \in F$; (ii) if d is any element of $\mathscr{L}$ such that $d<a$ for all $a \in F$, then $d<c$; we denote it by $\bigwedge_{a\in F} a$ whenever it exists. If F is a finite set, say $F=\{a_1,\cdots,a_n\}$, we often write $\bigwedge_{i=1}^n a_i$ or $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ instead of $\bigwedge_{a\in F} a$. The partially ordered set $\mathscr{L}$ is called a *lattice* if

(11)
(i) 0 and 1 exist in $\mathscr{L}$ and $0\neq 1$,
(ii) $\bigvee_{a\in F} a$ and $\bigwedge_{a\in F} a$ exist for all finite subsets F of $\mathscr{L}$.

Suppose that $\mathscr{L}$ is a lattice. Given any element a of $\mathscr{L}$, an element a' of $\mathscr{L}$ is said to be a *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$. a is then a complement of a'. $\mathscr{L}$ is said to be *complemented* if, given any element, there exists at least one complement of it. It is obvious that 0 and 1 have the unique complements 1 and 0, respectively. A lattice $\mathscr{L}$ is said to be *distributive* if for any three elements a, b, c of $\mathscr{L}$, the identities

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

are satisfied. A complemented distributive lattice is called a *Boolean algebra.* A *Boolean σ-algebra* $\mathscr{L}$ is a Boolean algebra in which $\bigwedge_{a\in F} a$ and $\bigvee_{a\in F} a$ exist for every countable subset F of $\mathscr{L}$.

Every element in a Boolean algebra has a unique complement. Suppose in fact that $\mathscr{L}$ is a Boolean algebra and that a is an element with two complements a_1 and a_2. Then, one has

$$a_1 = a_1 \wedge (a \vee a_2) = (a_1 \wedge a) \vee (a_1 \wedge a_2) = a_1 \wedge a_2 < a_2;$$

similarly, $a_2 < a_1$, so that $a_1 = a_2$. The unique complement of a is denoted by a'. Using the standard manipulations of set theory it is easy to show that $(\bigwedge_{a \in F} a)' = \bigvee_{a \in F} a'$ and $(\bigvee_{a \in F} a)' = \bigwedge_{a \in F} a'$ for any finite subset F of $\mathscr{L}$. If $\mathscr{L}$ is a Boolean σ-algebra, then the same identities remain valid even when F is countably infinite. If $\mathscr{L}$ is any Boolean algebra and a, b are elements in it with $a < b$, $c = a' \wedge b$ is the unique element of $\mathscr{L}$ such that $a \wedge c = 0$ and $a \vee c = b$; c is called *the complement of* a *in* b. Since $c = b \wedge a'$, $c < a'$ (cf. Birkhoff [1] and Sikorski [1] for the general theory of Boolean algebras and σ-algebras).

A *homomorphism* of a Boolean algebra $\mathscr{L}_1$ into a Boolean algebra $\mathscr{L}_2$ is a map h of $\mathscr{L}_1$ into $\mathscr{L}_2$ such that (i) $h(0) = 0$, $h(1) = 1$; (ii) $h(a') = h(a)'$ for all a in $\mathscr{L}_1$; (iii) $h(a \vee b) = h(a) \vee h(b)$, $h(a \wedge b) = h(a) \wedge h(b)$ for all a, b in $\mathscr{L}_1$. If h is a homomorphism and $a < b$, then $h(a) < h(b)$. An *isomorphism* of $\mathscr{L}_1$ onto $\mathscr{L}_2$ is a homomorphism h of $\mathscr{L}_1$ onto $\mathscr{L}_2$ such that $h(a) = 0$ if and only if $a = 0$; in this case h is also one-one.

The class of *all* subsets of any set is a Boolean algebra under set inclusion and set complementation. However, obviously this is not the most general Boolean algebra since infinite unions and intersections exist in it. Suppose now that X is a topological space. The class of subsets of X which are both open and closed (open-closed) is obviously a Boolean algebra. A well known theorem of Stone [1] asserts that every Boolean algebra is isomorphic to one such and that, if we require the topological space to be compact Hausdorff as well as totally disconnected, it is essentially uniquely determined by the Boolean algebra. We recall that a compact space is said to be *totally disconnected* if every open subset of it can be written as a union of open-closed subsets. We shall call a compact Hausdorff totally disconnected space a *Stone space.*

Let $\mathscr{L}$ be a Boolean algebra. A subset $\mathscr{M}$ of $\mathscr{L}$ is called a *dual ideal* if the following properties are satisfied:

(i) $0 \notin \mathscr{M}$,
(ii) if $a \in \mathscr{M}$ and $a < b$, then $b \in \mathscr{M}$,
(iii) if $a, b \in \mathscr{M}$, then $a \wedge b \in \mathscr{M}$.

$\mathscr{M}$ is said to be *maximal* if it is properly contained in no other dual ideal.

The naturalness of the notion of maximal dual ideals can be seen in the following way. Let X be a Stone space and $\mathscr{L} = \mathscr{L}(X)$ the Boolean algebra of all open-closed subsets of X. Then, for any $x \in X$, the collection $\mathscr{M}(x)$, where

$$\mathscr{M}(x) = \{A : A \in \mathscr{L}, x \in A\},$$

is easily seen to be a maximal dual ideal; it is also easy to check that the correspondence $x \to \mathscr{M}(x)$ is one-one if we notice that X is Hausdorff. The concept of maximal dual ideals is central in the proof of Stone's theorem.

Suppose that $\mathscr{L}$ is an arbitrary Boolean algebra. Using Zorn's lemma one can show easily that maximal dual ideals of $\mathscr{L}$ exist. Let $X=X(\mathscr{L})$ be the set of all maximal dual ideals of $\mathscr{L}$. For any $a \in \mathscr{L}$ we define X_a by

$$X_a = \{\mathscr{M} : \mathscr{M} \in X, a \in \mathscr{M}\},$$

$X_0 = \varnothing$, the null set, and $X_1 = X$. We shall say that a subset $A \subseteq X$ is *open* if A is the union of sets of the form X_a. This definition defines the structure of a topology on X called the *Stone topology*. We now have:

Theorem 1.1 (*Stone* [1]). *Let $\mathscr{L}$ be a Boolean algebra and let $X=X(\mathscr{L})$ be the space of all maximal dual ideals of $\mathscr{L}$. Then, equipped with the Stone topology, X becomes a Stone space. The map $a \to X_a$ is then an isomorphism of $\mathscr{L}$ with the Boolean algebra of all open-closed subsets of X. X is determined by $\mathscr{L}$, among the class of Stone spaces, up to a homeomorphism. More generally, let X and Y be Stone spaces and let $\mathscr{L}(X)$ and $\mathscr{L}(Y)$ be their respective Boolean algebras of open-closed subsets. If u is any isomorphism of $\mathscr{L}(Y)$ onto $\mathscr{L}(X)$, there exists a homeomorphism h of X onto Y such that*

$$u(A) = h^{-1}(A) \qquad (A \in \mathscr{L}(Y)); \tag{12}$$

moreover, h is uniquely determined by (12).

This theorem is very well known and we do not give its proof. The reader may consult the books of Birkhoff [1], Sikorski [1], and the paper of Stone [1] for the proof.

Corollary 1.2. *Let X be a Stone space and let $\mathscr{L}=\mathscr{L}(X)$ be the Boolean algebra of open-closed subsets of X. If $t \to D_t(-\infty < t < \infty)$ is any one-parameter group of automorphisms of $\mathscr{L}$, there exists a unique one-parameter group $t \to h_t$ of homeomorphisms of X onto itself such that for all t and $A \in \mathscr{L}$, $D_t(A) = h_t^{-1}(A)$.*

Proof. Theorem 1.1 ensures the existence and uniqueness of each h_t. If t_1, t_2 are real, then $h_{t_1+t_2}$ and $h_{t_1} \circ h_{t_2}$ induce the same automorphism $D_{t_1+t_2}$ of $\mathscr{L}$, so that $h_{t_1+t_2} = h_{t_1} \circ h_{t_2}$.

The theorem of Stone shows that there is essentially no distinction between an abstract Boolean algebra and a Boolean algebra of sets. If one deals with Boolean σ-algebras, the situation becomes somewhat less straightforward. We shall now describe the modifications necessary when one replaces Boolean algebras by Boolean σ-algebras.

If $\mathscr{L}_1$ and $\mathscr{L}_2$ are Boolean σ-algebras, and h a map of $\mathscr{L}_1$ into $\mathscr{L}_2$, h is called a *σ-homomorphism* if (i) $h(0)=0$, $h(1)=1$; (ii) $h(a')=h(a)'$ for all $a \in \mathscr{L}_1$; and (iii) if F is any subset of $\mathscr{L}_1$ which is finite or countably infinite, $h(\bigvee_{a\in F} a) = \bigvee_{a \in F} h(a)$ and $h(\bigwedge_{a\in F} a) = \bigwedge_{a\in F} h(a)$. Suppose $\mathscr{L}_1$

and $\mathscr{L}_2$ are two Boolean σ-algebras and h a σ-homomorphism of $\mathscr{L}_1$ onto $\mathscr{L}_2$. The set $\mathscr{N}=\{a : a \in \mathscr{L}_1, h(a)=0\}$ is a subset of $\mathscr{L}_1$ with the properties: (i) $0 \in \mathscr{N}$, $1 \notin \mathscr{N}$; (ii) if $a \in \mathscr{N}$ and $b<a$, then $b \in \mathscr{N}$; (iii) if F is a countable subset of $\mathscr{N}$, $\bigvee_{a\in F} a \in \mathscr{N}$. $\mathscr{N}$ is called the *kernel* of h. Suppose conversely $\mathscr{L}$ is a Boolean σ-algebra and $\mathscr{N}$ a subset of $\mathscr{L}$ with properties (i) to (iii) listed above. We shall say that elements a and b of $\mathscr{L}$ are equivalent, $a \sim b$, if $a \wedge b'$ and $b \wedge a'$ are in $\mathscr{N}$. It is easily verified that $\sim$ is an equivalence relation. Let $\overline{\mathscr{L}}$ be the set of all equivalence classes, and for any a in $\mathscr{L}$, let $\bar{a}$ denote the unique equivalence class containing a. We define $\bar{a}<\bar{b}$ whenever there are elements a in $\bar{a}$, and b in $\bar{b}$ such that $a<b$. It is then easily shown that $\overline{\mathscr{L}}$ is a Boolean σ-algebra whose zero and unit elements are, respectively, $\bar{0}$ and $\bar{1}$, and that the map $a \to \bar{a}$ is a σ-homomorphism of $\mathscr{L}$ onto $\overline{\mathscr{L}}$ with kernel $\mathscr{N}$. We write $\overline{\mathscr{L}}=\mathscr{L}/\mathscr{N}$.

Theorem 1.3 (*Loomis* [1]). *Let $\mathscr{L}$ be a Boolean σ-algebra. Then there exists a set X, a σ-algebra $\mathscr{S}$ of subsets of X, and a σ-homomorphism h of $\mathscr{S}$ onto $\mathscr{L}$.*

Proof. Let X be a Stone space such that the lattice $\mathscr{L}'=\mathscr{L}(X)$ of open-closed subsets of X is isomorphic to the Boolean algebra $\mathscr{L}$. Let $\mathscr{S}$ denote the smallest σ-algebra of subsets of X containing $\mathscr{L}'$. We denote by $\cup$ and $\cap$ the operations of set union and set intersection for subsets of X, and by $\vee$ and $\wedge$ the lattice-theoretic operations in $\mathscr{L}$ and $\mathscr{L}'$.

If $A_1, A_2, \cdots$ is any sequence of sets in $\mathscr{L}'$, then $\bigvee_n A_n=A$ exists in $\mathscr{L}'$ since $\mathscr{L}'$ is isomorphic to $\mathscr{L}$ and $\mathscr{L}$ is a σ-algebra. Since A is the smallest element of $\mathscr{L}'$ containing all the A_n, it follows that the set $A-\bigcup_n A_n$ cannot contain any element of $\mathscr{L}'$ as a subset. The sets in $\mathscr{L}'$ form a base for the topology of X and hence we conclude that $A-\bigcup_n A_n$ cannot contain any nonnull open set. Since $\bigcup_n A_n$ is open, this shows that $A-\bigcup_n A_n$ is a closed nondense set.

Consider now the class $\mathscr{S}_1$ of all sets $A \in \mathscr{S}$ with the property that for some B in $\mathscr{L}'$, $(A-B) \cup (B-A)$ is of the first category. If B_1 and B_2 are elements of $\mathscr{L}'$ such that $(A-B_i) \cup (B_i-A)$ is of the first category ($i=1, 2$), then it will follow that $(B_1-B_2) \cup (B_2-B_1)$ is of the first category, which is not possible (by the category theorem of Baire) unless $B_1=B_2$. Thus, for any A in $\mathscr{S}_1$ there exists a *unique* $B=h_1(A)$ in $\mathscr{L}'$ such that $(A-B) \cup (B-A)$ is of the first category. Clearly $\mathscr{L}' \subseteq \mathscr{S}_1$, and for $A \in \mathscr{L}'$, $h_1(A)=A$.

We claim that $\mathscr{S}_1$ is a σ-algebra. Since

$$(A-B) \cup (B-A) = (A'-B') \cup (B'-A')$$

(primes denoting complementation in X), we see that for any A in $\mathscr{S}_1$,

A' is in $\mathscr{S}_1$ and $h_1(A')=h_1(A)'$. Suppose $A_1, A_2, \cdots$ is any sequence in $\mathscr{S}_1$. Write $B_n=h_1(A_n)$, $A=\bigcup_n A_n$, $B=\bigvee_n B_n$, $B_0=\bigcup_n B_n$. By what we said above, $B-B_0$ is a closed nondense set. Moreover, as $B_0 \subseteq B$, we have

$$\begin{aligned}(A-B) \cup (B-A) &\subseteq \{(A-B_0) \cup (B_0-A)\} \cup (B-B_0)\\ &\subseteq \bigcup_n \{(A_n-B_n) \cup (B_n-A_n)\} \cup (B-B_0).\end{aligned}$$

As all members of the right side are of the first category, this proves that $A \in \mathscr{S}_1$ and $h_1(A)=\bigvee_n h_1(A_n)$. In a similar fashion we can show that $\bigcap_n A_n$ lies in $\mathscr{S}_1$ and $h_1(\bigcap_n A_n)=\bigwedge_n h_1(A_n)$.

The conclusions of the preceding paragraph show that $\mathscr{S}_1$, is a Boolean σ-algebra $\subseteq \mathscr{S}$. Since $\mathscr{S}_1$ contains $\mathscr{L}'$, $\mathscr{S}_1=\mathscr{S}$. Moreover, we see at the same time that h_1 is a σ-homomorphism of $\mathscr{S}$ onto $\mathscr{L}'$. If we write $h=k \circ h_1$ where k is an isomorphism of $\mathscr{L}'$ onto $\mathscr{L}$, then h is a σ-homomorphism of $\mathscr{S}$ onto $\mathscr{L}$.

Remark. Let $\mathscr{S}$ be the σ-algebra of Borel sets on the unit interval [0,1], $\mathscr{N}$ the class of Borel sets of Lebesgue measure 0, and $\mathscr{L}=\mathscr{S}/\mathscr{N}$. Then $\mathscr{L}$ is a Boolean σ-algebra. We can obviously define Lebesgue measure λ as a countably additive function $\tilde{\lambda}$ on $\mathscr{L}$; $\tilde{\lambda}$ is strictly positive in the sense that for any $a \neq 0$ of $\mathscr{L}$, $\tilde{\lambda}(a)$ is positive. From this it follows that any family of mutually disjoint elements of $\mathscr{L}$ is countable. On the other hand, since $\mathscr{S}$ is countably generated, so is $\mathscr{L}$. However, any σ-algebra of subsets of some space X which is countably generated can be proved to have atoms, that is, minimal elements. Since $\mathscr{L}$ does not have atoms, $\mathscr{L}$ cannot be isomorphic to any σ-algebra of sets.

4. FUNCTIONS

We now take up the second question raised in section 2, namely, the problem of describing the calculus of functions on a set X entirely in terms of the Boolean σ-algebra of subsets of X with respect to which all these functions are measurable. The results are summarized in theorems 1.4 and 1.6 of this section.

Let X be any set of points x and $\mathscr{S}$ a Boolean σ-algebra of subsets of X. A function f from X into a complete separable metric space Y is said to be $\mathscr{S}$-measurable if $f^{-1}(E) \in \mathscr{S}$ for all Borel sets $E \subseteq Y$. If f is $\mathscr{S}$-measurable, the mapping $E \to f^{-1}(E)$ is a σ-homomorphism of the σ-algebra $\mathscr{B}(Y)$ of Borel subsets of Y into $\mathscr{S}$. Suppose now $\mathscr{L}$ is an abstract Boolean σ-algebra. We shall then define a *Y-valued observable associated with $\mathscr{L}$* to be any σ-homomorphism of $\mathscr{B}(Y)$ into $\mathscr{L}$. If $Y=R^1$, the real line, we call these observables *real valued* and refer to them simply as *observables*. From our definition of σ-homomorphisms we see that a map $u(E \to u(E))$

of $\mathscr{B}(Y)$ into $\mathscr{L}$ is a Y-valued observable associated with $\mathscr{L}$ if and only if (i) $u(\varnothing)=0$, $u(Y)=1$; (ii) $u(Y-E)=u(E)'$ for all E in $\mathscr{B}(Y)$; (iii) if $E_1, E_2, \cdots$ is any sequence of Borel sets in Y, $u(\bigcup_n E_n)=\bigvee_n u(E_n)$ and $u(\bigcap_n E_n)=\bigwedge_n u(E_n)$.

Theorem 1.4 *Let X be a set, $\mathscr{S}$ a Boolean σ-algebra of subsets of X and h a σ-homomorphism of $\mathscr{S}$ onto a Boolean σ-algebra $\mathscr{L}$. Suppose further that $u(E \to u(E))$ is any (real valued) observable associated with $\mathscr{L}$. Then there exists an $\mathscr{S}$-measurable real valued function f defined on X such that*

$$u(E) = h(f^{-1}(E)) \tag{13}$$

for all Borel sets $E \subseteq R^1$. f is essentially unique in the sense that if g is any $\mathscr{S}$-measurable real valued function defined on X such that $u(E)=h(g^{-1}(E))$ for all Borel sets $E \subseteq R^1$, the set $\{x : x \in X, f(x) \neq g(x)\}$ belongs to the kernel of h.

Proof. We begin with a simple observation. Suppose A and B are two subsets of X in $\mathscr{S}$ such that $A \subseteq B$, and c any element of $\mathscr{L}$ such that $h(A) < c < h(B)$. Then we can select a set C in $\mathscr{S}$ such that $A \subseteq C \subseteq B$ and $h(C)=c$. In fact, since h maps $\mathscr{S}$ onto $\mathscr{L}$, there exists C_1 in $\mathscr{S}$ such that $h(C_1)=c$. If we define $C=(C_1 \cap B) \cup A$, then $A \subseteq C \subseteq B$ while

$$h(C) = (h(C_1) \wedge h(B)) \vee h(A) = (c \wedge b) \vee a = c.$$

We now come to the proof of theorem 1.4. Let $r_1, r_2, \cdots$ be any distinct enumeration of the rational numbers in R^1 and let D_i be the interval $\{t : t \in R^1, t < r_i\}$. Evidently, $u(D_i) < u(D_j)$ whenever $r_i < r_j$. We shall now construct sets $A_1, A_2, \cdots$ in $\mathscr{S}$ such that (a) $h(A_i)=u(D_i)$ for all $i=1, 2, 3, \cdots$; (b) $A_i \subseteq A_j$ whenever $r_i < r_j$. Let A_1 be any set in $\mathscr{S}$ such that $h(A_1)=u(D_1)$. Suppose $A_1, A_2, A_3, \cdots, A_n$ in $\mathscr{S}$ have been constructed such that (i) $h(A_i)=u(D_i)$ for $i=1, 2, \cdots, n$; (ii) $A_i \subseteq A_j$ whenever $r_i < r_j$, $1 \leq i, j \leq n$. We shall construct A_{n+1} as follows. Let $(i_1, i_2, \cdots, i_n)$ be the permutation of $(1, 2, \cdots, n)$ such that $r_{i_1} < r_{i_2} < \cdots < r_{i_n}$. Then, there exists a unique k such that $r_{i_k} < r_{n+1} < r_{i_{k+1}}$ (we define $r_{i_0} = -\infty$ and $r_{i_{n+1}} = +\infty$), and by the observation made in the preceding paragraph, we can select A_{n+1} in $\mathscr{S}$ such that $A_{i_k} \subseteq A_{n+1} \subseteq A_{i_{k+1}}$ (we define $A_{i_0} = \varnothing$, $A_{i_{n+1}} = X$). The collection $\{A_1, A_2, \cdots, A_{n+1}\}$ then has the same properties relative to $r_1, r_2, \cdots, r_{n+1}$ as $\{A_1, \cdots, A_n\}$ had relative to $r_1, r_2, \cdots, r_n$. It thus follows by induction that there exists a sequence $A_1, A_2 \cdots$ of sets in $\mathscr{S}$ with the properties (a) and (b). As

$$h\Big(\bigcap_j A_j\Big) = \bigwedge_j u(D_j) = u\Big(\bigwedge_j D_j\Big) = 0,$$

we may, by replacing A_k by $A_k - \bigcap_j A_j$ if necessary, assume that

$\bigcap_j A_j = \varnothing$. Further $h(\bigcup_j A_j) = \bigvee_j u(D_j) = u(\bigvee_j D_j) = 1$ so that $h(N) = 0$, where $N = X - \bigcup_j A_j$. We now define a function f as follows:

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in N, \\ \inf\{r_j : x \in A_j\} & \text{if} \quad x \in \bigcup_j A_j. \end{cases} \tag{14}$$

Clearly, f is finite everywhere. Moreover, for any k,

$$f^{-1}(D_k) \cap (X - N) = \bigcup_{j : r_j < r_k} A_j,$$

so that f is $\mathscr{S}$-measurable. Further,

$$\begin{aligned} h(f^{-1}(D_k)) &= h\Big(\bigcup_{j : r_j < r_k} A_j\Big) \\ &= \bigvee_{j : r_j < r_k} u(D_j) \\ &= u(D_k), \end{aligned}$$

so that $h(f^{-1}(E)) = u(E)$ whenever $E = D_k$ for some k. Since the class of all E for which this equation is valid is a Boolean σ-algebra, we conclude that $h(f^{-1}(E)) = u(E)$ for all Borel sets E.

It remains to examine the uniqueness. Let g be any real valued $\mathscr{S}$-measurable function on X such that $h(g^{-1}(E)) = u(E)$ for all Borel sets E. Then, if we write D_k' for $R^1 - D_k$,

$$\begin{aligned} M &= \{x : x \in X, f(x) \neq g(x)\} \\ &= \bigcup_k \{(f^{-1}(D_k) \cap g^{-1}(D_k')) \bigcup (f^{-1}(D_k') \cap g^{-1}(D_k))\}, \end{aligned}$$

so that

$$\begin{aligned} h(M) &= \bigvee_k \{u(D_k) \wedge u(D_k')\} \\ &= 0. \end{aligned}$$

This shows that M belongs to the kernel of h. This completes the proof of the theorem.

Lemma 1.5. *Let X be a set, $\mathscr{S}$ a σ-algebra of subsets of X and f an $\mathscr{S}$-measurable mapping of X into R^n. Suppose $\mathscr{S}^{\sim} = \{f^{-1}(F) : F \in \mathscr{B}(R^n)\}$. Then to any $\mathscr{S}^{\sim}$-measurable real function c on X there corresponds a real valued Borel function $c^{\sim}$ on R^n such that $c(x) = c^{\sim}(f(x))$ for all $x \in X$.*

Proof. Since c is $\mathscr{S}^{\sim}$-measurable, there exists a sequence c_n $(n = 1, 2, \cdots)$ of $\mathscr{S}^{\sim}$-measurable functions such that (i) each c_n takes only finitely many values; (ii) $c_n(x) \to c(x)$ for all $x \in X$. For any n, let $A_{n1}, A_{n2}, \cdots, A_{nk_n}$ be disjoint subsets of X whose union is X such that c_n is a constant, say a_{ni}, on A_{ni}, the a_{ni} being distinct for $i = 1, 2, \cdots, k_n$. Since c_n is $\mathscr{S}^{\sim}$-measurable, $A_{ni} \in \mathscr{S}^{\sim}$; so there exists a Borel set B_{ni} of R^n such that $A_{ni} = f^{-1}(B_{ni})$

($i=1, 2, \cdots, k_n$). Replacing B_{ni} by $B_{ni}-\bigcup_{j<i} B_{nj}$ if necessary, we may assume that the B_{ni} are disjoint. Let us define the function $c_n^{\sim}$ on R^n as follows:

$$c_n^{\sim}(t) = \begin{cases} a_{ni} & \text{if} \quad t \in B_{ni}, \\ 0 & \text{if} \quad t \notin \bigcup_i B_{ni}. \end{cases}$$

Clearly $c_n^{\sim}$ is Borel and $c_n(x)=c_n^{\sim}(f(x))$ for all x in X. Let us define $c^{\sim}$ on R^n as follows:

$$c^{\sim}(t) = \begin{cases} \lim_{n\to\infty} c_n^{\sim}(t) \text{ if the limit exists,} \\ 0 \text{ otherwise.} \end{cases}$$

Clearly $c^{\sim}$ is Borel. Since $c_n(x)=c_n^{\sim}(f(x))$ and $\lim c_n(x)$ exists and is equal to $c(x)$ for all x in X, $c^{\sim}(f(x))=c(x)$ for all x in X.

Let $\mathscr{L}$ be a Boolean σ-algebra. $\mathscr{L}_1 \subseteq \mathscr{L}$ is said to be a *sub-σ-algebra* if (i) $0, 1 \in \mathscr{L}_1$; (ii) if $a \in \mathscr{L}_1$, then $a' \in \mathscr{L}_1$; (iii) if $a_1, a_2, \cdots$ are in $\mathscr{L}_1$, then $\bigvee_n a_n$ and $\bigwedge_n a_n$ are in $\mathscr{L}_1$. A sub-σ-algebra $\mathscr{L}_1$ is said to be *separable* if there exists a countable subset D of $\mathscr{L}$ such that $\mathscr{L}_1$ is the smallest sub-σ-algebra of $\mathscr{L}$ containing D.

Theorem 1.6. (i) *Let $\mathscr{L}$ be a Boolean σ-algebra and $u(E \to u(E))$ an observable associated with $\mathscr{L}$. Then the range $\mathscr{L}_u=\{u(E) : E \in \mathscr{B}(R^1)\}$ of u is a separable Boolean sub-σ-algebra of $\mathscr{L}$. Conversely, if $\mathscr{L}_1$ is a separable Boolean sub-σ-algebra of $\mathscr{L}$, there exists an observable u associated with $\mathscr{L}$ such that $\mathscr{L}_1$ is the range of u.*

(ii) *Let u_i ($i=1, 2, \cdots, n$) be observables associated with $\mathscr{L}$, and $\mathscr{L}_i$ ($i=1, 2, \cdots, n$) their respective ranges. Suppose $\mathscr{L}_0$ is the smallest sub-σ-algebra of $\mathscr{L}$ containing all the $\mathscr{L}_i$. Then there exists a unique σ-homomorphism u of $\mathscr{B}(R^n)$ (the σ-algebra of Borel subsets of the n-dimensional space R^n) onto $\mathscr{L}_0$ such that for any Borel set E of R^1, $u_i(E)=u(p_i^{-1}(E))$, where p_i is the projection $(t_1, t_2, \cdots, t_n) \to t_i$ of R^n onto R^1. If φ is any real valued Borel function on R^n, the map $E \to u(\varphi^{-1}(E))(E \in \mathscr{B}(R^1))$ is an observable associated with $\mathscr{L}$ whose range is contained in $\mathscr{L}_0$. Conversely, if $v(E \to v(E))$ is any observable associated with $\mathscr{L}$ such that the range of v is contained in $\mathscr{L}_0$, there exists a real valued Borel function φ on R^n such that $v(E)=u(\varphi^{-1}(E))$ for all E.*

Proof. If u is an observable with range $\mathscr{L}_u$, $\mathscr{L}_u$ is obviously the smallest sub-σ-algebra of $\mathscr{L}$ containing all the $u(E)$, where E is any open interval of R^1 with rational end points. This shows that $\mathscr{L}_u$ is separable.

Suppose conversely that $\mathscr{L}_1 \subseteq \mathscr{L}$ is a separable sub-σ-algebra of $\mathscr{L}$. By theorem 1.3 there exists a set X, a σ-algebra $\mathscr{S}$ of subsets of X, and a σ-homomorphism h of $\mathscr{S}$ onto $\mathscr{L}$. Let $\{A_n : n=1, 2, \cdots\}$ be a countable family of sets of $\mathscr{S}$ such that $\mathscr{L}_1$ is the smallest sub σ-algebra of $\mathscr{L}$

containing all the $h(A_n)$. We denote by $\mathscr{S}_0$ the smallest σ-algebra of subsets of X containing all the A_n. The function

$$c : x \rightarrow (\chi_{A_1}(x), \chi_{A_2}(x), \cdots, \chi_{A_n}(x), \cdots)$$

(where χ_A denotes the function which is 1 on A, and 0 on $X-A$) is $\mathscr{S}_0$-measurable from X into the compact metric space Y which is the product of countably many copies of the 2-point space consisting of 0 and 1. Moreover, it is obvious that each A_n is of the form $c^{-1}(F)$ for some Borel set $F \subseteq Y$, and hence $\mathscr{S}_0 = \{c^{-1}(F) : F \text{ Borel in } Y\}$. Now, by a classical theorem (Kuratowski [1]), there exists a Borel isomorphism d of Y onto R^1, so that the function $c_1 : x \rightarrow d(c(x))$ is an $\mathscr{S}_0$-measurable real valued function and $\mathscr{S}_0 = \{c_1^{-1}(E) : E \text{ Borel in } R^1\}$. If we now define, for any Borel set E of R^1, $u(E)$ by the equation

$$u(E) = h(c_1^{-1}(E)),$$

then u is an observable associated with $\mathscr{L}$ whose range is $\mathscr{L}_1$. This proves (i).

We now come to the proof of (ii). Suppose $u_1, u_2, \cdots, u_n$ are observables associated with $\mathscr{L}$, having ranges $\mathscr{L}_1, \cdots, \mathscr{L}_n$, respectively. Each $\mathscr{L}_i$ is separable and hence $\mathscr{L}_0$, the smallest sub-σ-algebra of $\mathscr{L}$ containing all the $\mathscr{L}_i$, is also separable. Let X, $\mathscr{S}$, and h have the same significance as in the proof of (i). By theorem 1.4, there exists a real valued $\mathscr{S}$-measurable function f_i on X such that $u_i(E) = h(f_i^{-1}(E))$ for all Borel subsets E of R^1. Let $\tilde{f}$ be the map $x \rightarrow (f_1(x), \cdots, f_n(x))$ of X into R^n. Then $\tilde{f}$ is $\mathscr{S}$-measurable. The map $u : F \rightarrow h(\tilde{f}^{-1}(F))$ is then a σ-homomorphism of $\mathscr{B}(R^n)$ into $\mathscr{L}$ such that $u_i(E) = u(p_i^{-1}(E))$ for all $E \in \mathscr{B}(R^1)$. Since $\mathscr{B}(R^n)$ is the smallest σ-algebra of subsets of R^n containing all the sets $p_i^{-1}(E)$, it is clear that the range of u is $\mathscr{L}_0$. The uniqueness of u is obvious.

For any real Borel function φ on R^n, $E \rightarrow u(\varphi^{-1}(E))$ is an observable associated with $\mathscr{L}$ whose range is obviously contained in $\mathscr{L}_0$. Suppose now that v is an observable associated with $\mathscr{L}$ whose range $\mathscr{L}_v \subseteq \mathscr{L}_0$. If we use the notations of the previous paragraph, and define $\mathscr{S}^\sim$ by

$$\mathscr{S}^\sim = \{\tilde{f}^{-1}(F) : F \in \mathscr{B}(R^n)\},$$

then h maps $\mathscr{S}^\sim$ onto $\mathscr{L}_0$. Applying theorem 1.4 to $\mathscr{S}^\sim$ and v, we infer the existence of a real valued $\mathscr{S}^\sim$-measurable Borel function c on X such that $h(c^{-1}(E)) = v(E)$ for all Borel sets E of R^1. By lemma 1.5, since c is $\mathscr{S}^\sim$-measurable, there exists a real valued Borel function φ on R^n such that $c(x) = \varphi(\tilde{f}(x))$ for all $x \in X$. If now E is any Borel set on the line, we have:

$$\begin{aligned} u(\varphi^{-1}(E)) &= h(\tilde{f}^{-1}(\varphi^{-1}(E))) \\ &= h(c^{-1}(E)) \\ &= v(E). \end{aligned}$$

This proves (ii) and completes the proof of the theorem.