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27 Borwein/Choi/Rooney/Weirathmueller The Riemann Hypothesis

Peter Borwein, Stephen Choi, Brendan Rooney
and Andrea Weirathmueller (Eds.)

The Riemann Hypothesis

A Resource for the Afficionado and Virtuoso Alike

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For Pinot

- P. B.

For my parents, my lovely wife, Shirley, my daughter, Priscilla, and son, Matthew

- S. C.

For my parents Tom and Katrea

- B. R.

For my family

- A. W.

Preface

Since its inclusion as a Millennium Problem, numerous books have been written to introduce the Riemann hypothesis to the general public. In an average local bookstore, it is possible to see titles such as John Derbyshire's *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, Dan Rockmore's *Stalking the Riemann Hypothesis: The Quest to Find the Hidden Law of Prime Numbers*, and Karl Sabbagh's *The Riemann Hypothesis: The Greatest Unsolved Problem in Mathematics*.

This book is a more advanced introduction to the theory surrounding the Riemann hypothesis. It is a source book, primarily composed of relevant original papers, but also contains a collection of significant results. The text is suitable for a graduate course or seminar, or simply as a reference for anyone interested in this extraordinary conjecture.

The material in Part I (Chapters 1-10) is mostly organized into independent chapters and one can cover the material in many ways. One possibility is to jump to Part II and start with the four expository papers in Chapter 11. The reader who is unfamiliar with the basic theory and algorithms used in the study of the Riemann zeta function may wish to begin with Chapters 2 and 3. The remaining chapters stand on their own and can be covered in any order the reader fancies (obviously with our preference being first to last). We have tried to link the material to the original papers in order to facilitate more in-depth study of the topics presented.

We have divided Part II into two chapters. Chapter 11 consists of four expository papers on the Riemann hypothesis, while Chapter 12 contains the original papers that developed the theory surrounding the Riemann hypothesis.

Presumably the Riemann hypothesis is very difficult, and perhaps none of the current approaches will bear fruit. This makes selecting appropriate papers problematical. There is simply a lack of profound developments and attacks on the full problem. However, there is an intimate connection between the prime number theorem and the Riemann hypothesis. They are connected theoretically and historically, and the Riemann hypothesis may be thought of as a grand generalization of the prime number theorem. There is a large body of theory on the prime number theorem and a progression of solutions. Thus we have chosen various papers that give proofs of the prime number theorem.

While there have been no successful attacks on the Riemann hypothesis, a significant body of evidence has been generated in its support. This evidence is often computational; hence we have included several papers that focus on, or use computation of, the Riemann zeta function. We have also included Weil's proof of the Riemann hypothesis for function fields (Section 12.8) and the deterministic polynomial primality test of Argawal at al. (Section 12.20).

Acknowledgments. We would like to thank the community of authors, publishers, and libraries for their kind permission and assistance in republishing the papers included in Part II. In particular, "On Newman's Quick Way to the Prime Number Theorem" and "Pair Correlation of Zeros and Primes in Short Intervals" are reprinted with kind permission of Springer Science and Business Media, "The Pair Correlation of Zeros of the Zeta Function" is reprinted with kind permission of the American Mathematical Society, and "On the Difference $\pi(x) - \mathrm{Li}(x)$" is reprinted with kind permission of the London Mathematical Society.

We gratefully acknowledge Michael Coons, Karl Dilcher, Ron Ferguson and Alexa van der Waall for many useful comments and corrections that help make this a better book.

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Notation

The notation in this book is standard. Specific symbols and functions are defined as needed throughout, and the standard meaning of basic symbols and functions is assumed. The following is a list of symbols that appear frequently in the text, and the meanings we intend them to convey.

$\Rightarrow$	"If ..., then ..." in natural language
$\in$	membership in a set
$\#\{A\}$	the cardinality of the set A
$:=$	defined to be
$x \equiv y \pmod p$	x is congruent to y modulo p
$[x]$	the integral part of x
$\{x\}$	the fractional part of x
$\|x\|$	the absolute value of x
$x!$	for $x \in \mathbb{N}$, $x! = x \cdot (x-1) \cdots 2 \cdot 1$
(n, m)	the greatest common divisor of n and m
$\phi(x)$	Euler's totient function evaluated at x
$\log(x)$	the natural logarithm, $\log_e(x) = \ln(x)$
$\det(A)$	the determinant of matrix A
$\pi(x)$	the number of prime numbers $p \leq x$
$\mathrm{Li}(x)$	the logarithmic integral of x, $\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}$
$\sum$	summation
$\prod$	product
$\to$	tends toward
x^+	toward x from the right
x^-	toward x from the left
$f'(x)$	the first derivative of $f(x)$ with respect to x
$\Re(x)$	the real part of x
$\Im(x)$	the imaginary part of x

$\overline{x}$	the complex conjugate of x
$\arg(x)$	the argument of a complex number x
$\Delta_C \arg(f(x))$	the number of changes in the argument of $f(x)$ along the contour C
$\mathbb{N}$	the set of natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	the set of integers
$\mathbb{Z}/p\mathbb{Z}$	the ring of integers modulo p
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{C}$	the set of complex numbers
$f(x) = O(g(x))$	$\|f(x)\| \leq A\|g(x)\|$ for some constant A and all values of $x > x_0$ for some x_0
$f(x) = o(g(x))$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$
$f \ll g$	$\|f(x)\| \leq A\|g(x)\|$ for some constant A and all values of $x > x_0$ for some x_0
$f \ll_\varepsilon g$	$\|f(x)\| \leq A(\varepsilon)\|g(x)\|$ for some given function $A(\varepsilon)$ and all values of $x > x_0$ for some x_0
$f(x) = \Omega(g(x))$	$\|f(x)\| \geq A\|g(x)\|$ for some constant A and all values of $x > x_0$ for some x_0
$f \sim g$	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

Part I

Introduction to the Riemann Hypothesis

1

Why This Book

One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation [133].

Bernhard Riemann, 1859

The above comment appears in Riemann's memoir to the Berlin Academy of Sciences (Section 12.2). It seems to be a passing thought, yet it has become, arguably, the most central problem in modern mathematics.

This book presents the Riemann hypothesis, connected problems, and a taste of the related body of theory. The majority of the content is in Part II, while Part I contains a summary and exposition of the main results. It is targeted at the educated nonexpert; most of the material is accessible to an advanced mathematics student, and much is accessible to anyone with some university mathematics.

Part II is a selection of original papers. This collection encompasses several important milestones in the evolution of the theory connected to the Riemann hypothesis. It also includes some authoritative expository papers. These are the "expert witnesses" who offer the most informed commentary on the Riemann hypothesis.

1.1 The Holy Grail

The Riemann hypothesis has been the Holy Grail of mathematics for a century and a half. Bernhard Riemann, one of the extraordinary mathematical talents of the nineteenth century, formulated the problem in 1859. The hypothesis makes a very precise connection between two seemingly unrelated mathematical objects (namely, prime numbers and the zeros of analytic functions). If solved, it would give us profound insight into number theory and, in particular, the nature of prime numbers.

Why is the Riemann hypothesis so important? Why is it the problem that many mathematicians would sell their souls to solve? There are a number of great old unsolved problems in mathematics, but none of them has quite the stature of the Riemann hypothesis. This stature can be attributed to a variety of causes ranging from mathematical to cultural. As with the other old great unsolved problems, the Riemann hypothesis is clearly very difficult. It has resisted solution for 150 years and has been attempted by many of the greatest minds in mathematics.

The problem was highlighted at the 1900 International Congress of Mathematicians, a conference held every four years and the most prestigious international mathematics meeting. David Hilbert, one of the most eminent mathematicians of his generation, raised 23 problems that he thought would shape twentieth century mathematics. This was somewhat self-fulfilling, since solving a Hilbert problem guaranteed instant fame and perhaps local riches. Many of Hilbert's problems have now been solved.

Being one of Hilbert's 23 problems was enough to guarantee the Riemann hypothesis centrality in mathematics for more than a century. Adding to interest in the hypothesis is a million-dollar bounty as a "Millennium Prize Problem" of the Clay Mathematics Institute. That the Riemann hypothesis should be listed as one of seven such mathematical problems (each with a million-dollar prize associated with its solution) indicates not only the contemporary importance of a solution, but also the importance of motivating a new generation of researchers to explore the hypothesis further.

Solving any of the great unsolved problems in mathematics is akin to the first ascent of Everest. It is a formidable achievement, but after the conquest there is sometimes nowhere to go but down. Some of the great problems have proven to be isolated mountain peaks, disconnected from their neighbors. The Riemann hypothesis is quite different in this regard. There is a large body of mathematical speculation that becomes fact if the Riemann hypothesis is solved. We know many statements of the form "if the Riemann hypothesis, then the following interesting mathematical statement", and this is rather different from the solution of problems such as the Fermat problem.

The Riemann hypothesis can be formulated in many diverse and seemingly unrelated ways; this is one of its beauties. The most common formulation is that certain numbers, the zeros of the "Riemann zeta function", all lie on a certain line (precise definitions later). This formulation can, to some extent, be verified numerically.

In one of the largest calculations done to date, it was checked that the first ten trillion of these zeros lie on the correct line. So there are ten trillion pieces of evidence indicating that the Riemann hypothesis is true and not a single piece of evidence indicating that it is false. A physicist might be overwhelmingly pleased with this much evidence in favour of the hypothesis, but to some mathematicians this is hardly evidence at all. However, it is interesting ancillary information.

In order to prove the Riemann hypothesis it is required to show that all of these numbers lie in the right place, not just the first ten trillions. Until such a proof is provided, the Riemann hypothesis cannot be incorporated into the body of mathematical facts and accepted as true by mathematicians (even though it is probably true!). This is not just pedantic fussiness. Certain mathematical phenomena that appear true, and that can be tested in part computationally, are false, but only false past computational range (This is seen in Sections 12.12, 12.9, and 12.14).

Accept for a moment that the Riemann hypothesis is the greatest unsolved problem in mathematics and that the greatest achievement any young graduate student could aspire to is to solve it. Why isn't it better known? Why hasn't it permeated public consciousness in the way black holes and unified field theory have, at least to some extent? Part of the reason for this is that it is hard to state rigorously, or even unambiguously. Some undergraduate mathematics is required in order for one to be familiar enough with the objects involved to even be able to state the hypothesis accurately. Our suspicion is that a large proportion of professional mathematicians could not precisely state the Riemann hypothesis if asked.

> *If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?*
>
> Attributed to **David Hilbert**

1.2 Riemann's Zeta and Liouville's Lambda

The Riemann zeta function is defined, for $\Re(s) > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.2.1}$$

The Riemann hypothesis is usually given as follows: the nontrivial zeros of the Riemann zeta function lie on the line $\Re(s) = \frac{1}{2}$. There is already, of course, the problem that the above series doesn't converge on this line, so one is already talking about an analytic continuation.

Our immediate goal is to give as simple (equivalent) a statement of the Riemann hypothesis as we can. Loosely, the statement is, "the number of integers with an even number of prime factors is the same as the number of integers with an odd number of prime factors." This is made precise in terms of the Liouville function, which gives the parity of the number of prime factors of a positive integer.

Definition 1.1. *The Liouville function is defined by*

$$\lambda(n) = (-1)^{\omega(n)},$$

where $\omega(n)$ is the number of, not necessarily distinct, prime factors of n, counted with multiplicity.

So $\lambda(2) = \lambda(3) = \lambda(5) = \lambda(7) = \lambda(8) = -1$ and $\lambda(1) = \lambda(4) = \lambda(6) = \lambda(9) = \lambda(10) = 1$ and $\lambda(x)$ is completely multiplicative (i.e., $\lambda(xy) = \lambda(x)\lambda(y)$ for any $x, y \in \mathbb{N}$) taking only values ± 1. (Alternatively, one can define λ as the completely multiplicative function with $\lambda(p) = -1$ for any prime p.)

The following connections between the Liouville function and the Riemann hypothesis were explored by Landau in his doctoral thesis of 1899.

Theorem 1.2. *The Riemann hypothesis is equivalent to the statement that for every fixed $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{\lambda(1) + \lambda(2) + \cdots + \lambda(n)}{n^{\frac{1}{2}+\varepsilon}} = 0.$$

This translates to the following statement: The Riemann hypothesis is equivalent to the statement that an integer has equal probability of having an odd number or an even number of distinct prime factors (in the precise sense given above). This formulation has inherent intuitive appeal.

We can translate the equivalence once again. The sequence

$$\{\lambda(i)\}_{i=1}^{\infty} = \{1, -1, -1, 1, -1, 1, -1, -1, 1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, \ldots\}$$

behaves more or less like a random sequence of 1's and -1's in that the difference between the number of 1's and -1's is not much larger than the square root of the number of terms.

This is an elementary and intuitive statement of the Riemann hypothesis. It is an example of the many diverse reformulations given in Chapter 5. It is relatively easy to formulate, and lends itself to an attractive picture (see Figure 1.1).

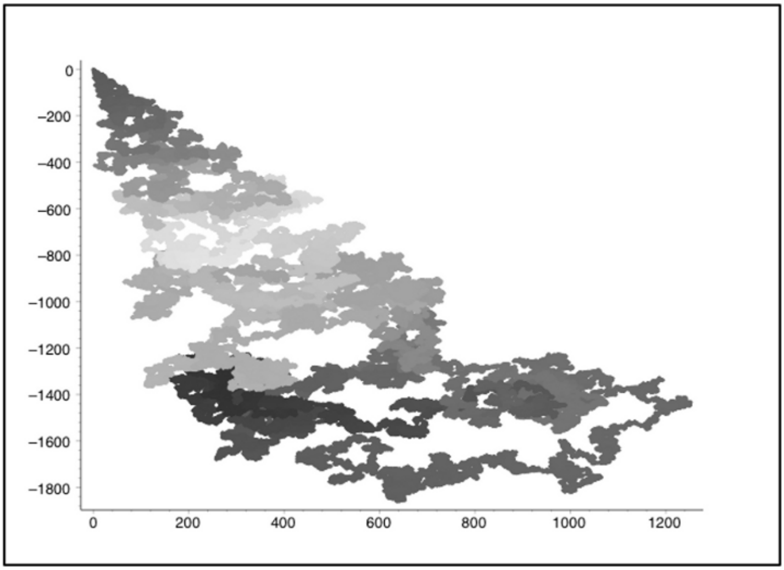

Fig. 1.1. $\{\lambda(i)\}_{i=1}^{\infty}$ plotted as a "random walk," by walking $(\pm 1, \pm 1)$ through pairs of points of the sequence.

1.3 The Prime Number Theorem

The prime number theorem is a jewel of mathematics. It states that the number of primes less than or equal to n is approximately $n/\log n$, and was conjectured by Gauss in 1792, on the basis of substantial computation and insight. One can view the Riemann hypothesis as a precise form of the prime number theorem, in which the rate of convergence is made specific (see Section 5.1).

Theorem 1.3 (The Prime Number Theorem). *Let $\pi(n)$ denote the number of primes less than or equal to n. Then*

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\log(n)} = 1.$$

As with the Riemann hypothesis, the prime number theorem can be formulated in terms of the Liouville lambda function, a result also due to Landau in his doctoral thesis of 1899.

Theorem 1.4. *The prime number theorem is equivalent to the statement*

$$\lim_{n\to\infty} \frac{\lambda(1) + \lambda(2) + \cdots + \lambda(n)}{n} = 0.$$

So the prime number theorem is a relatively weak statement of the fact that an integer has equal probability of having an odd number or an even number of distinct prime factors

The prime number theorem was first proved independently by de la Vallée Poussin (see Section 12.4) and Hadamard (see Section 12.3) around 1896, although Chebyshev came close in 1852 (see Section 12.1). The first and easiest proofs are analytic and exploit the rich connections between number theory and complex analysis. It has resisted trivialization, and no really easy proof is known. This is especially true for the so-called elementary proofs, which use little or no complex analysis, just considerable ingenuity and dexterity. The primes arise sporadically and, apparently, relatively randomly, at least in the sense that there is no easy way to find a large prime number with no obvious congruences. So even the amount of structure implied by the prime number theorem is initially surprising.

Included in the original papers in Part II are a variety of proofs of the prime number theorem. Korevaar's expository paper (see Section 12.16) is perhaps the most accessible of these. We focus on the prime number theorem not only for its intrinsic interest, but also because it represents a high point in the quest for resolution of the Riemann hypothesis. Now that we have some notion of what the Riemann hypothesis is, we can move on to the precise analytic formulation.

2

Analytic Preliminaries

The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics [64].

G. H. Hardy, 1940

In this chapter we develop some of the more important, and beautiful, results in the classical theory of the zeta function. The material is mathematically sophisticated; however, our presentation should be accessible to the reader with a first course in complex analysis. At the very least, the results should be meaningful even if the details are elusive.

We first develop the functional equation for the Riemann zeta function from Riemann's seminal paper, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (see Section 12.2), as well as some basic properties of the zeta function. Then we present part of de la Vallée Poussin's proof of the prime number theorem (see Section 12.4); in particular, we prove that $\zeta(1+it) \neq 0$ for $t \in \mathbb{R}$. We also develop some of main ideas to numerically verify the Riemann hypothesis; in particular, we prove that $N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$, where $N(T)$ is the number of zeros of $\zeta(s)$ up to height T. Finally, we give a proof of Hardy's result that there are infinitely many zeros of $\zeta(s)$ on the critical line, that is, the line $\left\{\frac{1}{2} + it, t \in \mathbb{R}\right\}$, from Section 12.5.

2.1 The Riemann Zeta Function

The Riemann hypothesis is a precise statement, and in one sense what it means is clear, but what it's connected with, what it implies, where it comes from, can be very unobvious [138].

M. Huxley

Defining the Riemann zeta function is itself a nontrivial undertaking. The function, while easy enough to define formally, is of sufficient complexity that such a statement would be unenlightening. Instead we “build” the Riemann zeta function in small steps. By and large, we follow the historical development of the zeta function from Euler to Riemann. This development sheds some light on the deep connection between the zeta function and the prime numbers. We begin with the following example of a Dirichlet series.

Let $s = \sigma + it$ $(\sigma, t \in \mathbb{R})$ be a complex number. Consider the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots . \tag{2.1.1}$$

This series is the first building block of the Riemann zeta function. Notice that if we set $s = 1$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots ,$$

the well-known harmonic series, which diverges. Also notice that whenever $\Re(s) \leq 1$, the series will diverge. It is also easy to see that this series converges whenever $\Re(s) > 1$ (this follows from the integral test). So this Dirichlet series defines an analytic function in the region $\Re(s) > 1$. We initially define the Riemann zeta function to be

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2.1.2}$$

for $\Re(s) > 1$.

Euler was the first to give any substantial analysis of this Dirichlet series. However, Euler confined his analysis to the real line. Euler was also the first to evaluate, to high precision, the values of the series for $s = 2, 3, \ldots, 15, 16$. For example, Euler established the formula,

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

Riemann was the first to make an intensive study of this series as a function of a complex variable.

Euler’s most important contribution to the theory of the zeta function is the Euler product formula. This formula demonstrates explicitly the connection between prime numbers and the zeta function. Euler noticed that every positive integer can be uniquely written as a product of powers of different primes. Thus, for any $n \in \mathbb{N}$, we may write

$$n = \prod_{p_i} p_i^{e_i},$$

where the p_i range over all primes, and the e_i are nonnegative integers. The exponents e_i will vary as n varies, but it is clear that if we consider each $n \in \mathbb{N}$ we will use every possible combination of exponents $e_i \in \mathbb{N} \cup \{0\}$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right),$$

where the infinite product is over all the primes. On examining the convergence of both the infinite series and the infinite product, we easily obtain the Euler product formula:

Theorem 2.1 (Euler Product Formula). *For $s = \sigma + it$ and $\sigma > 1$, we have*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{2.1.3}$$

The Euler product formula is also called the analytic form of the fundamental theorem of arithmetic. It demonstrates how the Riemann zeta function encodes information on the prime factorization of integers and the distribution of primes.

We can also recast one of our earlier observations in terms of the Euler product formula. Since convergent infinite products never vanish, the Euler product formula yields the following theorem.

Theorem 2.2. *For all $s \in \mathbb{C}$ with $\Re(s) > 1$, we have $\zeta(s) \neq 0$.*

We have seen that the Dirichlet series (2.1.2) diverges for any s with $\Re(s) \leq 1$. In particular, when $s = 1$ the series is the harmonic series. Consequently, the Dirichlet series (2.1.2) does not define the Riemann zeta function outside the region $\Re(s) > 1$. We continue to build the zeta function, as promised. However, first we note that our definition of the zeta function, valid for $\Re(s) > 1$, actually uniquely determines the values of $\zeta(s)$ for all $s \in \mathbb{C}$. This is a consequence of the fact that $\zeta(s)$ is analytic for $\Re(s) > 1$, and continues analytically to the entire plane, with one exceptional point, as explained below.

Recall that analytic continuation allows us to "continue" an analytic function on one domain to an analytic function of a larger domain, uniquely, under certain conditions. (Specifically, given functions f_1, analytic on domain D_1, and f_2, analytic on domain D_2, such that $D_1 \cap D_2 \neq \varnothing$ and $f_1 = f_2$ on $D_1 \cap D_2$, then $f_1 = f_2$ on $D_1 \cup D_2$.) So if we can find a function, analytic on $\mathbb{C}\backslash\{1\}$, that agrees with our Dirichlet series on any domain, D, then we succeed in defining $\zeta(s)$ for all $s \in \mathbb{C}\backslash\{1\}$.

In his 1859 memoir, Riemann proves that the function $\zeta(s)$ can be continued analytically to an analytic function over the whole complex plane, with the exception of $s = 1$, and at $s = 1$, $\zeta(s)$ has a simple pole, with residue 1.

We now define the Riemann zeta function. Following convention, we write $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ when $s \in \mathbb{C}$.

Definition 2.3. *The Riemann zeta function $\zeta(s)$ is the analytic continuation of the Dirichlet series* (2.1.2) *to the whole complex plane, minus the point $s = 1$.*

Defining the zeta function in this way is concise and correct, but its properties are quite unclear. We continue to build the zeta function by finding the analytic continuation of $\zeta(s)$ explicitly. To start with, when $\Re(s) > 1$, we write

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = s \sum_{n=1}^{\infty} n \int_n^{n+1} x^{-s-1} dx.$$

Let $x = [x] + \{x\}$, where $[x]$ and $\{x\}$ are the integral and fractional parts of x, respectively.

Since $[x]$ is always the constant n for any x in the interval $[n, n+1)$, we have

$$\zeta(s) = s \sum_{n=1}^{\infty} \int_n^{n+1} [x] x^{-s-1} dx = s \int_1^{\infty} [x] x^{-s-1} dx.$$

By writing $[x] = x - \{x\}$, we obtain

$$\begin{aligned} \zeta(s) &= s \int_1^{\infty} x^{-s} dx - s \int_1^{\infty} \{x\} x^{-s-1} dx \\ &= \frac{s}{s-1} - s \int_1^{\infty} \{x\} x^{-s-1} dx, \qquad \sigma > 1. \end{aligned} \tag{2.1.4}$$

We now observe that since $0 \leq \{x\} < 1$, the improper integral in (2.1.4) converges when $\sigma > 0$ because the integral $\int_1^{\infty} x^{-\sigma-1} dx$ converges. Thus the improper integral in (2.1.4) defines an analytic function of s in the region $\Re(s) > 0$. Therefore, the meromorphic function on the right-hand side of (2.1.4) gives the analytic continuation of $\zeta(s)$ to the region $\Re(s) > 0$, and the term $\frac{s}{s-1}$ gives the simple pole of $\zeta(s)$ at $s = 1$ with residue 1.

Equation (2.1.4) extends the definition of the Riemann zeta function only to the larger region $\Re(s) > 0$. However, Riemann used a similar argument to obtain the analytic continuation to the whole complex plane. He started from the classical definition of the gamma function Γ.

We recall that the gamma function extends the factorial function to the entire complex plane with the exception of the nonpositive integers. The usual definition of the *gamma function*, $\Gamma(s)$, is by means of Euler's integral

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$$

but this applies only for $\Re(s) > 0$. Weierstrass' formula

$$\frac{1}{s\Gamma(s)} := e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}$$

where γ is Euler's constant, applies in the whole complex plane. The Γ function is analytic on the entire complex plane with the exception of $s = 0, -1, -2, \ldots$, and the residue of $\Gamma(s)$ at $s = -n$ is $\frac{(-1)^n}{n!}$. Note that for $s \in \mathbb{N}$ we have $\Gamma(s) = (s-1)!$.

We have

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} e^{-t} t^{\frac{s}{2}-1} dt$$

for $\sigma > 0$. On setting $t = n^2 \pi x$, we observe that

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^{\infty} x^{\frac{s}{2}-1} e^{-n^2 \pi x} dx.$$

Hence, with some care on exchanging summation and integration, for $\sigma > 1$,

$$\begin{aligned}\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \int_0^{\infty} x^{\frac{s}{2}-1} \left(\sum_{n=1}^{\infty} e^{-n^2 \pi x}\right) dx \\ &= \int_0^{\infty} x^{\frac{s}{2}-1} \left(\frac{\vartheta(x) - 1}{2}\right) dx,\end{aligned}$$

where

$$\vartheta(x) := \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}$$

is the *Jacobi theta function*. The functional equation (also due to Jacobi) for $\vartheta(x)$ is

$$x^{\frac{1}{2}} \vartheta(x) = \vartheta(x^{-1}),$$

and is valid for $x > 0$. This equation is far from obvious; however, the proof lies beyond our focus. The standard proof proceeds using Poisson summation, and can be found in Chapter 2 of [22].

Finally, using the functional equation of $\vartheta(x)$, we obtain

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left\{\frac{1}{s(s-1)} + \int_1^{\infty} \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right) \cdot \left(\frac{\vartheta(x) - 1}{2}\right) dx\right\}. \tag{2.1.5}$$

Due to the exponential decay of $\vartheta(x)$, the improper integral in (2.1.5) converges for *every* $s \in \mathbb{C}$ and hence defines an entire function in $\mathbb{C}$. Therefore, (2.1.5) gives the analytic continuation of $\zeta(s)$ to the whole complex plane, with the exception of $s = 1$.

Theorem 2.4. *The function*

$$\zeta(s) := \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left\{ \frac{1}{s(s-1)} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}} \right) \cdot \left(\frac{\vartheta(x) - 1}{2} \right) dx \right\}$$

is meromorphic with a simple pole at $s = 1$ *with residue* 1.

We have now succeeded in our goal of continuing the Dirichlet series (2.1.2) that we started with, to $\zeta(s)$, a meromorphic function on $\mathbb{C}$. We can now consider all complex numbers in our search for the zeros of $\zeta(s)$. We are interested in these zeros because they encode information about the prime numbers. However, not all of the zeros of $\zeta(s)$ are of interest to us. Surprisingly, we can find, with relative ease, an infinite number of zeros, all lying outside of the region $0 \leq \Re(s) \leq 1$. We refer to these zeros as the *trivial zeros* of $\zeta(s)$ and we exclude them from the statement of the Riemann hypothesis.

Before discussing the zeros of $\zeta(s)$ we develop a functional equation for it. Riemann noticed that formula (2.1.5) not only gives the analytic continuation of $\zeta(s)$, but can also be used to derive a functional equation for $\zeta(s)$. He observed that the term $\frac{1}{s(s-1)}$ and the improper integral in (2.1.5) are invariant under the substitution of s by $1 - s$. Hence we have the following functional equation:

Theorem 2.5 (The Functional Equation). *For any* s *in* $\mathbb{C}$,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

For convenience, and clarity, we will define the function as

$$\xi(s) := \frac{s}{2}(s-1)\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s). \tag{2.1.6}$$

In view of (2.1.5), $\xi(s)$ is an entire function and satisfies the simple functional equation

$$\xi(s) = \xi(1-s). \tag{2.1.7}$$

This shows that $\xi(s)$ is symmetric around the vertical line $\Re(s) = \frac{1}{2}$.

We now have developed the zeta function sufficiently to begin considering its various properties; in particular, the location of its zeros. There are a few assertions we can make based on the elementary theory we have already presented.

We begin our discussion by isolating the trivial zeros of $\zeta(s)$. Recall that the only poles of $\Gamma(s)$ are simple and situated at $s = 0, -1, -2, \ldots$. It follows from (2.1.5) that $\zeta(s)$ has simple zeros at $s = -2, -4, \ldots$ (the pole $s = 0$ of $\Gamma(\frac{s}{2})$ is canceled by the term $\frac{1}{s(s-1)}$). These zeros, arising from the poles of the gamma function, are termed the *trivial zeros*. From the functional equation and Theorem 2.2, all other zeros, the *nontrivial zeros*, lie in the vertical strip $0 \leq \Re(s) \leq 1$. In view of equation (2.1.6), the nontrivial zeros of $\zeta(s)$ are precisely the zeros of $\xi(s)$, and hence they are symmetric about the vertical line $\Re(s) = \frac{1}{2}$. Also, in view of (2.1.5), they are symmetric about the real axis, $t = 0$. We summarize these results in the following theorem.

Theorem 2.6. *The function $\zeta(s)$ satisfies the following*

1. *$\zeta(s)$ has no zero for $\Re(s) > 1$;*
2. *the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;*
3. *$\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;*
4. *the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$;*
5. *the zeros of $\xi(s)$ are precisely the nontrivial zeros of $\zeta(s)$.*

The strip $0 \leq \Re(s) \leq 1$ is called the *critical strip* and the vertical line $\Re(s) = \frac{1}{2}$ is called the *critical line*.

Riemann commented on the zeros of $\zeta(s)$ in his memoir (see the statement at the start of Chapter 1). From his statements the Riemann hypothesis was formulated.

Conjecture 2.7 (The Riemann Hypothesis). *All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.*

Riemann's eight-page memoir has legendary status in mathematics. It not only proposed the Riemann hypothesis, but also accelerated the development of analytic number theory. Riemann conjectured the asymptotic formula for the number, $N(T)$, of zeros of $\zeta(s)$ in the critical strip with $0 \leq \Im(s) < T$ to be

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

(proved by von Mangoldt in 1905). In particular, there are infinitely many nontrivial zeros. Additionally, he conjectured the product representation of $\xi(s)$ to be

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \tag{2.1.8}$$

where A, B are constants and ρ runs over all the nontrivial zeros of $\zeta(s)$ (proved by Hadamard in 1893).

2.2 Zero-free Region

One approach to the Riemann hypothesis is to expand the zero-free region as much as possible. However, the proof that the zero-free region includes the vertical line $\Re(s) = 1$ (i.e., $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$) is already nontrivial. In fact, this statement is equivalent to the prime number theorem, namely

$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty$$

(a problem that required a century of mathematics to solve). Since we wish to focus our attention here on the analysis of $\zeta(s)$, we refer the reader to proofs of this equivalence in Sections 12.3, 12.4, and 12.16.

Theorem 2.8. *For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.*

Proof. In order to prove this result we follow the 1899 approach of de la Vallée Poussin (see Section 12.4). Recall that when $\sigma > 1$, the zeta function is defined by the Dirichlet series (2.1.2) and that the Euler product formula gives us

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{2.2.1}$$

where $s = \sigma + it$. Taking logarithms of each side of (2.2.1), we obtain

$$\log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right).$$

Using the Taylor expansion of $\log(1-x)$ at $x = 0$, we have

$$\begin{aligned} \log \zeta(s) &= \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-sm} \\ &= \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-\sigma m} p^{-imt} \\ &= \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-\sigma m} e^{-imt \log p}. \end{aligned} \tag{2.2.2}$$

It follows that the real part of $\log \zeta(s)$ is

$$\Re(\log \zeta(s)) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-\sigma m} \cos(mt \log p). \tag{2.2.3}$$

Note that by (2.2.3),

$$\begin{aligned}
&3\Re(\log\zeta(\sigma)) + 4\Re(\log\zeta(\sigma+it)) + \Re(\log\zeta(\sigma+2ti)) \\
&= 3\sum_p\sum_{m=1}^{\infty} m^{-1}p^{-\sigma m} + 4\sum_p\sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}\cos(mt\log p) \\
&\quad + \sum_p\sum_{m=1}^{\infty} m^{-1}p^{-\sigma m}\cos(2mt\log p) \\
&= \sum_p\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{p^{\sigma m}}\Big(3+4\cos(mt\log p)+\cos(2mt\log p)\Big).
\end{aligned}$$

Using $\log W = \log|W| + i\arg(W)$ and the elementary inequality

$$2(1+\cos\theta)^2 = 3 + 4\cos\theta + \cos(2\theta) \geqslant 0, \tag{2.2.4}$$

valid for any $\theta \in \mathbb{R}$, we obtain

$$3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2ti)| \geqslant 0,$$

or equivalently,

$$|\zeta(\sigma)|^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \geqslant 1. \tag{2.2.5}$$

Since $\zeta(\sigma)$ has a simple pole at $\sigma = 1$ with residue 1, the Laurent series of $\zeta(\sigma)$ at $\sigma = 1$ is

$$\zeta(\sigma) = \frac{1}{1-\sigma} + a_0 + a_1(\sigma-1) + a_2(\sigma-1)^2 + \cdots = \frac{1}{1-\sigma} + g(\sigma),$$

where $g(\sigma)$ is analytic at $\sigma = 1$. Hence, for $1 < \sigma \le 2$, we have $|g(\sigma)| \le A_0$, for some $A_0 > 0$, and

$$|\zeta(\sigma)| = \frac{1}{1-\sigma} + A_0.$$

Now we will show that $\zeta(1+it) \neq 0$ using inequality (2.2.5). To obtain a contradiction, suppose that there is a zero on the line $\sigma = 1$. So $\zeta(1+it) = 0$ for some $t \in \mathbb{R}$, $t \neq 0$. Then by the mean-value theorem,

$$\begin{aligned}
|\zeta(\sigma+it)| &= |\zeta(\sigma+it) - \zeta(1+it)| \\
&= |\sigma-1||\zeta'(\sigma_0+it)|, \qquad 1 < \sigma_0 < \sigma, \\
&\le A_1(\sigma-1),
\end{aligned}$$

where A_1 is a constant depending only on t. Also, when σ approaches 1 we have $|\zeta(\sigma+2it)| < A_2$, where A_2 again depends only on t. Note that in (2.2.5) the degree of the term $\sigma - 1$, which is 4, is greater than that of the term $\frac{1}{\sigma-1}$, which is 3. So for fixed t, as σ tends to 1^+, we have