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Eugene F. Milone
and William J. F. Wilson

Solar System Astrophysics

Planetary Atmospheres and the Outer Solar System

Springer

Eugene F. Milone
Dept. Physics & Astronomy
University of Calgary, Calgary
2500 University Drive NW
Calgary, Alberta T2N 1N4
Canada
milone@ucalgary.ca

Willam J. F. Wilson
Dept. Physics & Astronomy
University of Calgary, Calgary
2500 University Drive NW
Calgary, Alberta T2N 1N4
Canada
wjfwilso@ucalgary.ca

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Preface

This work is appearing in two parts because its mass is the result of combining detailed exposition and recent scholarship. Book I, dealing mainly with the inner solar system, and Book II, mainly on the outer solar system, represent the combined, annually updated, course notes of E. F. Milone and W. J. F. Wilson for the undergraduate course in solar system astrophysics that has been taught as part of the Astrophysics Program at the University of Calgary since the 1970s. The course, and so the book, assumes an initial course in astronomy and first-year courses in mathematics and physics. The relevant concepts of mathematics, geology, and chemistry that are required for the course are introduced within the text itself.

Solar System Astrophysics is intended for use by second- and third-year astrophysics majors, but other science students have also found the course notes rewarding. We therefore expect that students and instructors from other disciplines will also find the text a useful treatment. Finally, we think the work will be a suitable resource for amateurs with some background in science or mathematics. Most of the mathematical formulae presented in the text are derived in logical sequences. This makes for large numbers of equations, but it also makes for relatively clear derivations. The derivations are found mainly in Chapters 2–6 in Book I, subtitled, *Background Science and the Inner Solar System*, and in Chapters 10 and 11 in Book II subtitled, *Planetary Atmospheres and the Outer Solar System*. Equations are found in the other chapters as well but these contain more expository material and recent scholarship than some of the earlier chapters. Thus, Chapters 8 and 9, and 12–16 contain some useful derivations, but also much imagery and results of modern studies.

The first volume starts with a description of historical perceptions of the solar system and universe, in narrowing perspective over the centuries, reflecting the history (until the present century, when extra-solar planets again have begun to broaden our focus). The second chapter treats the basic concepts in the geometry of the circle and of the sphere, reviewing and extending material from introductory astronomy courses, such as spherical coordinate transformations. The third chapter then reviews basic mechanics and two-body systems, orbital description, and the computations of ephemerides, then progresses to the restricted three-body and n-body cases, and concludes with a discussion of perturbations. The fourth chapter treats the core of the solar system, the Sun, and is not a bad introduction to solar or stellar astrophysics; the place of the Sun in the galaxy and in the context of other stars is described, and radiative transport, optical depth, and limb-darkening are introduced. In Chapter 5, the structure and composition of the Earth are discussed, the Adams–Williamson equation is derived, and its use for determining the march of pressure and density with radius described. In Chapter 6, the thermal structure and energy transport through the Earth are treated, and in this chapter the basic ideas of thermodynamics are put to use. Extending the discussion of the Earth's interior, Chapter 7 describes the rocks and minerals in the Earth and their crystalline structure. Chapter 8 treats the Moon, its structures, and its origins, making use of the developments of the preceding chapters. In Chapter 9, the surfaces of the other terrestrial planets are described, beginning with Mercury. In each of the three sections of this chapter, a brief historical discussion is followed by descriptions of modern ground-based and space mission results, with some of the spectacular imagery of Venus and Mars. The chapter concludes with a description of the evidence for water and surface modification on Mars. This concludes the discussion of the inner solar system.

The second volume begins in Chapter 10 with an extensive treatment of the physics and chemistry of the atmosphere and ionosphere of the Earth and an introduction to meteorology, and this discussion is extended to the atmospheres of Venus and Mars. Chapter 11 treats the magnetospheres of the inner planets, after a brief exposition of electromagnetic theory. In Chapter 12, we begin to treat the outer solar system, beginning with the gas giants. The structure, composition, and particle environments around these planets are discussed, and this is continued in Chapter 13, where the natural satellites and rings of these objects are treated in detail, with abundant use made of the missions to the outer planets. In Chapter 14, we discuss comets, beginning with an historical introduction that highlights the importance of comet studies to the development of modern astronomy. It summarizes the ground- and space-based imagery and discoveries, but makes use of earlier derivations to discuss cometary orbits. This chapter ends with the demise of comets and the physics of meteors. Chapter 15 treats the study of meteorites and the remaining small bodies of the solar system, the asteroids (*aka* minor planets, planetoids), and the outer solar system "Kuiper Belt" objects, and the closely related objects known as centaurs, plutinos, cubewanos, and others, all of which are numbered as asteroids. The chapter ends with discussions of the origin of the solar system and of debris disks around other stars, which point to widespread evidence of the birth of other planetary systems. Finally, in Chapter 16, we discuss the methods and results of extra-solar planet searches, the distinctions among stars, brown dwarfs, and planets, and we explore the origins of planetary systems in this wider context.

At the end of each chapter we have a series of challenges. Instructors may use these as homework assignments, each due two weeks after the material from that chapter were discussed in class; *we* did! The general reader may find them helpful as focusing aids.

January 22, 2008 *E. F. Milone* & *W. J. F. Wilson*

Acknowledgments

These volumes owe their origin to more than 30 years of solar system classes in the Astrophysics Program at the University of Calgary, called, at various times, Geophysics 375, Astrophysics 301, 309, and 409. Therefore, we acknowledge, first, the students who took these courses and provided feedback. It is also a pleasure to thank the following people for their contributions:

David Mouritsen, formerly of Calgary and now in Toronto, provided for Chapter 1 and our covers an image of his original work of art, an interpretation of Kepler's *Mysterium Cosmographicum*, in which the orbits of the planets are inscribed within solid geometric figures.

In Chapter 3, the latest version of David Bradstreet's software package, *Binary Maker 3* was used to create an image to illustrate restricted three-body solutions.

University of Calgary Professor Emeritus Alan Clark gave us an image of an active region and detailed comments on the solar physics material of Chapter 4; Dr Rouppe van der Voort of the University of Oslo provided high-quality images of two other active region figures, for Chapter 4; the late Dr Richard Tousey of the US Naval Research Laboratory provided slides of some of the images, subsequently scanned for Chapter 4; limb-darkened spectral distribution plots were provided by Dr Robert L. Kurucz, of the Harvard-Smithsonian Center for Astrophysics; Dr. Charles Wolff, of Goddard Space Flight Center, NASA, reviewed the solar oscillations sections and provided helpful suggestions.

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NASA's online photo gallery provided many of the images in Chapters 8, 9, 12, 13, 14, and some of those in Chapter 15; additional images were provided by the Naval Research Laboratory (of both the Sun and the

Moon). Some of these and other images involved work by other institutions, such as the U.S. Geological Survey, the Jet Propulsion Laboratory, Arizona State Univ., Cornell, the European Space Agency, the Italian Space Agency (ASI), CalTech, Univ. of Arizona, Space Science Institute, Boulder, the German Air and Space Center (DLR), Brown University, the Voyagers and the Cassini Imaging Teams, CICLOPS, the Hubble Space Telescope, University of Maryland, the Minor Planet Center, Applied Physics Laboratory of the Johns Hopkins University, and the many individual sources, whether cited in captions or not, who contributed their talents to producing these images.

Dr. John Trauger provided a high resolution UV image of Saturn and its auroras for Chapter 12.

Dr William Reach, Caltech, provided an infrared mosaic image of Comet Schwassmann–Wachmann 3, and Mr John Mirtle of Calgary provided many of the comet images for Chapter 14, including those of Comets 109P/Swift–Tuttle, C/1995 O1 (Hale–Bopp), C/Hyakutake, Lee, C/Ikeya–Zhang, Brorsen–Metcalfe, and Machholz; Dr. Michael J. Mumma of NASA's GSFC provided an important correction, and Professor Michael F. A'Hearn of the University of Maryland critiqued the comet content of Chapter 14.

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Dr Charles Lineweaver, University of New South Wales, provided a convincing illustration for the brown dwarf desert, illustrated in Chapter 16; University of Calgary graduate student Michael Williams provided several figures from his MSc thesis for Chapter 16.

Mr Alexander Jack assisted in updating and improving the readability of equations and text in some of the early chapters, and he and Ms Veronica Jack assisted in developing the tables of the extra-solar planets and their host stars for Chapter 16.

In addition, we thank the many authors, journals, and publishers who have given us permission to use their figures and tabular material or adaptations thereof, freely. Finally, it is also a pleasure to thank Springer editors Dr Hans Koelsch, Dr Harry Blom, and their associate, Christopher Coughlin, for their support for this project.

Contents

Planetary Atmospheres and the Outer Solar System

10. Planetary Atmospheres

10.1 Atmospheric Constituents

The constituents of a planetary atmosphere are determined in a general way by the likelihood that a given constituent will be retained, rather than lost by evaporation into space over long periods of time. This likelihood is determined by three factors: (1) the equilibrium temperature of the planet, because the hotter the atmosphere, the larger the mean kinetic energy of its molecules,

$$E = \tfrac{3}{2}\, kT \tag{10.1}$$

where $k = 1.380658(12) \times 10^{-23}$ J/K is the Boltzmann constant, T is the equilibrium temperature in kelvin, and the number in parentheses is the uncertainty in the last two digits of the constant; (2) the molecular weight, m, of each atmospheric constituent, because equipartition of energy requires that more massive particles have smaller mean velocities,

$$E = \tfrac{1}{2} m v^2 \tag{10.2}$$

and (3) the planet's escape velocity,

$$v_{\mathrm{esc}} = \sqrt{\frac{2GM}{R}} \tag{10.3}$$

where M and R are the planetary mass and radius, respectively. In a region of atmosphere in thermal equilibrium, the distribution of the number of molecules vs. the speed at which they are moving at any instant is said to be *Maxwellian* (See Schlosser et al. 1991/4, Fig. 18.1).

The *molecular weight* of any species of molecule can be written

$$m = \mu m_{\mathrm{u}} \tag{10.4}$$

where $m_{\mathrm{u}} = (1/12)[m(^{12}\mathrm{C})] = 1.6605402(10) \times 10^{-27}$ kg is the unit atomic mass, and the dimensionless quantity μ is expressed in units of m_{u}. It is not uncommon to refer to μ as the *molecular weight*, the units of m_{u} being understood.

From (10.1), (10.2) and (10.4), the root mean square speed of any constituent can be written in terms of μ as

$$v = \sqrt{\frac{3kT}{m}} = 157.94\sqrt{\frac{T}{\mu}}\ \mathrm{m/s} \tag{10.5}$$

A very massive planet will retain even the lightest gases over a wide range of temperatures. Quite generally, if $v/v_{\rm esc} = 1/3$, the atmosphere can be expected to be lost in weeks; if $v/v_{\rm esc} = 1/4$, 10,000 years; if $v/v_{\rm esc} = 1/5$, 10^8 years; and if $v/v_{\rm esc} = 1/6$, it can be considered retained for eons, or billions of years (Gy), assuming no major departure from the assumed equilibrium planetary temperature over this interval of time.

From (10.3) and (10.5), the lower limit for the molecular weight of a molecule that can be *retained* is

$$\mu \geq \left(157.94\frac{\sqrt{T}}{(v_{\rm esc}/6)}\right)^2 = 8.980 \times 10^5 \frac{T}{v_{\rm esc}^2} \tag{10.6}$$

From (6.31) of Milone & Wilson (2008), the equilibrium temperature for an assumed black body, rapidly rotating planet is:

$$T = \{\mathfrak{L}_\odot(1-A)/[16\pi\sigma r^2]\}^{\frac{1}{4}} = 1.078 \times 10^8\,[(1-A)/r^2]^{\frac{1}{4}}\ \mathrm{K}$$

where A is the bolometric albedo, $\sigma = 5.671\times10^{-8}$ W m^{-2} K^{-4} is the Stefan–Boltzmann constant, r is the distance from the sun in meters, and the solar luminosity is $\mathfrak{L}_\odot = 3.845 \times 10^{26}$ W (Cox 2000, p. 340).

For a particular molecule to be retained over a significant fraction of the solar system's lifetime (assuming the equilibrium temperature correctly indicates the effective mean temperature of the planet, and there is no significant change in this temperature over this interval),

$$\mu \geq 8.980 \times 10^5 \times 1.078 \times 10^8 \frac{[(1-A)/r^2]^{1/4}}{2GM/R} = 7.254 \times 10^{23}\frac{R}{M}\left(\frac{1-A}{r^2}\right)^{1/4} \tag{10.7}$$

where r, R, M, and $G = 6.6726\times10^{-11}$ N m^2/kg^2 are in SI units. The corresponding numerical constant for a slowly (more precisely, synchronously) rotating planet is greater by a factor $2^{\frac{1}{4}} = 1.189$. Finally, with R and M in units of the Earth's radius and mass (6.378×10^6 m, 5.974×10^{24} kg, respectively) and a in au, (10.7) becomes

$$\mu \geq 2.002\frac{R/R_\oplus}{M/M_\oplus}\left(\frac{1-A}{a^2}\right)^{1/4} \tag{10.8}$$

As an example, for the Earth, $A \approx 0.307$, $\mu >\sim 2 \times 0.91 = 1.82$, provided T is correctly given by (10.7); i.e., $< T_\oplus >= T_{\rm eq} = 254$ K. In point of fact, the

Earth is slightly warmer because of its atmosphere, so that $< T_\oplus > = 288\,\mathrm{K}$. Using this value directly in (10.6), one obtains, $\mu = 2.07$.

Thus, the Earth is marginally unable to retain hydrogen at present, but should have retained helium ($\mu \sim 4$). It does not, in fact, have large amounts of helium in its atmosphere, suggesting that the Earth's surface was much hotter in the distant past, perhaps as a result of energy of accretion (or reconstitution, if the Mars-sized impact event suggested for the origin of the Moon did in fact occur).

See Schlosser et al. (1991/4, pp. 94–97) for further discussion of the species of molecules retainable in planetary atmospheres.

One of the more interesting concepts in planetary physics is the probability of escape of gases from the sub-solar region, where the instantaneous temperature is much higher than the global or even hemispherical average. How do we know if the evaporation from this location alone is the determining factor?

The answer probably lies in the *mean free path* of a molecule on the surface of a planet. The term refers to the average separation of molecules before colliding with other molecules. If a molecule receives sufficient energy for it to escape, how long does it take before it collides with another molecule? The problem and the solutions to it are described in, for example, Jeans' (1952) *Kinetic Theory of Gases.* The probability of collision depends strongly on the atmospheric density. For a particular molecule, the distance, d, traveled before a collision with some other, stationary, molecule may be written

$$d = \frac{1}{\pi n R^2} \tag{10.9}$$

where n is the number density [the number of particles (in this case, molecules) per unit volume], and R is a characteristic distance about equal to the mean radius of a molecule. The time it takes to cover this distance depends on the speed of the molecule. The average mean free path of a molecule, the speeds of which are expected to correspond to a Maxwellian distribution, is

$$\frac{1}{\sqrt{2}\pi n R^2} \tag{10.10}$$

This number is about 6×10^{-8} m for Earth's atmosphere, or about $320R$ (Jeans 1952 pp. 44–49).

Thus, a great many collisions may occur prior to escape, the angle of alteration of direction in each collision being a further variable, and, at each collision, energy may be lost as well as gained; obviously, it will be easier to escape in a less dense environment.

Next we discuss the structure of atmospheres, and the behavior of pressure with height.

10.2 Atmospheric Structure

10.2.1 Pressure Variation with Height

For an atmosphere to be a reasonably permanent feature of a planet, the atmosphere must be in hydrostatic equilibrium, i.e., it should neither accelerate outward nor collapse inward. Here, we will assume that the atmosphere can be approximated by spherical symmetry, e.g., only the radial components of force vectors are non-zero.

Figure 10.1 shows a free-body diagram for a small parcel of gas in such an atmosphere, in the form of a short cylinder of thickness dr, cross-section A, volume $\mathrm{d}V = A\mathrm{d}r$, density ρ, and mass $\mathrm{d}m = \rho \mathrm{d}V$, located at a distance r from the center of the planet. The unit vector, $\hat{r}$, signifies the positive radial direction; subscripts r or $r + \mathrm{d}r$ specify radial location.

For hydrostatic equilibrium, the forces acting on this pillbox-shaped parcel must add to zero (N.B: in this vector equation each term carries its own sign):

$$\vec{F}_r + \vec{F}_{r+\mathrm{d}r} + \mathrm{d}\vec{F}_\mathrm{g} = 0 \tag{10.11}$$

where from Figure 10.1, and with $M(r)$ being the total mass interior to radius r,

$$\mathrm{d}\vec{F}_\mathrm{g} = -g\,\mathrm{d}m\,\hat{\mathbf{r}} = -g\rho\,\mathrm{d}V\,\hat{\mathbf{r}} = -g\rho A\,\mathrm{d}r\,\hat{\mathbf{r}} = -\frac{GM(r)}{r^2}\rho A\,\mathrm{d}r\,\hat{\mathbf{r}} \tag{10.12}$$

$$\vec{F}_r = P_r A\hat{\mathrm{r}} \tag{10.13}$$

$$\vec{F}_{r+\mathrm{d}r} = -P_{r+\mathrm{d}r}A\hat{\mathrm{r}} = -(P_r + \mathrm{d}P)A\hat{\mathrm{r}} \tag{10.14}$$

Substituting (10.12)–(10.14) into (10.11) gives

$$P_r A - (P_r + \mathrm{d}P)A - g\rho A\mathrm{d}r = 0 \tag{10.15}$$

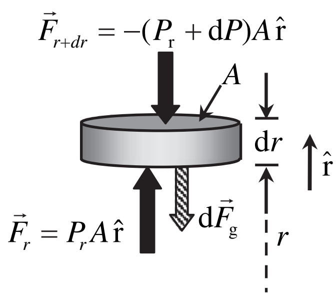

Fig. 10.1. Mechanical equilibrium of a parcel of air in a stationary atmosphere

or, solving for dP,

$$\mathrm{d}P = -\frac{\mathrm{d}F_{\mathrm{g}}}{A} = -g\rho\,\mathrm{d}r \tag{10.16}$$

The quantities g and ρ are positive, so dP is negative when dr is positive, i.e., pressure decreases outward in the atmosphere. The *pressure gradient* is the rate of change of pressure with distance, so

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -g\rho \tag{10.17}$$

also known as the *equation of hydrostatic equilibrium.*

The pressure can also be expressed in terms of the perfect gas law, which provides an *equation of state*:

$$P = nkT \tag{10.18}$$

Here, n is again the number density, related to the density and the mean molecular weight of the atmosphere through the relation

$$n = \rho/m \tag{10.19}$$

Then

$$P = \rho kT/m = \rho kT/\mu m_{\mathrm{u}}. \tag{10.20}$$

From (10.17) and (10.20), the pressure gradient is then

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\mu m_{\mathrm{u}} g P}{kT} \tag{10.21}$$

from which we obtain the differential equation

$$\frac{\mathrm{d}P}{P} = -\frac{\mu m_{\mathrm{u}} g}{kT}\mathrm{d}r \tag{10.22}$$

The integrated solution to (10.22) is

$$\ln\frac{P}{P_0} = -\frac{\mu m_{\mathrm{u}} g}{kT}(r - r_0) \tag{10.23}$$

or

$$\frac{P}{P_0} = \exp\left\{-\frac{\mu m_{\mathrm{u}} g}{kT}(r - r_0)\right\} \tag{10.24}$$

where r_0 is a reference height (e.g., the base of the atmosphere) and P_0 is the pressure at $r = r_0$.

The pressure *scale height* is defined as

$$H = \frac{kT}{\mu m_{\mathrm{u}} g} \tag{10.25}$$

k is related to the universal gas constant, R = 8.314472(15) J mol^{-1} K^{-1}, by R = kN_A, where N_A = 6.02214179(30) × 10^{23} mol^{-1} is Avogadro's number. Then

$$H = \frac{RT}{\mu m_u N_A g} = (10^3 \text{mol kg}^{-1}) \frac{RT}{\mu g} \tag{10.25a}$$

The factor in parentheses in (10.25a) may be combined with R to obtain a new constant (commonly and somewhat confusingly also written R), R = 8314.472(15) J kg^{-1} K^{-1}; then

$$H = \frac{RT}{\mu g} \tag{10.26}$$

Then, letting $h = r - r_0$, we get the *pressure scale height equation*,

$$P = P_0 e^{-h/H} \tag{10.27}$$

Equation (10.27) has been used to compute the mean molecular weight for outer planet atmospheres. In particular, the occultation of a star by Jupiter or another planet provides such an opportunity. The geometry, technique, and many results are reviewed by Elliott and Olkin (1996). As the planet progressively covers the star, astronomers at various sites on the Earth measure the optical depth along the line traversed by the starlight through the planet's atmosphere as a function of time. The *optical depth* is a measure of the absorptive/scattering properties of an atmosphere; an optical depth of one corresponds to the distance required for the transmitted light to decrease in intensity by a factor e. The curve of optical depth vs. time can be converted to a curve of refractive index vs. depth vertically into the atmosphere. Refractive index in turn depends on the number density of absorbers, n, as does the atmospheric pressure by (10.18), $P = nkT$, so that the index of refraction demonstrates similar behavior to pressure as the occultation proceeds. The refractivity scale height allows a determination of the pressure scale height, and from that the mean molecular weight of the atmosphere. In practice, the temperature and therefore pressure scale height vary with height through the atmosphere, and this variation is not generally known, so the profiles are compared to those predicted from atmospheric models with a number of parameters that are adjusted to achieve best fits. An example of the usefulness of an occultation for determining an atmosphere's mean molecular weight is given in Ch. 12.1.1.

10.2.2 Temperature Variation with Height

The temperature structure of a planetary atmosphere, as for the interior, is reached through a consideration of the heat flow.

We consider an atmosphere in which heat flow is dominated by *adiabatic convection*, i.e., parcels of air convect without exchanging heat with their surroundings.

The adiabatic relation between pressure and density in an ideal gas is

$$P = \text{const } \rho^\gamma \text{ or } PV^\gamma = \text{const} \tag{10.28}$$

where $\gamma = c_P/c_V$ is a quantity known as the *ratio of specific heats* (discussed below). Writing n = N/V, where N is the number of particles in volume V, we may solve the perfect gas law (10.18) for V to obtain

$$V = \frac{NkT}{P} \tag{10.29}$$

Then from (10.28) and (10.29),

$$P\left(\frac{NkT}{P}\right)^\gamma = \text{const} \tag{10.30}$$

or

$$(Nk)^\gamma T^\gamma P^{1-\gamma} = \text{const} \tag{10.31}$$

Finally,

$$T = \text{const}' \, P^{(\gamma-1)/\gamma} \tag{10.32}$$

We have shown that P depends on the altitude above the ground; we can now expect T to have such a dependence also.

The *heat capacity*, C, of a system is the heat input per unit temperature increase, i.e., the heat required to raise the temperature of the system by 1°:

$$C \equiv \frac{\mathrm{d}Q}{\mathrm{d}T} \tag{10.33}$$

Two types of heat capacity are particularly useful:

1. Heat capacity at constant pressure, P,

$$C_P \equiv \left(\frac{\mathrm{d}Q}{\mathrm{d}T}\right)_{P=\text{const}}$$

2. Heat capacity at constant volume, V,

$$C_V \equiv \left(\frac{\mathrm{d}Q}{\mathrm{d}T}\right)_{V=\mathrm{const}}$$

If heat is added to a gas, the change shows up as an increase in internal energy and as work done,

$$\mathrm{d}Q = \mathrm{d}U + P\,\mathrm{d}V \tag{10.34}$$

In (10.34), known as the differential form of the first law of thermodynamics, $\mathrm{d}Q$ is the heat entering the system, $\mathrm{d}U$ is the change in the internal energy of the gas, and $P\ \mathrm{d}V$ is the work done by the gas on its surroundings.

We now apply (10.34) to three different processes in a planetary atmosphere.

First, in an *adiabatic* process, such as occurs during adiabatic convection, no heat enters or leaves a parcel of gas as it convects, i.e., $\mathrm{d}Q = 0$. Equation (10.34) then shows that, in an adiabatic process, any work done by expansion of the gas is carried out at the expense of internal energy:

$$P\,\mathrm{d}V = -\mathrm{d}U \tag{10.35}$$

Second, if a process occurs at constant volume (referred to as an *isochoric* process), then no work is done and (10.34) gives

$$\mathrm{d}Q = \mathrm{d}U \tag{10.36}$$

that is, the heat goes into raising the internal energy because no expansion is permitted. The definition of C_V then gives

$$C_V \equiv \left(\frac{\mathrm{d}Q}{\mathrm{d}T}\right)_{V=\mathrm{const}} = \frac{\mathrm{d}U}{\mathrm{d}T} \tag{10.37}$$

for a process at constant volume.

Third, in an *isobaric* process, when the pressure is constant and the volume is allowed to change, (10.34) gives

$$C_P \equiv \left(\frac{\mathrm{d}Q}{\mathrm{d}T}\right)_{P=\mathrm{const}} = \frac{\mathrm{d}U}{\mathrm{d}T} + P\frac{\mathrm{d}V}{\mathrm{d}T} \tag{10.38}$$

Now noting from (10.37) that $\mathrm{d}U/\mathrm{d}T$ equals C_V, we arrive at

$$C_P = C_V + P\frac{\mathrm{d}V}{\mathrm{d}T} \tag{10.39}$$

We now write the equation of state, (10.18), $P = nkT$, as

$$PV = NRT \tag{10.40}$$

where now N is the number of mols[1] and R is the *molar gas constant*,[2] which has the value

$$\begin{aligned} 8.314472(15)\,\mathrm{J\ mol^{-1}\ K^{-1}} &\cong 1.987\ \mathrm{kcal\ kmol^{-1}\ K^{-1}} \\ &\cong 0.08208\ \mathrm{L\ atm\ mol^{-1}\ K^{-1}} \end{aligned}$$

We can differentiate (10.40) to get

$$P\frac{\mathrm{d}V}{\mathrm{d}T} + V\frac{\mathrm{d}P}{\mathrm{d}T} = NR \tag{10.41}$$

Then for a process at constant pressure,

$$P\frac{\mathrm{d}V}{\mathrm{d}T} = NR \tag{10.42}$$

and (10.39) becomes

$$C_P = C_V + NR \tag{10.43}$$

We now define the *specific heat capacity*, also called the *specific heat*, c, by either

$$c \equiv \frac{C}{N} = \frac{1}{N}\frac{\mathrm{d}Q}{\mathrm{d}T} \qquad \text{molar specific heat}$$

or

$$c \equiv \frac{C}{m} = \frac{1}{m}\frac{\mathrm{d}Q}{\mathrm{d}T} \qquad \text{specific heat per unit mass}$$

depending on context. Equation (10.43) then shows that the molar specific heats are related by

$$c_P = c_V + R \tag{10.44}$$

The specific heat of an ideal gas depends on the number of degrees of freedom, g, of its particles. In general, the molar specific heats are given by

$$c_V = (g/2)R, \qquad c_P = [(g/2) + 1]R$$

[1] Or *moles*, gram-molecular weights or the mass equivalent of Avogadro's number ($6.02214179(30) \times 10^{23}$) of molecules of this species.

[2] Or *universal gas constant*. See also Section 10.2.1.

so the ratio of specific heats is

$$\gamma \equiv \frac{c_P}{c_V} = \frac{\left(\frac{g}{2}+1\right)R}{\left(\frac{g}{2}\right)R} = \frac{g+2}{g} \tag{10.45}$$

In an ideal, monatomic gas the particles have only the three translational degrees of freedom in the x, y, and z directions, so $g = 3$ and

$$c_V = (3/2)R, \quad c_P = (5/2)R$$

For the light diatomic gases H_2, N_2, CO and O_2, the molecules have at least five degrees of freedom: three translational and two rotational (about the two axes perpendicular to the long axis of the molecule). If temperatures are high enough, vibration adds a sixth. Thus at the lower temperatures,

$$c_V = (5/2)R, \quad c_P = (7/2)R$$

Consequently,

$$\gamma = \frac{5}{3} \quad \text{for a monatomic gas}$$

$$\text{and} \qquad \gamma = \frac{7}{5} = 1.4 \quad \text{for a diatomic gas}$$

For polyatomic gases and chemically active gases (e.g., CO_2, NH_3, CH_4, Cl_2, and Br_2), C_p and C_V, and thus c_p and c_V, vary with temperature in a different way for each gas.

From (10.41) we have

$$P\,\mathrm{d}V = NR\,\mathrm{d}T - V\,\mathrm{d}P \tag{10.46}$$

and from (10.37),

$$\mathrm{d}U = C_V\ \mathrm{d}T \tag{10.47}$$

Then (10.34) becomes

$$\mathrm{d}Q = (C_V + NR)\mathrm{d}T - V\,\mathrm{d}P \tag{10.48}$$

In the adiabatic case, $\mathrm{d}Q = 0$, and, using the relation $C_P - C_V = NR$ from (10.43), we find

$$C_P\,\mathrm{d}T = V\,\mathrm{d}P \tag{10.49}$$

When a parcel of gas rises through a displacement $\mathrm{d}r$, the pressure change within the parcel is given by (10.16). Substituting from (10.16) and replacing r by the distance, h, above a reference level in the atmosphere then gives

$$C_P\,\mathrm{d}T = -V\,g\,\rho\,\mathrm{d}h \tag{10.50}$$

Dividing by $V\rho = m$ and using the specific heat per unit mass, $c_P = C_P/m$, we get

$$\mathrm{d}T/\mathrm{d}h = -g/c_P = -\Gamma \tag{10.51}$$

where Γ is known as the *adiabatic lapse rate*. If the air is dry, the symbol Γ_d is used. Equation (10.51) shows the explicit dependence of the temperature on altitude for an adiabatically convecting atmosphere that we mentioned earlier in this section.

In the Earth's atmosphere, $g = 9.81\,\mathrm{m/s^2}$, $c_P = 1005\,\mathrm{J/kg\,K}$, and

$$\Gamma_\mathrm{d} = 0.00976\,\mathrm{K/m} = 9.76\,\mathrm{K/km} \tag{10.51a}$$

Later we will compare the temperature structures of the atmospheres of the Earth, Venus, and Mars, and the *mixing ratio* or relative abundance by volume of specific constituents of the atmosphere.

For a wet atmosphere,

$$c_P = c_{P(\text{water vapor})}\,w + c_{P\,(\text{dry air})}\,(1 - w)$$

where w is the ratio of the masses of water vapor to dry air for a given volume of air.

Because $c_{P(\text{moist air})} > c_{P(\text{dry air})}$, the variation of T with height is smaller for moist air than for dry air. A lucid and topical expansion of this topic can be found in Seinfeld and Pandis (1998, esp., Chapter 14).

10.3 Circulation in the Atmosphere

10.3.1 Centrifugal and Coriolis Forces

In a rotating frame, such as a planet, there are "apparent" or "virtual" forces and motions that arise solely because of the motion of the frame. These are often referred to as *fictitious forces*.

(In this section, **boldface type** in equations is used to denote vectors.) Consider a displacement vector, $\mathbf{r}$, fixed in a frame of reference, Σ', which is rotating with an angular velocity, $\boldsymbol{\Omega}$, with respect to an inertial (non-rotating) frame, Σ, as illustrated in Figure 10.2. By virtue of the rotation of Σ', the tip of the vector $\mathbf{r}$ will appear to be moving with a linear velocity

$$\mathbf{v}_\Sigma = \boldsymbol{\Omega} \times \mathbf{r} \tag{10.52}$$

when viewed from frame Σ.

Fig. 10.2. A vector, **r**, fixed in a rotating frame of reference, Σ′, and viewed from an inertial frame, Σ

The quantity $\boldsymbol{\Omega} \times \mathbf{r}$ is a vector cross-product, and is itself a vector, the magnitude of which equals the product of the magnitudes of the two vectors times the sine of the angle between them:

$$|\boldsymbol{\Omega} \times \mathbf{r}| = \Omega\, r \sin\theta \tag{10.53}$$

and the direction of which is perpendicular to both vectors, as given by the "right-hand rule" (see Figure 10.3):

Point the fingers of your right hand along the first vector in the cross-product, then orient your hand so you can curl your fingers from the first vector to the second vector. Your thumb now points in the direction of the cross-product.

If in Figure 10.2 the vector **r** changes with time as viewed from frame Σ′, the effect as viewed from frame Σ will be

$$(\mathrm{D}\mathbf{r})_\Sigma = (\mathrm{D}\mathbf{r})_{\Sigma'} + \boldsymbol{\Omega} \times \mathbf{r} \tag{10.54}$$

where D is the operator $D = \mathrm{d}/\mathrm{d}t$.

Equation (10.54) gives the relationship between the velocity vectors,

$$\mathbf{v}_\Sigma = \mathbf{v}_{\Sigma'} + \boldsymbol{\Omega} \times \mathbf{r} \tag{10.55}$$

In the example above, $\mathbf{v}_{\Sigma'} = 0$.

In fact, the rate of change of any vector, **A**, as seen in Σ is the rate of change in Σ′ plus the cross-product with the angular velocity vector, $\boldsymbol{\Omega}$:

$$(\mathrm{D}\mathbf{A})_\Sigma = (\mathrm{D}\mathbf{A})_{\Sigma'} + \boldsymbol{\Omega} \times \mathbf{A} \tag{10.56}$$

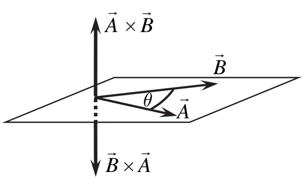

Fig. 10.3. The vector cross-product

Thus the variation of the velocity vector is, from either (10.55) or (10.56),

$$(\mathrm{D}\,\mathbf{v}_\Sigma)_\Sigma = (\mathrm{D}\,\mathbf{v}_\Sigma)_{\Sigma'} + \boldsymbol{\Omega} \times \mathbf{v}_\Sigma \tag{10.57}$$

The quantity on the LHS is the acceleration observed by an observer in the inertial frame, Σ.

Substituting (10.55) into (10.57) gives

$$(\mathrm{D}\mathbf{v}_\Sigma)_\Sigma = (\mathrm{D}\mathbf{v}_{\Sigma'})_{\Sigma'} + 2(\boldsymbol{\Omega} \times \mathbf{v}_{\Sigma'}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{10.58}$$

Then defining $\mathbf{a} = (\mathrm{D}\mathbf{v}_\Sigma)_\Sigma$, $\mathbf{a}' = (\mathrm{D}\mathbf{v}_{\Sigma'})_{\Sigma'}$, and $\mathbf{v}' = \mathbf{v}_{\Sigma'}$, we obtain

$$\mathbf{a} = \mathbf{a}' + 2\boldsymbol{\Omega} \times \mathbf{v}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{10.59}$$

where primes denote quantities measured in the rotating frame. If our vectors actually describe the position, velocity, and acceleration of a particle, then multiplying by the particle's mass, we arrive at the force equation:

$$\mathbf{F} = \mathbf{F}' + 2m\boldsymbol{\Omega} \times \mathbf{v}' + m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{10.60}$$

or, from the standpoint of an observer in the moving frame,

$$\mathbf{F}' = \mathbf{F} - 2m\boldsymbol{\Omega} \times \mathbf{v}' - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{10.61}$$

The second term on the RHS,

$$\mathbf{F}'_{\mathbf{C}or} = -2m\boldsymbol{\Omega} \times \mathbf{v}' \tag{10.62}$$

is the *Coriolis force*, and the third,

$$\mathbf{F}'_{\mathbf{C}} = -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \tag{10.63}$$

the *centrifugal force*. The Coriolis and centrifugal forces are so-called *fictitious forces* that are apparent to a non-inertial observer rotating with the planet. Some experimentation with the right-hand rule, and remembering the minus sign in the equation, will show that the centrifugal force is directed perpendicularly outward from the rotation axis of the planet.

The practical consequences of these terms will be examined next.

10.3.2 Physical Effects of the Centrifugal and Coriolis Forces

10.3.2.1 The Centrifugal Force Figure 10.4 shows a parcel of air in the atmosphere of a rotating planet. The view is that of an inertial observer in space, but we analyze the forces as experienced by a non-inertial observer on the planet's surface. The gravitational force on the parcel, $F_g = mg$ (where m is the mass of the parcel), acts directly toward the center of the planet, whereas applying the right-hand rule in Figure 10.4 shows that the centrifugal force, $\mathbf{F}'_\mathbf{C}$, acts radially outward from the planet's rotation axis. (Here we retain the primed notation for "fictitious" forces that exist only in the rotating frame.)

From Figure 10.4, the apparent weight of the parcel (i.e., the upward force needed to prevent the parcel from falling) is

$$(F_g)_{\text{eff}} = F_g - (F'_C)_\perp = F_g - F'_C \cos\lambda \tag{10.64}$$

where λ is the latitude of the parcel. Then using (10.52) and (10.63),

$$(F_g)_{\text{eff}} = mg - m\frac{v^2}{r}\cos\lambda = m\left(g - \frac{v^2}{r}\cos\lambda\right) \equiv mg_{\text{eff}} \tag{10.65}$$

where

$$g_{\text{eff}} \equiv g - \frac{v^2}{r}\cos\lambda \tag{10.66}$$

is the effective gravitational acceleration (or *effective gravity*) at the parcel's location. Rotation thus reduces the apparent weight and apparent acceleration due to gravity on a rotating planet.

The component $(F'_C)_{||}$ in Figure 10.4 causes the parcel of air to drift toward the equator, creating an equatorial bulge. The consequent *rotational*

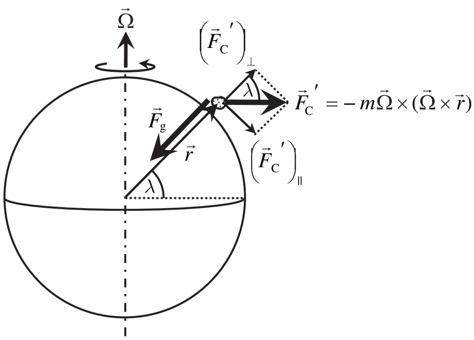

Fig. 10.4. The centrifugal force, $\vec{F}'_C$, on a parcel of air in the atmosphere of a rotating planet, and its components parallel and perpendicular to the planetary surface. λ is the latitude of the parcel

flattening can be seen in images of Jupiter and Saturn as in Figures 12.1 and 12.9, for example. The process is self-limiting, because the bulge in turn creates pressure gradient forces that oppose $(F'_{\mathrm{C}})_{||}$. $(F'_{\mathrm{C}})_{||}$ is larger at a given latitude for faster-rotating planets, so faster-rotating planets are more rotationally flattened.

10.3.2.2 The Coriolis Force From (10.62), the Coriolis force on any object depends on its velocity, $\mathbf{v}'$, relative to the rotating reference frame of the planet. For a planet with a solid surface, e.g., a terrestrial planet, we can take $\mathbf{v}'$ as relative to the horizon plane at a point on the surface.

Some experimentation applying the right-hand rule (Section 10.3.1) to the cross-product in (10.62) combined with appropriate diagrams (see Figure 10.5 for an example) shows that:

1. An object at rest in the rotating frame (e.g., at rest relative to the Earth's surface) experiences no Coriolis force.
2. An object exactly on the equator experiences no Coriolis force when moving due north or south, because $\mathbf{\Omega} \;||\; \mathbf{v}'$. If moving in any other compass direction, it experiences no Coriolis force parallel to the surface because $\mathbf{\Omega}$ and $\mathbf{v}'$ are both in the plane of the surface.
3. In both hemispheres, an object moving toward the (nearest) pole experiences a Coriolis force toward the east, and an object moving toward the equator experiences a Coriolis force toward the west.

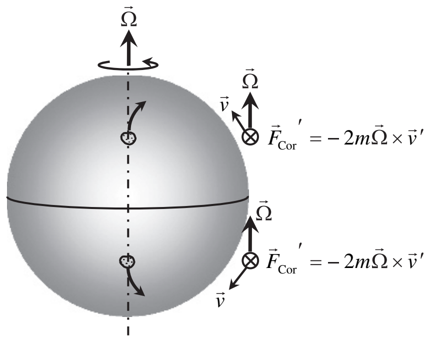

Fig. 10.5. The vectors on the right show the planetary angular velocity, $\vec{\Omega}$, the velocity, $\vec{v}$, and the Coriolis force, $\vec{F}'_{\mathrm{Cor}}$ (into the page) on a poleward-moving parcel of air on the limb of a rotating planet, as viewed from space. The *curved arrows* on the central longitude show the resulting motion of the parcel of air *relative to the planet's surface*, in the absence of other constraints. The northern parcel veers right (east) and the southern veers left (also east)

4. In both hemispheres, an object moving due east experiences a Coriolis force perpendicularly outward from the planet's rotation axis; the component parallel to the surface therefore causes the object to veer toward the equator. For an object moving due west, the Coriolis force is perpendicularly inward toward the rotation axis and the object veers toward the pole.

Case (3), above, can be understood intuitively when analyzed in the inertial frame (e.g., in the view from space shown in Figure 10.5). Consider objects at rest relative to the surface of the rotating planet. The linear speed eastward of such objects when measured in the inertial frame is greater for objects closer to the equator because they are further from the rotational axis. In the absence of other constraints, when the object moves poleward it finds itself above points on the ground that are closer to the rotational axis and moving eastward more slowly than itself. Consequently, it drifts toward the east relative to the ground below. An object moving toward the equator finds itself moving over points on the ground that are moving faster than itself, and it therefore drifts toward the west relative to the ground below.

Points (3) and (4) can be summarized by saying that the Coriolis force causes moving objects to veer toward the right in the northern hemisphere and toward the left in the southern, regardless of their direction of travel.

10.3.3 Pressure Gradient Force

Pressure varies horizontally as well as vertically in planetary atmospheres, and the resulting horizontal forces drive winds and atmospheric circulation. Occasionally these winds can be extreme. In Hurricane Wilma (2005), a Category 5 hurricane in the northwestern Caribbean, the surface atmospheric pressure at peak intensity was 1004 mb outside the hurricane and 882 mb at the center of the eye, the latter being the lowest value ever found for an Atlantic-basin hurricane since record-keeping began in 1851. The resulting peak sustained surface wind speed was 160 knots, or 300 km/h. Fortunately, these peak values occurred over open water, but Wilma was still very intense during landfall (Pasch et al. 2006).

In Figure 10.6, a small parcel of air of volume $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ is situated in a region of a planetary atmosphere containing a pressure gradient. We place the lower front left corner at (x,y,z), so the right-hand face is at $x+\mathrm{d}x$, the back face at $y+\mathrm{d}y$, and the top face at $z+\mathrm{d}z$. The pressure gradient vector, expressed as $\vec{\nabla}P$, where the del (or grad) operator is defined as

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$