SEMI-MARKOV RISK MODELS FOR FINANCE, INSURANCE AND RELIABILITY

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PREFACE

This book aims to give a complete and self-contained presentation of semi-Markov models with finitely many states, in view of solving real life problems of risk management in three main fields: Finance, Insurance and Reliability providing a useful complement to our first book (Janssen and Manca (2006)) which gives a theoretical presentation of semi-Markov theory. However, to help assure the book is self-contained, the first three chapters provide a summary of the basic tools on semi-Markov theory that the reader will need to understand our presentation. For more details, we refer the reader to our first book (Janssen and Manca (2006)) whose notations, definitions and results have been used in these four first chapters.

Nowadays, the potential for theoretical models to be used on real-life problems is severely limited if there are no good computer programs to process the relevant data. We therefore systematically propose the basic algorithms so that effective numerical results can be obtained. Another important feature of this book is its presentation of both homogeneous and non-homogeneous models. It is well known that the fundamental structure of many real-life problems is non-homogeneous in time, and the application of homogeneous models to such problems gives, in the best case, only approximated results or, in the worst case, nonsense results.

This book addresses a very large public as it includes undergraduate and graduate students in mathematics and applied mathematics, in economics and business studies, actuaries, financial intermediaries, engineers and operation researchers, but also researchers in universities and rd departments of banking, insurance and industry.

Readers who have mastered the material in this book will see how the classical models in our three fields of application can be extended in a semi-Markov environment to provide better new models, more general and able to solve problems in a more adapted way. They will indeed have a new approach giving a more competitive knowledge related to the complexity of real-life problems.

Let us now give some comments on the contents of the book.

As we start from the fact that the semi-Markov processes are the children of a successful marriage between renewal theory and Markov chains, these two topics are presented in Chapter 2.

The full presentation of Markov renewal theory, Markov random walks and semi-Markov processes, functionals of (J-X) processes and semi-Markov random walks is given in Chapter 3 along with a short presentation of non-homogeneous Markov and semi-Markov processes.

Chapter 4 is devoted to the presentation of discrete time semi-Markov processes, reward processes both in undiscounted and discounted cases, and to their numerical treatment.
Chapter 5 develops the Cox-Ross-Rubinstein or binomial model and semi-Markov extension of the Black and Scholes formula for the fundamental problem of option pricing in finance, including Greek parameters. In this chapter, we must also mention the presence of an option pricing model with arbitrage possibility, thus showing how to deal with a problem stock brokers are confronted with daily.
Chapter 6 presents other general finance and insurance semi-Markov models with the concepts of exchange and dated sums in stochastic homogeneous and non-homogeneous environments, applications in social security and multiple life insurance models.
Chapter 7 is entirely devoted to insurance risk models, one of the major fields of actuarial science; here, too, semi-Markov processes and diffusion processes lead to completely new risk models with great expectations for future applications, particularly in ruin theory.
Chapter 8 presents classical and semi-Markov models for reliability and credit risk, including the construction of rating, a fundamental tool for financial intermediaries.
Finally, Chapter 9 concerns the important present day problem of pension evolution, which is clearly a time non-homogeneous problem. As we need here more than one time variable, we introduce the concept of generalised non-homogeneous semi-Markov processes. A last section develops generalised non homogeneous semi-Markov models for salary line evolution.
Let us point out that whenever we present a semi-Markov model for solving an applied problem, we always summarise, before giving our approach, the classical existing models. Therefore the reader does not have to look elsewhere for supplementary information; furthermore, both approaches can be compared and conclusions reached as to the efficacy of the semi-Markov approach developed in this book.
It is clear that this book can be read by sections in a variety of sequences, depending on the main interest of the reader. For example, if the reader is interested in the new approaches for finance models, he can read the first four chapters and then immediately Chapters 5 and 6, and similarly for other topics in insurance or reliability.
The authors have presented many parts of this book in courses at several universities: Université Libre de Bruxelles, Vrije Universiteit Brussel, Université de Bretagne Occidentale (EURIA), Universités de Paris 1 (La Sorbonne) and Paris VI (ISUP), ENST-Bretagne, Université de Strasbourg, Universities of Roma (La Sapienza), Firenze and Pescara.
Our common experience in the field of solving some real problems in finance, insurance and reliability has joined to create this book, taking into account the remarks of colleagues and students in our various lectures. We hope to convince

potential readers to use some of the proposed models to improve the way of modelling real-life applications.

Jacques Janssen Raimondo Manca

Keywords :

Semi-Markov processes, Homogeneous, Non homogeneous, Risk Management, Finance, Insurance, Reliability, Models, Numerical results, Real Applications.

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Chapter 1

PROBABILITY TOOLS FOR STOCHASTIC MODELLING

In this chapter, the reader will find a short summary of the basic probability tools useful for understanding of the following chapters. A more detailed version including proofs can be found in Janssen and Manca (2006). We will focus our attention on stochastic processes in discrete time and continuous time defined by sequences of random variables.

1 THE SAMPLE SPACE

The basic concrete notion in probability theory is that of the *random experiment*, that is to say an experiment for which we cannot predict in advance the outcome. With each random experiment, we can associate the so-called *elementary events* ω, and the set of all these events Ω is called the *sample space*. Some other subsets of Ω will represent possible *events*. Let us consider the following examples.

Example 1.1 If the experiment consists in the measurement of the lifetime of an integrated circuit, then the sample space is the set of all non-negative real numbers $\mathbb{R}^+$. Possible events are $[a,b],(a,b),[a,b),(a,b]$ where for example the event $[a,b)$ means that the lifetime is at least *a* and strictly inferior to *b*.

Example 1.2 An insurance company is interested in the number of claims per year for its portfolio. In this case, the sample space is the set of natural numbers $\mathbb{N}$.

Example 1.3 A bank is to invest in some shares; so the bank looks to the history of the value of different shares. In this case, the sample space is the set of all non-negative real numbers $\mathbb{R}^+$.

To be useful, the set of all possible events must have some properties of stability so that we can generate new events such as:

(i) *the complement* A^c : $A^c = \{\omega \in \Omega : \omega \notin A\}$, (1.1)

(ii) *the union* $A \bigcup B$: $A \bigcup B = \{\omega : \omega \in A \text{ or} \omega \in B\}$, (1.2)

(iii) *the intersection* $A \bigcap B$: $A \bigcap B = \{\omega : \omega \in A, \omega \in B\}$. (1.3)

More generally, if $(A_n, n \geq 1)$ represents a sequence of events, we can also consider the following events:

$$\bigcup_{n\geq 1} A_n, \ \bigcap_{n\geq 1} A_n \tag{1.4}$$

representing respectively the *union* and the *intersection* of all the events of the given sequence. The first of these two events occurs iff at least one of these events occurs and the second iff all the events of the given sequence occur. The set Ω is called the *certain event* and the set $\varnothing$ the *empty event*. Two events A and B are said to be *disjoint* or *mutually exclusive* iff

$$A \cap B = \varnothing. \tag{1.5}$$

Event A *implies* event B iff

$$A \subset B. \tag{1.6}$$

In **Example 1.3**, the event "the value of the share is between "50$ and 75$" is given by the set $\left[50, 75\right]$.

2 PROBABILITY SPACE

Given a sample space Ω, the set of all possible events will be noted by $\Im$, supposed to have the structure of a σ*-field* or a σ*-algebra.*

Definition 2.1 *The family* $\Im$ *of subsets of* Ω *is called a* σ*-field or a* σ*-algebra iff the following conditions are satisfied*:

(i) $\Omega, \varnothing$ *belong* to $\Im$,

(ii) Ω *is stable under denumerable intersection*:

$$A_n \in \Im, \forall n \geq 1 \Rightarrow \bigcap_{n\geq 1} A_n \in \Im, \tag{2.1}$$

(iii) $\Im$ *is stable for the complement set operation*

$$A \in \Im \Rightarrow A^c \in \Im, \ A^c = \Omega - A. \tag{2.2}$$

Then, using the well-known de Morgan's laws saying that

$$\left(\bigcup_{n\geq 1} A_n\right)^c = \bigcap_{n\geq 1} A_n^c, \ \left(\bigcap_{n\geq 1} A_n\right)^c = \bigcup_{n\geq 1} A_n^c, \tag{2.3}$$

it is easy to prove that a σ*-algebra* $\Im$ is also stable under denumerable union:

$$A_n \in \Im, \forall n \geq 1 \Rightarrow \bigcup_{n\geq 1} A_n \in \Im. \tag{2.4}$$

Any couple $(\Omega, \Im)$ where $\Im$ is a σ-algebra is called a *measurable space*.

The next definition concerning the concept of *probability measure* or simply *probability* is an idealization of the concept of the *frequency* of an event. Let us consider a random experiment called E with which is associated the couple

$(\Omega,\Im)$; if the set A belongs to $\Im$ and if we can repeat the experiment E n times, under the same conditions of environment, we can count how many times A occurs. If $n(A)$ represents this number of occurrences, the *frequency* of the event A is defined as

$$f(A)=\frac{n(A)}{n}. \tag{2.5}$$

In general, this number tends to become stable for large values of n.
The notion of frequency satisfies the following elementary properties:

(i) $(A,B\in\Im, A\cap B=\varnothing \Rightarrow f(A\cup B)=f(A)+f(B),$ (2.6)

(ii) $f(\Omega)=1$, (2.7)

(iii) $A,B\in\Im,\Rightarrow f(A\cup B)=f(A)+f(B)-f(A\cap B),$ (2.8)

(iv) $A\in\Im\Rightarrow f(A^c)=1-f(A).$ (2.9)

To have a useful mathematical model for the theoretical idealization of the notion of frequency, we now introduce the following definition.

Definition 2.2 a) *The triplet* $(\Omega,\Im,P)$ *is called a probability space if* Ω *is a non-void set of elements,* $\Im$ *a* σ*-algebra of subsets of* Ω *and P an application from* $\Im$ *to* $[0,1]$ *such that:*

(i)
$$\begin{aligned}&(A_n,n\ge1),A_n\in\Im,n\ge1:\left(i\ne j\Rightarrow A_i\cap A_j=\phi\right)\\&\Rightarrow P\left(\bigcup_{n\ge1}A_n\right)=\sum_{n=1}^{\infty}P(A_n)\ (\sigma-\text{additivity of } P),\end{aligned} \tag{2.10}$$

(ii) $P(\Omega)=1.$ (2.11)

b) *The application P satisfying conditions* (2.10) *and* (2.11) *is called a probability measure or simply probability.*

Remark 2.1 1) The sequence of events $(A_n,n\ge1)$ satisfying the condition that

$$(A_n,n\ge1),A_n\in\Im,n\ge1:i\ne j\Rightarrow A_i\cap A_j=\varnothing \tag{2.12}$$

is called *mutually exclusive*.
2) The relation (2.11) assigns the value 1 for the probability of the entire sample space Ω. There may exist events A' strictly subsets of Ω such that

$$P\left(A'\right)=1. \tag{2.13}$$

In this case, we say that A is *almost sure* or that the statement defining A is true *almost surely* (in short a.s.) or holds for almost all ω.
From axioms (2.10) and (2.11), we can deduce the following properties:

Property 2.1 (i) If $A,B\in\Im$, then

$$P(A\cup B)=P(A)+P(B)-P(A\cap B). \tag{2.14}$$

(ii) If $A\in\Im$, then

$$P(A^c) = 1 - P(A). \tag{2.15}$$

(iii) $P(\emptyset) = 0.$ (2.16)

(iv) If $(B_n, n \geq 1)$ is a sequence of disjoint elements of $\Im$ forming a partition of Ω, then for all A belonging to $\Im$,

$$P(A) = \sum_{n=1}^{\infty} P(A \cap B_n). \tag{2.17}$$

(v) *Continuity property of P*: if $(A_n, n \geq 1)$ is an increasing (decreasing) sequence of elements of $\Im$, then

$$P\left(\bigcup_{n\geq 1} A_n\right) = \lim_n P(A_n);\ \left(P\left(\bigcap_{n\geq 1} A_n\right) = \lim_n P(A_n)\right). \tag{2.18}$$

Remark 2.2 a) Boole's inequality asserts that if $(A_n, n \geq 1)$ is a sequence of events, then

$$P\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n\geq 1} P(A_n). \tag{2.19}$$

b) From (2.14), it is clear that we also have

$$A \subset B \Rightarrow P(A) \leq P(B). \tag{2.20}$$

Example 2.1 a) *The discrete case*

When the sample space Ω is *finite* or *denumerable*, we can set

$$\Omega = \{\omega_1, ..., \omega_j, ...\} \tag{2.21}$$

and select for $\Im$ the set of all the subsets of Ω, represented by 2^{Ω}.

Any probability measure P can be defined with the following sequence:

$$(p_j, j \geq 1),\ p_j \geq 0, j \geq 1,\ \sum_{j\geq 1} p_j = 1 \tag{2.22}$$

so that

$$P(\{w_j\}) = p_j, j \geq 1. \tag{2.23}$$

On the probability space $(\Omega, 2^{\Omega}, P)$, the probability assigned for an arbitrary event A = $\{\omega_{k_1}, ..., \omega_{k_l}\}, k_j \geq 1, j = 1, ..., l, k_i \neq k_j$ if $i \neq j$ is given by

$$P(A) = \sum_{j=1}^{l} p_{k_j}. \tag{2.24}$$

b) *The continuous case*

Let Ω be the real set $\mathbb{R}$; It can be proven (Halmos (1974)) that there exists a minimal σ-algebra generated by the set of intervals:

$$\beta = \{(a,b),[a,b],[a,b),(a,b],a,b \in \mathbb{R}, a \le b\}. \tag{2.25}$$

It is called the *Borel* σ*-algebra* represented by β and the elements of β are called *Borel sets*.

Given a probability measure *P* on (Ω, β), we can define the real function *F*, called the distribution function related to *P*, as follows.

Definition 2.3 *The function F from* $\mathbb{R}$ *to* $[0,1]$ *defined by*

$$P((-\infty, x]) = F(x), x \in \mathbb{R} \tag{2.26}$$

is called the distribution function related to the probability measure P.

From this definition and the basic properties of *P*, we easily deduce that:

$$\begin{aligned} &P((a,b]) = F(b) - F(a),\ P((a,b)) = F(b-) - F(a), \\ &P([a,b)) = F(b-) - F(a-),\ P([a,b]) = F(b) - F(a-). \end{aligned} \tag{2.27}$$

Moreover, from (2.26), any function F from $\mathbb{R}$ to $[0,1]$ is a distribution function (in short d.f.) iff it is a non-decreasing function satisfying the following conditions:

F is right continuous at every point x_0,

$$\lim_{x \uparrow x_0} F(x) = F(x_0), \tag{2.28}$$

and moreover

$$\lim_{x \to +\infty} F(x) = 1, \lim_{x \to -\infty} F(x) = 0. \tag{2.29}$$

If the function *F* is derivable on $\mathbb{R}$ with *f* as derivative, we have

$$F(x) = \int_{-\infty}^{x} f(y)dy, x \in \mathbb{R}. \tag{2.30}$$

The function *f* is called the density function associated with the d.f. *F* and in the case of the existence of such a Lebesgue integrable function on $\mathbb{R}$, *F* is called *absolutely continuous*.

From the definition of the concept of integral, we can give the intuitive interpretation of *f* as follows; given the small positive real number Δx, we have

$$P(\{x, x + \Delta x\}) \approx f(x)\Delta x. \tag{2.31}$$

Using the Lebesgue-Stieltjes integral, it can be seen that it is possible to define a probability measure *P* on $(\mathbb{R}, \beta)$ starting from a d.f. *F* on $\mathbb{R}$ by the following definition of *P*:

$$P(A) = \int_A dF(x), \forall A \in \Im. \tag{2.32}$$

In the absolutely continuous case, we get

$$P(A) = \int_A f(y)dy. \tag{2.33}$$

Remark 2.3 In fact, it is also possible to define the concept of d.f. in the discrete case if we set, without loss of generality, on $(N_0, 2^{N_0})$, the measure P defined from the sequence (2.22). Indeed, if for every positive integer k, we set

$$F(k) = \sum_{j=1}^{k} p_j \tag{2.34}$$

and generally, for any real x,

$$F(x) = \begin{cases} 0, x \leq 0, \\ F(k), x \in [k, k+1), \end{cases} \tag{2.35}$$

then, for any positive integer k, we can write

$$P(\{1, ..., k\}) = F(k) \tag{2.36}$$

and so calculate the probability of any event.

3 RANDOM VARIABLES

Let us suppose the probability space $(\Omega, \Im, P)$ and the measurable space (E, ψ) are given.

Definition 3.1 *A random variable* (*in short r.v.*) *with values in E is an application X from* Ω *to E such that*

$$\forall B \in \psi : X^{-1}(B) \in \Im, \tag{3.1}$$

where $X^{-1}(B)$ *is called the inverse image of the set B defined by*

$$X^{-1}(B) = \{\omega : X(\omega) \in B\}, X^{-1}(B) \in \Im. \tag{3.2}$$

Particular cases

a) If $(E, \psi) = (\mathbb{R}, \beta)$, X is called a *real random variable*.

b) If $(E, \psi) = (\overline{\mathbb{R}}, \overline{\beta})$, where $\overline{\mathbb{R}}$ is the *extended real line* defined by $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ and $\overline{\beta}$ the *extended Borel* σ*-field* of $\overline{\mathbb{R}}$, that is the minimal σ-field containing all the elements of β and the extended intervals

$$\begin{gathered} [-\infty, a), (-\infty, a], [-\infty, a], (-\infty, a), \\ [a, +\infty), (a, +\infty], [a, +\infty], (a, +\infty), \ a \in \mathbb{R}, \end{gathered} \tag{3.3}$$

then X is called a *real extended value random variable*.

c) If $E = \mathbb{R}^n (n > 1)$ with the product σ-field $\beta^{(n)}$ of β, X is called an *n-dimensional real random variable*.

d) If $E = \overline{\mathbb{R}}^{(n)}$ (n>1) with the product σ-field $\overline{\beta}^{(n)}$ of $\overline{\beta}$, X is called a real *extended n-dimensional real random variable*.

A random variable *X* is called *discrete* or *continuous* according as *X* takes at most a denumerable or a non-denumerable infinite set of values.

Remark 3.1 In *measure theory*, the only difference is that condition (2.11) is no longer required and in this case the definition of a r.v. given above gives the notion of *measurable function*. In particular a measurable function from $(\mathbb{R},\beta)$ to $(\mathbb{R},\beta)$ is called a *Borel function*.

Let *X* be a real r.v. and let us consider, for any real *x*, the following subset of Ω: $\{\omega: X(\omega)\le x\}$.

As, from relation (3.2),

$$\{\omega: X(\omega)\le x\} = X^{-1}((-\infty,x]), \tag{3.4}$$

it is clear from relation (3.1) that this set belongs to the σ-algebra $\Im$. Conversely, it can be proved that the condition

$$\{\omega: X(\omega)\le x\}\in\Im, \tag{3.5}$$

valid for every *x* belonging to a dense subset of $\mathbb{R}$, is sufficient for *X* being a real random variable defined on Ω. The probability measure *P* on $(\Omega,\Im)$ induces a probability measure μ on $(\mathbb{R},\beta)$ defined as

$$\forall B\in\beta: \mu(B) = P(\{\omega: X(\omega)\in B\}). \tag{3.6}$$

We say that μ is the induced probability measure on $(\mathbb{R},\beta)$, called the *probability distribution* of the r.v. *X*. Introducing the distribution function related to μ, we get the next definition.

Definition 3.2 *The distribution function of the r.v. X, represented by* F_X, *is the function from* $\mathbb{R}\to[0,1]$ *defined by*

$$F_X(x) = \mu((-\infty,x]) = P(\{\omega: X(\omega)\le x\}). \tag{3.7}$$

In short, we write

$$F_X(x) = P(X\le x). \tag{3.8}$$

This last definition can be extended to the multi-dimensional case with a r.v. *X* being an n-dimensional real vector: $X=(X_1,...,X_n)$, a measurable application from $(\Omega,\Im,P)$ to $(\mathbb{R}^n,\beta^n)$.

Definition 3.3 *The distribution function of the r.v.* $X=(X_1,...,X_n)$, *represented by* F_X, *is the function from* $\mathbb{R}^n$ *to* $[0,1]$ *defined by*

$$F_X(x_1,...,x_n) = P\left(\{\omega : X_1(\omega) \le x_1,...,X_n(\omega) \le x_n\}\right). \tag{3.9}$$

In short, we write

$$F_X(x_1,...,x_n) = P(X_1 \le x_1,...,X_n \le x_n). \tag{3.10}$$

Each component X_i (i=1,…,n) is itself a one-dimensional real r.v. whose d.f., called the *marginal d.f.*, is given by

$$F_{X_i}(x_i) = F_X(+\infty,...,+\infty,x_i,+\infty,...,+\infty). \tag{3.11}$$

The concept of random variable is *stable* under a lot of mathematical operations; so any Borel function of a r.v. X is also a r.v.
Moreover, if X and Y are two r.v., so are

$$\inf\{X,Y\},\sup\{X,Y\},X+Y,X-Y,X\cdot Y,\frac{X}{Y}, \tag{3.12}$$

provided, in the last case, that Y does not vanish.
Concerning the convergence properties, we must mention the property that, if $(X_n, n \ge 1)$ is a *convergent* sequence of r.v. − that is, for all $\omega \in \Omega$, the sequence $(X_n(\omega))$ converges to $X(\omega)$ −, then the limit X is also a r.v. on Ω. This convergence, which may be called the *sure convergence*, can be weakened to give the concept of *almost sure* (in short a.s.) *convergence* of the given sequence.

Definition 3.4 *The sequence* $(X_n(\omega))$ *converges a.s. to* $X(\omega)$ *if*

$$P\left(\{\omega : \lim X_n(\omega) = X(\omega)\}\right) = 1. \tag{3.13}$$

This last notion means that the possible set where the given sequence does not converge is a *null set*, that is a set N belonging to $\Im$ such that

$$P(N) = 0. \tag{3.14}$$

In general, let us remark that, given a null set, it is not true that every subset of it belongs to $\Im$ but of course if it belongs to $\Im$, it is clearly a null set (see relation (2.20)).
To avoid unnecessary complications, we will suppose from now on that any considered probability space is *complete*, This means that all the subsets of a null set also belong to $\Im$ and thus that their probability is zero.

4 INTEGRABILITY, EXPECTATION AND INDEPENDENCE

Let us consider a complete measurable space $(\Omega, \Im, \mu)$ and a real measurable variable X defined on Ω. To any set A belonging to $\Im$, we associate the r.v. I_A, called the indicator of A, defined as

$$I_A(\omega) = \begin{cases} 1, \omega \in A, \\ 0, \omega \notin A. \end{cases} \quad (4.1)$$

If there exists partition $(A_n, n \geq 1)$ with all its sets measurable such that

$$\omega \in A_n \Rightarrow X(\omega) = a_n (a_n \in \mathbb{R}), n \geq 1, \quad (4.2)$$

then X is called a *discrete* variable. If moreover, the partition is finite, it is said to be *finite*. It follows that we can write X in the following form:

$$X(\omega) = \sum_{n=1}^{\infty} a_n I_{A_n}(\omega). \quad (4.3)$$

Definition 4.1 *The integral of the discrete variable X is defined by*

$$\int_{\Omega} X d\mu = \sum_{n=1}^{\infty} a_n \mu(A_n), \quad (4.4)$$

provided that this series is absolutely convergent.

Of course, if X is integrable, we have the integrability of $|X|$ too and

$$\int_{\Omega} |X| d\mu = \sum_{n=1}^{\infty} |a_n| \mu(A_n). \quad (4.5)$$

To define in general the integral of a measurable function X, we first restrict ourselves to the case of a non-negative measurable variable X for which we can construct a monotone sequence $(X_n, n \geq 1)$ of discrete variables converging to X as follows:

$$X_n(\omega) = \sum_{k=1}^{\infty} \frac{k}{2^n} I_{\left\{\omega: \frac{k}{2^n} \leq X < \frac{k+1}{2^n}\right\}}. \quad (4.6)$$

Since for each n,

$$\begin{aligned} & X_n(\omega) \leq X_{n+1}(\omega), \\ & 0 \leq X(\omega) - X_n(\omega) \leq \frac{1}{2^n}, \end{aligned} \quad (4.7)$$

the sequence $(X_n, n \geq 1)$ of discrete variables converges monotonically to X on Ω.

Definition 4.2 *The non-negative measurable variable X is integrable on Ω iff the elements of the sequence $(X_n, n \geq 1)$ of discrete variables defined by relation* (4.6) *are integrable and if the sequence $\left(\int_{\Omega} X_n dP\right)$ converges.*

From this last definition, it follows that

$$E(X) = \lim E(X_n), \quad (4.8)$$

where

$$\int_{\Omega} X_n(\omega)d\mu = \sum_{k=1}^{\infty} \frac{k}{2^n}\mu\left(I_{\left\{\omega:\frac{k}{2^n}\le X<\frac{k+1}{2^n}\right\}}\right). \tag{4.9}$$

To extend the last definition without the non-negativity condition on *X*, let us introduce for an arbitrary variable *X*, the variables X^+ and X^- defined by

$$X^+(\omega) = \sup\{X(\omega),0\},\ X^-(\omega) = -\inf\{X(\omega),0\}, \tag{4.10}$$

so that

$$X = X^+ - X^-. \tag{4.11}$$

Definition 4.3 *The measurable variable X is integrable on* Ω *iff the non-negative variables* X^+ *and* X^- *defined by relation* (4.10) *are integrable and in this case*

$$\int_{\Omega} Xd\mu = \int_{\Omega} X^+d\mu - \int_{\Omega} X^-d\mu. \tag{4.12}$$

Remark 4.1 a) If the integral of *X* does not exist, it may however happen that

$$\int_{\Omega} X^+d\mu < \infty \left(\int_{\Omega} X^-d\mu < \infty\right), \int_{\Omega} X^-d\mu = \infty\left(\int_{\Omega} X^+d\mu \ = \infty\right). \tag{4.13}$$

In these two cases, we say that the integral of *X* is infinite; more precisely, we have

$$\int_{\Omega} Xd\mu = -\infty\left(\int_{\Omega} Xd\mu = +\infty\right). \tag{4.14}$$

If *A* is an element of the σ-algebra $\Im$, the integral on *A* is simply defined by

$$\int_{A} Xd\mu = \int_{A} X I_A d\mu. \tag{4.15}$$

Of course, *X* being a non-negative measurable variable with an infinite integral, it means that the approximation sequence (4.6) diverges to $+\infty$ for almost all ω.

Now let us consider a probability space $(\Omega,\Im,P)$ and a real random variable *X* defined on Ω. In this case, the concept of integrability is designed by *expectation* represented by

$$E(X) = \int_{\Omega} XdP\left(=\int XdP\right), \tag{4.16}$$

provided that this integral exists. The computation of the integral

$$\int_{\Omega} XdP\left(=\int XdP\right) \tag{4.17}$$

can be done using the induced measure μ on $(\mathbb{R},\beta)$,defined by relation (3.6) and then using the distribution function *F* of *X*. Indeed, we can write

$$E(X)\left(=\int_{\Omega} XdP\right)=\int_{R} Xd\mu, \tag{4.18}$$

and if F_X is the d.f. of X, it can be shown that

$$E(X)=\int_{R} xdF_X(x), \tag{4.19}$$

this last integral being a Lebesgue-Stieltjes integral. Moreover, if F_X is absolutely continuous with f_X as density, we get

$$E(X)=\int_{-\infty}^{+\infty} xf_x(x)dx. \tag{4.20}$$

If g is a Borel function, we also have (see for example Chung (2000), Royden (1963), Loeve (1963))

$$E(g(X))=\int_{-\infty}^{+\infty} g(x)dF_X \tag{4.21}$$

and with a density for X,

$$E(g(X))=\int_{-\infty}^{+\infty} g(x)f_X(x)dx. \tag{4.22}$$

The most important properties of the expectation are given in the next proposition.

Proposition 4.1 (i) *Linearity property of the expectation: If X and Y are two integrable r.v. and a,b two real numbers, then the r.v. aX+bY is also integrable and*

$$E(aX+bY)=aE(X)+bE(Y). \tag{4.23}$$

(ii) *If* $(A_n, n\geq 1)$ *is a partition of* Ω*, then*

$$E(X)=\sum_{n=1}^{\infty}\int_{A_n} XdP. \tag{4.24}$$

(iii) *The expectation of a non-negative r.v. is non-negative.*
(iv) *If X and Y are integrable r.v., then*

$$X\leq Y \Rightarrow E(X)\leq E(Y). \tag{4.25}$$

(v) *If X is integrable, so is* $|X|$ *and*

$$|E(X)|\leq E|X|. \tag{4.26}$$

(vi) *Dominated convergence theorem (Lebesgue: Let* $(X_n, n\geq 1)$ *be a sequence of r.v. converging a.s. to the r.v. X integrable, then all the r.v.* X_n *are integrable and moreover*

$$\lim E(X_n)=E(\lim X_n)\ (=E(X)). \tag{4.27}$$

(vii) *Monotone convergence theorem (Lebesgue): Let* $(X_n, n \geq 1)$ *be a non-decreasing sequence of non-negative r.v; then relation* (4.27) *is still true provided that* $+\infty$ *is a possible value for each member.*
(viii) *If the sequence of integrable r.v.* $(X_n, n \geq 1)$ *is such that*

$$\sum_{n=1}^{\infty} E\left(\left|X_n\right|\right) < \infty, \tag{4.28}$$

then the random series $\sum_{n=1}^{\infty} X_n$ *converges absolutely a.s. and moreover*

$$\sum_{n=1}^{\infty} E(X_n) = E\left(\sum_{n=1}^{\infty} X_n\right) \; (= E(X)), \tag{4.29}$$

where the r.v. is defined as the sum of the convergent series.

Given a r.v. *X*, *moments* are special cases of expectation.

Definition 4.4 *Let a be a real number and r a positive real number, then the expectation*

$$E\left(\left|X - a\right|^r\right) \tag{4.30}$$

is called the absolute moment of X, of order r, centred on a.

The moments are said to be centred moments of order *r* if *a*=*E*(*X*). In particular, for *r*=2, we get the *variance* of *X* represented by σ^2 (var(X)),

$$\sigma^2 = E\left(\left|X - m\right|^2\right). \tag{4.31}$$

Remark 4.2 From the linearity of the expectation (see relation (4.23)), it is easy to prove that

$$\sigma^2 = E(X^2) - (E(X))^2, \tag{4.32}$$

and so

$$\sigma^2 \leq E(X^2), \tag{4.33}$$

and more generally, it can be proven that the variance is the smallest moment of order 2 whatever the number *a* is.

The next property recalls inequalities for moments.

Proposition 4.2 (*Inequalities of Hölder and Minkowski*) (i) *Let X and Y be two r.v. such that* $\left|X\right|^p, \left|Y\right|^q$ *are integrable with*

$$1 < p < \infty, \; \frac{1}{p} + \frac{1}{q} = 1, \tag{4.34}$$

then:

$$|E(XY)| \le \left(E\left(|X^p|\right)\right)^{1/p} \cdot \left(E\left(|Y^q|\right)\right)^{1/q}. \tag{4.35}$$

(ii) *Let X and Y be two r.v. such that* $|X|^p, |Y|^p, 1 \le p < \infty$, *are integrable, then*

$$E\left(|X+Y|^p\right)^{1/p} \le \left(E\left(|X|^p\right)\right)^{1/p} + \left(E\left(|Y|^p\right)\right)^{1/p}. \tag{4.36}$$

If *p=2* in the first part of this last proposition, then relation (4.36) gives the *Cauchy-Schwarz inequality*

$$|E(XY)| \le \left(E\left(|X|^2\right)\right)^{1/2} \cdot \left(E\left(|Y|^2\right)\right)^{1/2}. \tag{4.37}$$

The last fundamental concept we will now introduce in this section is that of *stochastic independence*, or more simply *independence*.

Definition 4.5 *The events* $A_1,...,A_n,(n>1)$ *are stochastically independent or independent iff*

$$\forall m = 2,...,n, \forall n_k = 1,...,n : n_1 \ne n_2 \ne \cdots \ne n_k : P\left(\bigcap_{k=1}^{m} A_{n_k}\right) = \prod_{k=1}^{m} P(A_{n_k}). \tag{4.38}$$

For *n=2*, relation (4.38) reduces to

$$P(A_1 \cap A_2) = P(A_1)P(A_2). \tag{4.39}$$

Let us remark that piecewise independence of the events $A_1,...,A_n,(n>1)$ does not necessarily imply the independence of these sets and thus not the stochastic independence of these *n* events. As a counter example, let us suppose we drew a ball from an urn containing four balls called b_1, b_2, b_3, b_4 and let us consider the three following events:

$$A_1 = \{b_1, b_2\}, A_2 = \{b_1, b_3\}, A_3 = \{b_1, b_4\}. \tag{4.40}$$

Then assuming that the probability of having one ball is $1/4$, we get

$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = \frac{1}{4}, \tag{4.41}$$

but as

$$P(A_1 \cap A_2 \cap A_3) = \frac{1}{4} \tag{4.42}$$

too, we do not have the relation

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3), \tag{4.43}$$

and so we have proved that independence in pairs does not imply the independence of these three events.

We will now extend the concept of independence to random variables.

Definition 4.6 (i) *The n real r.v. $X_1, X_2, \ldots, X_n$ defined on the probability space $(\Omega, \Im, P)$ are said to be stochastically independent, or simply independent, iff for any Borel sets $B_1, B_2, \ldots, B_n$, we have*

$$P\left(\bigcap_{k=1}^{n}\{\omega : X_k(\omega) \in B_k\}\right) = \prod_{k=1}^{n} P\left(\{\omega : X_k(\omega) \in B_k\}\right). \tag{4.44}$$

(ii) *For an infinite family of r.v., independence means that the members of every finite subfamily are independent. It is clear that if $X_1, X_2, \ldots, X_n$ are independent, so are the r.v. $X_{i_1}, \ldots, X_{i_k}$ with $i_1 \neq \cdots \neq i_k$, $i_k = 1, \ldots, n, k = 2, \ldots, n$.*

From relation (4.44), we find that

$$P(X_1 \le x_1, \ldots, X_n \le x_n) = P(X_1 \le x_1) \cdots P(X_n \le x_n), \forall (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{4.45}$$

If the functions $F_X, F_{X_1}, \ldots, F_{X_n}$ are the distribution functions of the r.v. $X = (X_1, \ldots, X_n), X_1, \ldots, X_n$, we can write the preceding relation under the form

$$F_X(x_1, \ldots., x_n) = F_{X_1}(x_1) \cdot \cdots \cdot F_{X_n}(x_n), \forall (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{4.46}$$

It can be shown that this last condition is also sufficient for the independence of $X = (X_1, \ldots, X_n), X_1, \ldots, X_n$. If these d.f. have densities $f_X, f_{X_1}, \ldots, f_{X_n}$, relation (4.46) is equivalent to

$$f_X(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n), \forall (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{4.47}$$

In case of the integrability of the *n* real r.v $X_1, X_2, \ldots, X_n$, a direct consequence of relation (4.46) is that we have a very important property for the expectation of the product of *n* independent r.v.:

$$E\left(\prod_{k=1}^{n} X_k\right) = \prod_{k=1}^{n} E(X_k). \tag{4.48}$$

The notion of independence gives the possibility to prove the result called the *strong law of large numbers* which says that if $(X_n, n \ge 1)$ is a sequence of integrable independent and identically distributed r.v., then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} E(X). \tag{4.49}$$

The next section will present the most useful distribution functions for stochastic modelling.

5 MAIN DISTRIBUTION PROBABILITIES

Here we shall restrict ourselves to presenting the principal distribution probabilities related to real random variables.

5.1 The Binomial Distribution

Let us consider a random experiment E such that only two results are possible: a "success"(S) with probability p and a "failure (F) with *probability* $q = 1 - p$. If n independent trials are made in exactly the same experimental environment, the total number of trials in which the event S occurs may be represented by a random variable X whose distribution $(p_i, i = 0,...,n)$ with

$$p_i = P(X = i),\ i = 1,...,n \tag{5.1}$$

is called a *binomial distribution* with parameters (n,p). From basic axioms of probability theory seen before, it is easy to prove that

$$p_i = \binom{n}{i} p^i q^{n-i},\ i = 0,...,n, \tag{5.2}$$

a result from which we get

$$E(X) = np, \mathrm{var}(X) = npq. \tag{5.3}$$

The *characteristic function* and the *generating function, when it exists,* of X respectively defined by

$$\begin{aligned} \varphi_X(t) &= E(e^{itX}), \\ g_X(t) &= E(e^{tX}) \end{aligned} \tag{5.4}$$

are given by

$$\begin{aligned} \varphi_X(t) &= (pe^{it} + q)^n, \\ g_X(t) &= (pe^t + q)^n. \end{aligned} \tag{5.5}$$

Example 5.1 (*The Cox and Rubinstein financial model*) Let us consider a financial asset observed on n successive discrete time periods so that at the beginning of the first period, from time 0 to time 1, the asset starts from value S_0 and has at the end of this period only two possible values, uS_0 and dS_0 ($0<d<1, u>1$) respectively with probabilities p and q=1-p. The asset has the same type of evolution on each period and independently of the past. In period i, from time i–1 to time i, let us associate the r.v. $\xi_i, i = 1,...,n$ defined as follows:

$$\xi_i = \begin{cases} 1, \text{ with probability } p, \\ 0, \text{ with probability } q. \end{cases} \tag{5.6}$$

The value of the asset at the end of period n is given by the r.v. Y_n defined as

$$Y_n = u^{X_n} d^{n-X_n} \tag{5.7}$$

with

$$X_n = \xi_1 + \cdots + \xi_n. \tag{5.8}$$

It is clear that the r.v. X_n has a binomial distribution of parameters (n,p) and consequently, we get the distribution probability of Y_n: