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Giovanni Ferraro

The Rise and Development of the Theory of Series up to the Early 1820s

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Preface

The theory of series in the 17th and 18th centuries poses several interesting problems to historians. Indeed, mathematicians of the time derived numerous results that range from the binomial theorem to the Taylor formula, from the power series expansions of elementary functions to trigonometric series, from Stirling's series to series solution of differential equations, from the Euler–Maclaurin summation formula to the Lagrange inversion theorem, from Laplace's theory of generating functions to the calculus of operations, etc. Most of these results were, however, derived using methods that would be found unacceptable today, thus, if we look back to the theory of series prior to Cauchy without reconstructing internal motivations and the conceptual background, it appears as a corpus of manipulative techniques lacking in rigor whose results seem to be the puzzling fruit of the mind of a magician or diviner rather than the penetrating and complex work of great mathematicians.

For this reason, in this monograph, not only do I describe the entire complex of 17th- and 18th-century procedures and results concerning series, but also I reconstruct the implicit and explicit principles upon which they are based, draw attention to the underlying philosophy, highlight competing approaches, and investigate the mathematical context where the series theory originated. My aim is to improve the understanding of the framework of 17th- and 18th-century mathematics and avoid trivializing the complexity of historical development by bringing it into line with modern concepts and views and by tacitly assuming that certain results belong, in some unproblematic sense, to a unified theory that has come down to us today.

The initial and final points of my monograph require some clarification. The point of departure is the publication of a paper by Viète, *Variorum de rebus mathematicis responsorum. Liber VIII* (1593), where geometrical series are discussed and π is expressed in the form of an infinite product. Even though previous tracks of infinite series can be found, Viète's paper, when considered in the context of the new rising symbolic algebra, appears to be a step forward in a path –very slow to begin with, but that developed much more rapidly after 1650– that has made series an essential instrument in mathematics. The point of arrival is the early 1820s when Cauchy published *Cours d'analyse* and *Résumé des leçons données à l'École Royale Polytechnique sur le calcul infinitésimal*, which can be considered to mark the definitive abandonment of the 18th-century formal approach to the series theory.

My main arguments can be summarised as follows. The mathematicians who first used series were interested in their capacity to represent geometrical quantities and had an intuitive idea of convergence. They thought that a series represented a quantity and had a quantitative meaning if, and only if, it was convergent to this quantity. However, a distinction between finite and infinite sums was lacking, and this gave rise to formal procedures consisting of the infinite extension of finite procedures. In the works of mathematicians such as Newton and Leibniz, the quantitative and the formal aspect co-existed and formal manipulations were a tool for deriving convergent series.

As from the 1720s, several results began to upset the previously established balance between the quantitative and the formal. Mathematicians introduced recurrent series, which stressed the law of formation of coefficients, independently of the convergence of series. The attempt to improve the acceleration of series subsequently led to the emergence of asymptotic series, which showed the possibility of using divergent series to obtain appropriate approximations. Furthermore, the investigation of continued fractions and infinite products and certain applications of series (for instance, in numerical analysis and in number theory) increasingly stressed the formal aspects.

In this context, Euler offered a unitary interpretation of the complex of results concerning series, which even allowed the acceptance of those findings that did not form part of the early theory. A series was thought to be the result of a formal transformation of an analytical quantity expressed in a closed form. This transformation was considered sufficient to give a meaning to the series, even when the latter was not convergent. However, mathematicians were not free to invent transformations by a free creative act. They limited themselves to using the same transformations that were used in the original theory or at least were compatible with it. This seemed to guarantee that the new more formal conception was a generalization of earlier conception, which remained the essential basis from which all the parts of the series theory were subsequently generated.

The more formal Eulerian approach was widely predominant during the second part of the 18th-century for two main reasons. First, mathematicians who were critical of it were not able to eliminate the formal aspects of the early concept and found a really new theory: They always used the formal methodology that had led to asymptotic series and to the combinatorial use of series. Second, the formal concept of series contributed to the growth of mathematics. It led to many new discoveries and even to a new branch of analysis: the calculus of operations.

The formal approach became unsuited to most advanced mathematical research toward the end of the 18th century and the beginning of the 19th century. Applied mathematics encouraged investigations and introduction of new functions in analysis, but formal methodology was unable to treat quantities that were not elementary quantities and series that were not power series. The need to use trigonometric series to enable the analytical investigation of heat led Fourier to reject the formal concept of series and assert an entirely quantitative notion of series. Similarly, the need to introduce hypergeometric and gamma functions into analysis and to have an adequate analytical theory of them forced Gauss to highlight the quantitative meaning of the sum of series and to reject formal manipulations. The new approach based only upon convergence was the basis of Cauchy's treatises.

Given the purposes of this book, I cannot avoid dealing with some topics that are closely connected to series theory and are crucial to an understanding of its historical evolution: Not only do these include other infinite processes (continued fractions and infinite products) but also certain basic mathematical notions (quantity, numbers, functions) and the 18th-century concept of analysis.

This book is divided into four parts. The first part starts with a chapter devoted to the use of series prior to the rise of the calculus (Chapter 1), where I deal principally with Viète, Grégoire de Saint-Vincent, Mengoli, Wallis, and Gregory. I then move on to investigate the conception of the founders of the calculus (Leibniz in Chapter 2; Newton in Chapter 4). On the basis of this examination, and after discussing the contributions of Johann and Jacob Bernoulli (Chapters 3 and 5) and the notion of a quantity and of a number (Chapter 7), I offer an interpretative scheme of the early theory of series in Chapter 8. The first part also includes the appearance of Taylor series in Newton and Taylor (Chapter 6) and the rise of the problem of the sum of a divergent series in one of Grandi's writings and the ensuing debate in Leibniz, Varignon, Daniel Bernoulli, and Goldbach (Chapter 9).

In the second part, I illustrate the development of series theory from the 1720s to the 1750s. De Moivre's recurrent series and Bernoulli's method for solving equations are the subject of Chapter 10. Chapter 11 deals with the attempt to improve the acceleration of series and Stirling's series, the first example of asymptotic series. Chapter 12 examines the geometric conception of Colin Maclaurin. Most of the second part is devoted to Euler, "the master of all us," to use an expression that Libri [1846, 51] ascribes to Laplace. From 1730 to 1750, Euler obtained many important results, which I examine in Chapters 13 to 17. In particular, I shall concentrate on the problem of interpolation and some of Euler's first findings (Chapter 13), on Euler's derivation of the Euler–Maclaurin summation formula (Chapter 14), on issues connected to the interpretation of asymptotic series (Chapter 15), on the theory of infinite products and continued fractions (Chapter 16), and on the application of series to number theory (Chapter 17). Chapter 18 is a digression on some basic principles of analysis during the period from the 1740s to the 1810s, which is essential for understanding series theory in the second half of the 18th century. In particular, the relationship between analysis and geometry, the notion of a function, and the principle of generality of algebra are examined. In Chapter 19, I discuss some criticisms of certain procedures and how Euler rejected them by giving a merely formal interpretation of the notion of the sum.

The third part is devoted to the period when formal conception held undisputed sway. I begin by illustrating some of the greatest successes of the formal approach during the second part of the 18th century: the Lagrange inversion theorem, which is discussed in Chapter 20, the calculus of operations, examined in Chapter 21, and Laplace's theory of generating functions (the subject of Chapter 22). The problem of the representation of transcendental quantities and their analytical investigation is treated in Chapters 23, 24, and 25.

Integration by series and series solutions to differential equations were already known by the beginning of the calculus, but they underwent a remarkable development after Euler: Some examples from Euler, Laplace and Legendre are given in Chapter 26. I then deal with trigonometric series for which mathematicians applied the same procedure as that used for power series. This prevented them from being fully understood (Chapter 27).

The attempts to prove the binomial theorem, Lagrange's view of the Taylor theorem, and other significant developments that took place between the end of the 18th century and the beginning of 19th century are the subject matter of Chapter 28.

Chapters 29 and 30 focus on the problematic attempt of Legendre to enlarge the realm of accepted functions and to the emergence of techniques of inequalities in d'Alembert's and Lagrange's work.

The fourth and final part is devoted to the crisis in formal methods. It deals with Fourier's investigations of Fourier series (Chapter 31), Gauss's work on hypergeometric and gamma functions (Chapter 32), and Cauchy's contributions on series during the early 1820s (Chapter 33). The conceptions of these mathematicians differ from all other mathematicians discussed in this book since they belong to a new historical phase. However, the discussion of their approach allows me to illustrate some hypotheses about the abandonment of 18th-century series theory.

In order to write this monograph I have drawn on various papers of mine, in particular:

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Afragola, Italy
May 2007

Contents

Part I

From the beginnings of the 17th century to about 1720: Convergence and formal manipulation

In the first part of the present book, I examine the emergence of series theory and its development up to around 1720.

Series were introduced in mathematics mainly to solve geometric problems. Their use, which was initially rather sporadic, began to take on importance around 1650 and was crucial to the rise of the calculus. In Newton and Leibniz's times many results were obtained: They came to form an organic corpus of knowledge that constituted the early theory of series.[1]

From the very beginning mathematicians had an intuitive idea of convergence and they thought that convergent series[2] could represent geometric quantities.[3] However, convergence was not considered as a preliminary condition for handling series. Mathematicians did not distinguish between operations on infinite series and operations on finite series, and they formally manipulated infinite series by applying the same rules that were employed for finite sums. Therefore, series theory had a twofold aspect, the first based upon convergence, the second upon formal manipulation.

The formal aspect was not caused by the imprecision or vagueness of certain formulations and certain concepts; rather it was rooted in the basic notion of 17th- and 18th-century mathematics, the notion of quantity. Sometimes, especially in the first years, the formal aspect was almost hidden by the simplicity of the employed series and from the immediacy of geometric reference. At other times, the formal aspect appeared. This occurred mainly when the attention was focused on the problem of finding the development of given quantities. In early series theory, mathematicians thought

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[1] The term "theory" is, of course, not intended in the sense of a formal theory but instead as an organic set of principles, rules, methods, and logical deductions concerning a specific subject.

[2] Two terminological specifications are necessary. First, during the 17th and 18th centuries, mathematicians used the term "series" to denote both series and sequences. This could give rise to confusion. I shall therefore distinguish a series $\sum_{n=1}^{\infty} a_n$ from a sequence $\{a_n\}_{n=1}^{\infty}$. Second, prior to Cauchy, the term "convergence" usually denoted that the sequence a_n was decreasing and tended toward 0. I prefer to employ the term "convergence" in Cauchy's sense, namely a series is convergent if it has a finite sum, except for quotation and some particular cases that are explicitly indicated.

[3] I shall discuss the notion of quantity later on. At this moment I use the term "geometric quantity" to refer to lines or other geometrical objects connected to a curve, such as ordinate, abscissa, arc length, subtangent, normal, area between curves and axes, etc.

that the coexistence of the two aspects of series theory was guaranteed

(a) by the assumption that the expansion into series of a given quantity was convergent at least for an interval of values of the variable,

(b) by the possibility of postponing the investigation of convergence to the phase of application of a certain series to specific geometrical, mechanical or numerical problems.

Only in a few isolated cases did mathematicians recognize tensions or difficulties between convergence and formal manipulation.

Part I is divided into nine chapters. In the first chapter, I examine the earliest researches on series, infinite products, and continued fractions mainly by examining the works of Viète, Grégoire de Saint-Vincent, Mengoli, Wallis, Mercator, and James Gregory. In the following five chapters, I explore the early theory of series with particular attention to the relationship between convergence and formal manipulation and to the geometrical context in which the theory was originated. I concentrate upon the writings of Leibniz (Chapters 2 and 3), Johann Bernoulli (Chapter 3), Newton (Chapter 4), Jacob Bernoulli (Chapter 5), and Taylor (Chapter 6). This investigation provides the basis for an analysis of the notion of quantity and a comprehensive interpretation of early theory series (Chapters 7 and 8). Finally, Chapter 9 is devoted to the question of Grandi's series and the early debate on divergent series.

1 Series before the rise of the calculus

Even though series were occasionally found earlier, it is only from the 17th century that they began to be a topic of importance in mathematics. Their use mainly arose in the context of the problem of quadratures and rectifications of curves. During the 17th century, mathematicians attempted to find new methods for squaring curved lines, which avoided the difficulty of the so-called method of exhaustion.[4] This method, which had been one of the greatest successes of Greek geometry, made it possible to determine the area A of a given figure by means of a complex procedure divisible in two phases.

1. One or two sequences of polygons were constructed so that the areas of these polygons approximated to the given figure and suggested that the sought area A was equal to a certain area P.

2. One proved $A = P$ by means of a double *reductio ad absurdum* (namely, one showed that neither $A > P$ nor $A < P$ was true).

A classic example is the quadrature of the parabolic segment obtained by Archimedes[5]. As the first step in the proof, one considers the triangle ABC with area F, which is greater than one-half of the parabolic segment ACB with area P (see Fig. 1). Then, one considers the diameters B_1V_1 and B_1V_2 such that $AV_1 = AV_2 = \frac{AH}{2}$ and constructs the triangles AB_1C and BB_2C. These triangles are greater than one-half of the corresponding parabolic segments AB_1C and BB_2C. Moreover, both the triangles AB_1C and BB_2C are equal to $\frac{1}{8}F$ and, consequently, their sum is $\frac{1}{4}F$. The process can be continued so as to construct a sequence H_n such that

- H_n is a polygon formed by the sum of the triangles,
- at the nth step, the area of H_n is $\frac{1}{4^{n-1}}F$,
- the polygons H_n exhaust (namely, fill up entirely) the segment,
- the sum S_n of the areas of all the triangles up to the nth step is given by the finite geometric progression

$$S_n = F + \frac{F}{4} + \frac{F}{16} + \frac{F}{64} + \ldots + \frac{F}{4^{n-1}}. \tag{1}$$

After having shown that

$$S_n + \frac{1}{3}\frac{F}{4^{n-1}} = \frac{4}{3}F, \tag{2}$$

Archimedes proved that the area of the parabolic segment is $\frac{4}{3}F$ by reasoning as follows.

[4] The name is due to Grégoire de Saint-Vincent [1647, 740].

[5] See Archimedes [QA, 233–252].

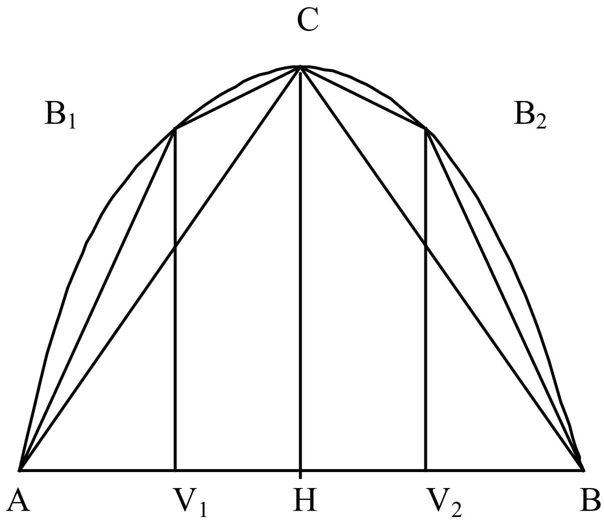

Fig. 1

- If $P > \frac{4}{3}F$, then $P - \frac{4}{3}F > 0$ and one can continue the exhaustion process until one obtains a sum S_n such that $P - S_n < P - \frac{3}{4}F$. Hence, $S_n > \frac{3}{4}F$, which contradicts formula (2).
- If $P < \frac{4}{3}F$, then $\frac{4}{3}F - P > 0$. Since the triangles constructed become increasingly smaller, at a certain step n, the area $\frac{F}{4^{n-1}}$ of the polygon H_n becomes less than $\frac{4}{3}F - P$. From (2), one obtains

$$\frac{4}{3}F - S_n = \frac{1}{3}\frac{F}{4^{n-1}} < \frac{F}{4^{n-1}} < \frac{4}{3}F - P.$$

  Hence, $S_n > P$, which is impossible.

During the 17th century, the method of exhaustion was always considered as a model of a rigorous mathematical reasoning, although it was thought to be too difficult, especially because of the double *reductio ad absurdum.* It was also thought to be too particular, since it was connected to specific properties of certain geometrical figures and the reasoning used in a specific case could not be used in others. In effect, the method of exhaustion was not a method of finding or discovery, but rather it was a method of justification of known results. Consequently, mathematicians searched for new methods that were easier and had a more general application. This led in a very natural way to the consideration of series and even infinite products and continued fractions. For instance, in the above-mentioned quadrature of the parabola, it is possible to avoid the double *reductio ad absurdum* by using the series $\sum_{n=0}^{\infty} \frac{1}{4^n}$. It is no wonder that series are found in many 17th-century works concerning the quadratures of curves and almost all the precursors of the calculus run up against series. In particular, the attempt to merge Cavalieri's geometrical method of indivisibles with the emerging use of algebra led to the investigation of several series.

* * *

Geometric series played a crucial role in earlier research on series. In the 1590s, geometric series[6] were mentioned in a work by Francois Viète, *Variorum de rebus mathematicis responsorum*, in which he tackled the problem of the quadrature of circle. In this paper Viète determined the sum of a geometric series $\sum_{i=1}^{\infty} a_i$. His starting point was Proposition 12 in Book 5 of Euclid's *Elements*: If any number of magnitudes are proportional, then one of the antecedents is to one of the consequents as the sum of the antecedents is to the sum of the consequents (see Euclid [E]). In modern symbols, if $s_n = \sum_{i=1}^{n} a_i$, then

$$a_1 : a_2 = (s_n - a_n) : (s_n - a_1).$$

Hence,

$$\frac{a_1 - a_2}{a_1} = \frac{a_1 - a_n}{s_n - a_n}.$$

By assuming that the terms of the geometric series were decreasing, Viète obtained

$$\frac{a_1 - a_2}{a_1} = \frac{a_1}{s}, \tag{3}$$

where $s = \sum_{i=1}^{\infty} a_i$. He justified (3) by stating that the magnitudes a_n were changed into nothing (*in nihil*) when the series was continued *ad infinitum*.[7] As an example, Viète considered the series

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3}$$

and explicitly noted that it fitted the Archimedean quadrature of the parabola.

[6]It worthwhile pointing out that geometric series had already appeared earlier. N. Oresme dealt with the nature and summation of geometric series in a manuscript, the *Quaestiones super geometriam Euclidis,* which was only published in 1961 (On Oresme's treatment of series, see Mazet [2003]). Oresme's results seem to have had little influence on the rise of series theory in 17th century.

[7]See Viète [1593, 397–398].

A few decades later, Grégoire de Saint-Vincent made geometric series a crucial instrument of his method of quadratures.[8] He wrote a remarkable treatise, the *Opus geometricum,* devoted to the quadrature of conics, which was published in 1647 though its essential aspects dated back to before 1625. Grégoire observed that the classic problems inherited from the Ancients had not been solved after many centuries; he therefore thought that new techniques and new methods needed to be discovered (*unde novas artes et methodo novas iudicam excogitandas*) to fill the lacunae of ancient geometry [1647, 51–52]. Such new methods were grounded precisely on infinite geometric series, which he discussed at length in the second book of the *Opus geometricum.*

Saint-Vincent defined a geometric series to be "a finite quantity divided by an uninterrupted sequence according to a given ratio" and distinguished series from progressions [1647, 54]. He used the term "progression" to mean both a finite sequence of the terms of a geometric series (which he understood as infinite) and the sum of this finite sequence. Saint-Vincent used the term "limit" to denote the sum of a geometric series and stated that the "limit" of a progression was the end of the series that the progression did not reach –even if it continued indefinitely; however, the progression could approach this limit more than any given quantity [1647, 54].

Saint-Vincent, as well Viète, had an intuitive but clear idea of what the sum of series was (whatever words they used to denote the sum). By using more recent terminology, we could state that, in their opinion, a series $\sum_{k=0}^{\infty} a_k$ had a sum S if the sequence of nth sums $S_n = \sum_{k=0}^{n} a_k$ was convergent to S; namely, if it approached S indefinitely when n increased so that the difference between S_n and S (in absolute value) became less than any given quantity. As we shall see below, this idea of the sum lay at the heart of the series theory during both the 17th century and, in a more complicated form, the 18th century.

Basing his argument on the concept of the sum, Saint-Vincent examined the famous paradox of Achilles and the turtle. He showed that Achilles gains on the turtle according to a decreasing geometric series, which has a finite sum. Therefore, Achilles does reach the turtle and one can also determine the point where the turtle is reached by summing the series [1647, 97–98].

Saint-Vincent obtained several results by applying geometric series.[9] One of the most interesting is the following proposition concerning the quadrature of the hyperbola:

> *Let AY and AX be the asymptote of the hyperbola HKM* [see Fig. 2]. *If the segment AX is divided into segments AB, AC,*

[8] On Grégrorie de Saint-Vincent, see Dhombres [1995].

[9] Saint-Vincent, in particular, determined the sums of $\sum_{n=0}^{\infty} q^n$ and $\sum_{n=0}^{\infty} q^{kn}$, for an integer k [1647, 115–149] and constructed two geometric series with different n-th terms but with the same sum [1647, 97–98].

CD, DE that are in continued proportion, then the areas $BCKH$, $CDLK$,$DEMN$ are equal[10] (Saint Vincent [1647, 586]).

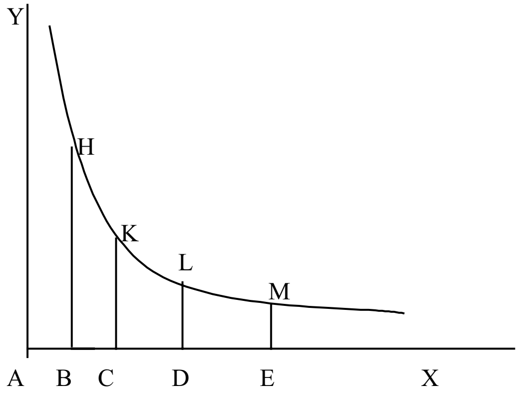

Fig. 2

Saint-Vincent did not employ the term "logarithm". This term had already been introduced at that time although it was used with a meaning that differed considerably from the modern one. Indeed, the word "logarithms" denoted the terms of an arithmetical progression that were matched with the terms of a geometric progression in sequence[11]. By using the term logarithm in this sense, Saint-Vincent's theorem can be formulated by stating that the areas $BCKH, CDLK, DEMN$ are the logarithms of the abscissas of the hyperbola HKM. This formulation was made explicit by de Sarasa in 1649.[12]

* * *

Pietro Mengoli was another mathematician who made a remarkable contribution to the rising theory of series. He was taught mathematics by Cavalieri and was influenced by Saint-Vincent and Torricelli[13]. In 1650,

[10] In other words, if the abscissa are in a geometric progression, then the areas are in an arithmetic progression.

[11] See Burn [2001, 4].

[12] As regards different historical interpretations of the actual contributions of Saint-Vincent and de Serasa to the study of natural logarithms, see Burn [2001].

[13] I point out that Torricelli gave a geometric proof of the sum of a geometric series in his *De dimensione Parabolae* [1644]. For Torricelli's proof, I refer to Panza [1992, 307–308]. A similar geometrical proof, given by Leibniz, is discussed in Chapter 2.

Mengoli published several results concerning series in *Novae quadraturae arithmeticae, seu de additione fractionum*, a treatise that stemmed from the examination of the Archimedean quadrature of parabola, as he stated in the introduction.[14] Mengoli based his argument upon two axioms:

1. If infinite magnitudes have an infinite extension, then one can take a certain number of these magnitudes such that they exceed any finite extension (In modern terms, if the sum of a series is infinite, then the partial sums become greater than any positive number) (Mengoli [1650, 18]).

2. If infinite magnitudes have a finite extension and if they are thought of as being arranged and gathered together to form another extension, then these two extensions are equal (that is to say, if a series with positive terms[15] converges to a finite number, then any rearrangement of the series converges to the same number) (Mengoli [1650, 19]).

From these axioms Mengoli derived various properties of the series of magnitudes. In particular,

a. if the sum of any number of a sequence of infinite quantities is bounded, then the series has a finite extension (in modern words, if the partial sums of a series are bounded, the series is convergent) (Mengoli [1650, 18]);

b. if a series has the finite extension S and A is a quantity less than S, then there is a finite number of the given magnitudes such that their sum exceeds A [namely, there exists a partial sum S_n of the series such that $S_n < A$ $(< S_{n+1})$] (Mengoli [1650, 19]).

Mengoli applied these axioms and properties to the determination of the sum of various numerical series by conceiving the numbers present in such series as specific values of geometric quantities. He represented the terms, partial sums and remainder of series by means of segments. In order to sum the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)},$$

Mengoli employed a relation, which he had proved in his *Novae quadraturae arithmeticae* [1650, 9] and which, using modern symbols, can be written as

$$\frac{a_2 - a_1}{a_1 a_2} + \frac{a_3 - a_2}{a_2 a_3} + \frac{a_4 - a_3}{a_3 a_4} + \ldots + \frac{a_n - a_{n-1}}{a_{n-1} a_n} = \frac{a_n - a_1}{a_1 a_n},$$

[14]On Mengoli's contribution to series theory, see Agostini [1941].

[15]Since Mengoli referred to geometrical quantities, he tacitly assumed that the terms of series were positive.

This formula makes it possible to establish that the partial sums of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ are

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}, \tag{4}$$

Since $\frac{n}{n+1} < 1$, the series has a finite extension S. This extension is precisely equal to 1. Indeed, if $S > 1$, then there should exist a partial sum S_n such that $S_n > 1$, which is impossible. Now, let $S < 1$ be. Since the numbers $\frac{n}{n+1}$ approach 1 indefinitely when n increases, the partial sums

$$S_n = \frac{n}{n+1}$$

would become greater than S when n is large enough. This is also impossible. Consequently, $S = 1$.

Similarly, Mengoli obtained the sum of many other series, such as

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4},$$
$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{11}{18},$$
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4},$$
$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)(2n+5)} = \frac{1}{12}.$$

Moreover, in the introduction to *Novae quadraturae arithmeticae,* Mengoli showed that the harmonic series did not converge.[16] In modern terms, his proof can be formulated as follows. Since

$$\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} > \frac{3}{n},$$

one has

$$\begin{aligned} S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \ldots \\ &= 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10} +\right) \cdots \\ &> 1 + \frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \frac{3}{12} + \ldots \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \\ &= 1 + S. \end{aligned}$$

[16] This result was not new (see Oresme [A]).

Consequently, S cannot be a finite quantity.

In the introduction to *Novae quadraturae arithmeticae,* Mengoli also took the series

$$\sum_{n=0}^{\infty} \frac{1}{n^2}$$

into consideration. He failed to calculate the sum of such a series, but this problem was subsequently tackled by Jakob Bernoulli, and became known as the Basel problem. It was considered a very interesting problem of pure mathematics and its solution was one of Euler's most important successes.

Mengoli also wrote *Geometriae speciosae elementa*[17] [1659] and *Circolo* [1672], where he rediscovered an infinite product expansion for $\pi/2$, which had already been found by Wallis in a way that I shall now go on to examine.

* * *

The use of series was worked on extensively by John Wallis. In *Arithmetica infinitorum* [1656] he tried to provide an arithmetical version of the method of indivisibles; this led him to deal with a large number of series by means of a peculiar methodology that had an enormous influence on later mathematicians.[18] As Maierù [1994, 118–119] noted, Wallis's treatment of series developed in a number of particular cases and makes use of the specific geometric properties of particular figures. To illustrate Wallis's method,[19] consider the problem of finding the area under the curves $y = x^k$ $(k = 1, 2, \ldots)$ and over the segment $[0, a]$ (see Fig. 3, where the curve $y = x^k$ is represented by means of PSR, $PQ = AB = a$, and $RQ = BC = a^k$). Following Cavalieri, Wallis regarded the figure PQR as consisting of an infinite number of parallel lines, every one of them having length equal to x^k. Therefore, if one divides the segment $PQ = AB = a$ into n pieces of length $h = \frac{a}{n}$, where n is infinite, the sum of these infinite lines is of the type

$$0^k + h^k + (2h)^k + (3h)^k + \ldots + (nh)^k, \ k = 1, 2, \ldots. \tag{5}$$

Similarly, the area of the rectangle is

$$a^k + a^k + a^k + \ldots + a^k = (nh)^k + (nh)^k + (nh)^k + \ldots + (nh)^k, \ k = 1, 2, \ldots.$$

The ratio between the parabola PQR and the rectangle $ABCD$ is

$$\frac{Area \text{ parabola } PSR}{Area \text{ rectangle } ABCD} = \frac{0^k + 1^k + 2^k + 3^k + \ldots + n^k}{n^k + n^k + n^k + n^k + \ldots + n^k}, \ k = 1, 2, \ldots. \tag{6}$$

This procedure led Wallis to consider the problem of determining the values

[17] For this work, I refer to Massa [1997].

[18] On Wallis's method of quadrature, see Scott [1938], Panza [1995, 135–176], and Maierù [1994], [1995], and [2000].

[19] See Wallis [1656, 1–52].

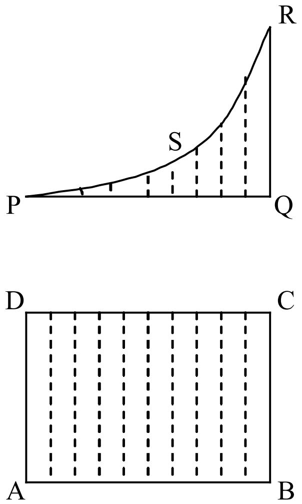

Fig. 3

of

$$\frac{0^k + 1^k + 2^k + 3^k + \ldots + n^k}{n^k + n^k + n^k + n^k + \ldots + n^k}, \tag{7}$$

for $n = \infty$ and $k = 1, 2, 3, \ldots$.[20] He stated

> The simplest method of investigation ... is to consider a certain number of individual cases, and to observe the emergent ratios, and to compare these with one another, so that a universal proposition may be established by induction. (Wallis [1656, 1])

He first considered the case $k = 1$ and observed that

$$\begin{aligned} \tfrac{0+1}{1+1} &= \tfrac{1}{2} \\ \tfrac{0+1+2}{2+2+2} &= \tfrac{1}{2} \\ \tfrac{0+1+2+3}{3+3+3+3} &= \tfrac{1}{2} \\ \tfrac{0+1+2+3+4}{4+4+4+4+4} &= \tfrac{1}{2}. \end{aligned}$$

[20]In modern terms, he sought

$$\lim_{n\to\infty} \frac{1^k + 2^k + 3^k + \ldots + n^k}{(n+1)n^k}.$$

The divergent series $\sum_j j^k$ appears in Wallis's work as intermediate steps during the analytical manipulation of geometrical entities.

By induction, Wallis asserted that

$$\frac{0+1+2+3+\ldots+n}{n+n+n+n+\ldots+n}=\frac{1}{2}.$$

Wallis then considered the case $k=2$. Since

$$\begin{array}{lll} \frac{0+1}{1+1} & =\frac{1}{2} & =\frac{1}{3}+\frac{1}{6} \\ \frac{0+1+4}{4+4+4} & =\frac{5}{12} & =\frac{1}{3}+\frac{1}{6\cdot 2} \\ \frac{0+1+4+9}{9+9+9+9} & =\frac{14}{36} & =\frac{1}{3}+\frac{1}{6\cdot 3} \\ \frac{0\ +\ 1\ +\ 4\ +\ 9+16}{16+16+16+16+16} & =\frac{30}{80} & =\frac{1}{3}+\frac{1}{6\cdot 4}, \end{array}$$

he stated

$$\frac{0^2+1^2+2^2+3^2+\ldots+n^2}{n^2+n^2+n^2+n^2+\ldots+n^2}=\frac{1}{3}+\frac{1}{6n}.$$

The ratio approached $\frac{1}{3}$ as the number of terms increased, and

$$\frac{0^2+1^2+2^2+3^2+\ldots+n^2}{n^2+n^2+n^2+n^2+\ldots+n^2}=\frac{1}{3}$$

for $n=\infty$. In case $k=3$, Wallis proceeded in a similar way and found

$$\frac{0^3+1^3+2^3+3^3+\ldots+n^3}{n^3+n^3+n^3+n^3+\ldots+n^3}=\frac{1}{4}+\frac{1}{4n}$$

and

$$\frac{0^3+1^3+2^3+3^3+\ldots+n^3}{n^3+n^3+n^3+n^3+\ldots+n^3}=\frac{1}{4}$$

for $n=\infty$. By generalizing these results, Wallis asserted[21]

$$\frac{0^k+1^k+2^k+3^k+\ldots+n^k}{n^k+n^k+n^k+n^k+\ldots+n^k}=\frac{1}{k+1}. \tag{8}$$

Wallis did not stop here. He continued to generalize in order to give a meaning to (8) even when $k\neq 1,2,3,\ldots$. He first stated that if the value 0 was assigned to k, then one obtained

$$\frac{0^0+1^0+2^0+3^0+\ldots+n^0}{n^0+n^0+n^0+n^0+\ldots+n^0}=\frac{1}{1}.$$

He then sought to justify the assignment of fractional values to k in the following way. If we denote[22] the series $0^k+1^k+2^k+3^k+\ldots+n^k$ by A_k,

[21]Of course, from this formula and (6), one can deduce that the area under the parabola $y=x^k$ from 0 to a is $\frac{a^{k+1}}{k+1}\left(=\sum_{j=0}^{n} j^k\right)$.

[22]The symbolism is mine.

and the reciprocal of their sums (the corresponding ratio, in Wallis's terms) by $b_k(= k+1)$, then formula (8) can be written in the form

$$\frac{A_k}{n^k + n^k + \ldots + n^k} = \frac{1}{b_k}.$$

Wallis observed that

$$\sqrt{0^4} + \sqrt{1^4} + \sqrt{2^4} + \sqrt{3^4} + \ldots = 0^2 + 1^2 + 2^2 + 3^2 + \ldots .$$

The terms $n^2 = \sqrt{n^4}$ of A_2 are the square roots of the terms of A_4 and, therefore, A_2 can be viewed as the series "interpolating" A_0 and A_4. The corresponding ratios of A_0, A_2, and A_4 are the numbers $b_0 = 1$, $b_2 = 3$, and $b_4 = 5$, which are in arithmetic progression (see table below).

$A_0 = 0^0 + 1^0 + 2^0 + 3^0 + \ldots + n^0$	->	$b_0 = 1$
$A_2 = 0^2 + 1^2 + 2^2 + 3^2 + \ldots + n^2$	->	$b_2 = 3$
$A_4 = 0^4 + 1^4 + 2^4 + 3^4 + \ldots + n^4$	->	$b_4 = 5$

At this point Wallis considered $A_{\sqrt{}} = \sqrt{0} + \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots$ and stated that it was the series interpolating A_0 and A_1 since it behaved with respect to A_0 and A_1 as A_2 behaved with respect to A_0 and A_4. By analogy, the value of

$$\frac{\sqrt{0} + \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n}}{\sqrt{n} + \sqrt{n} + \sqrt{n} + \sqrt{n} + \ldots + \sqrt{n}}$$

ought to be a number $\frac{1}{b}$ such that $b_0 = 1$, b and $b_1 = 2$ (namely, the corresponding ratios of $A_0 = 1$, $A_{\sqrt{}}$, and $A_1 = 2$) were in arithmetic progression. Hence, $b = \frac{1}{2} + 1$. By observing that for $k = \frac{1}{2}$, formula (8) becomes

$$\frac{0^{1/2} + 1^{1/2} + 2^{1/2} + 3^{1/2} + \ldots + n^{1/2}}{n^{1/2} + n^{1/2} + n^{1/2} + n^{1/2} + \ldots + n^{1/2}} = \frac{1}{\frac{1}{2} + 1},$$

Wallis concluded that $n^{\frac{1}{2}} = \sqrt{n}$.

Similarly, Wallis observed that

$$A_1 = \sqrt[3]{0^3} + \sqrt[3]{1^3} + \sqrt[3]{2^3} + \sqrt[3]{3^3} + \ldots = 0 + 1 + 2 + 3 + \ldots,$$
$$A_2 = \left(\sqrt[3]{0^3}\right)^2 + \left(\sqrt[3]{1^3}\right)^2 + \left(\sqrt[3]{2^3}\right)^2 + \left(\sqrt[3]{3^3}\right)^2 + \ldots = 0^2 + 1^2 + 2^2 + 3^2 + \ldots,$$

and that the terms $n = \sqrt[3]{n^3}$ of A_1 were the cube roots of the terms of A_3 and the terms $n^2 = \left(\sqrt[3]{n^3}\right)^2$ of A_2 were the squares of cube roots. For this reason A_1 and A_2 could be viewed as the series interpolating A_0 and A_3. The corresponding ratios of A_0, A_1, A_2, A_3 ($b_0 = 1$, $b_1 = 2$, $b_2 = 3$, $b_3 = 4$) were in arithmetical progression. Wallis then considered

$$\frac{\sqrt[3]{0} + \sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n}}{\sqrt[3]{n} + \sqrt[3]{n} + \sqrt[3]{n} + \sqrt[3]{n} + \ldots \sqrt[3]{n}} = \frac{1}{r} \qquad (9)$$

and

$$\frac{\left(\sqrt[3]{0}\right)^2+\left(\sqrt[3]{1}\right)^2+\left(\sqrt[3]{2}\right)^2+\left(\sqrt[3]{3}\right)^2+\ldots+\left(\sqrt[3]{n}\right)^2}{\left(\sqrt[3]{n}\right)^2+\left(\sqrt[3]{n}\right)^2+\left(\sqrt[3]{n}\right)^2+\left(\sqrt[3]{n}\right)^2+\ldots\left(\sqrt[3]{n}\right)^2}=\frac{1}{q} \tag{10}$$

and assumed that

$$\sqrt[3]{0}+\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{3} \quad \text{and} \quad \left(\sqrt[3]{0}\right)^2+\left(\sqrt[3]{1}\right)^2+\left(\sqrt[3]{2}\right)^2+\left(\sqrt[3]{3}\right)^2$$

behaved with respect to

$$A_0 \quad \text{and} \quad A_1$$

in the same way as A_1 and A_2 behaved with respect to A_0 and A_3. By analogy he concluded that b_0, r, q, b_1 had to be in arithmetical progression as b_0, b_1, b_2, b_3. Therefore,

$$r=\frac{4}{3} \quad \text{and} \quad q=\frac{5}{3}.$$

This made it possible to write (9) and (10) as

$$\frac{0^{1/3}+1^{1/3}+2^{1/3}+3^{1/3}+\ldots+n^{1/3}}{n^{1/3}+n^{1/3}+n^{1/3}+n^{1/3}+\ldots n^{1/3}}=\frac{1}{1+\frac{1}{3}}$$

and

$$\frac{0^{2/3}+1^{2/3}+2^{2/3}+3^{2/3}+\ldots+n^{2/3}}{n^{2/3}+n^{2/3}+n^{2/3}+n^{2/3}+\ldots n^{2/3}}=\frac{1}{1+\frac{2}{3}}.$$

Consequently, $\sqrt[3]{n}$ was equal to $n^{1/3}$ and $\left(\sqrt[3]{n}\right)^2$ was equal to $n^{2/3}$. In this way Wallis was able to find the meaning of the power x^α, where α was a rational number ($n^{\frac{l}{k}}=\sqrt[k]{n^l}$). He even considered the case in which α was an irrational and a negative number.

Wallis's analogical procedure (later known as Wallis's interpolation) was of great importance in the 18th century. It can be considered as an answer to the following problem:

> *Given a sequence P_k, defined for integral values of k, find the meaning of P_α where α is a nonintegral number.*

In the case Wallis considered, P_k were the sequences x^k and

$$\frac{0^k + 1^k + 2^k + 3^k + \ldots + n^k}{n^k + n^k + n^k + n^k + \ldots + n^k},$$

and the problem was reduced to the interpolation of the number sequence

$$\frac{1}{k+1}, \quad k = 0, 1, 2, \ldots.$$

From a modern point of view, this problem is meaningless. A modern mathematician attributes meaning to new operations, formulas, or symbols using appropriate definitions. Operations, formulas, and symbols do not have a "natural" meaning. Thus, if x^n is defined only for an integer value of n, then any meaning can be assigned to a new symbol such as $x^{1/2}$.

Wallis viewed the matter differently. New combinations of symbols, such as $x^{1/2}$ and x^0, were not introduced arbitrarily. Mathematical objects were not given by definition, but they existed in nature (or were an idealization of natural objects). It seemed obvious to them that $x^{1/2}$ and x^0 had a "natural" meaning and that mathematicians had to discover it. When a new symbol or a new object had to be introduced, mathematicians asked "What is the value (or the meaning) of the symbol?" and not "How shall we define it?"

For Wallis, interpolating x^n required investigating the objects x, x^2, ... and reconstructing the "nature" of these objects just as one reconstructed the nature of a physical phenomenon by interpolating physical data. When he met with the undefined symbolic notation $x^{1/2}$, he did not take $x^{1/2} = \sqrt{x}$ by a useful but arbitrary definition; rather he "discovered" that the true meaning[23] of $x^{1/2}$ was $\sqrt{x}$.

In *Arithmetica infinitorum*,[24] Wallis reduced the problem of the quadrature of the circle to determining the corresponding ratio of the series whose general term is $\zeta_p = \sqrt{R^2 - p^2a^2}$. To do this, he considered the series whose general terms are

$$(R^2 - p^2a^2)^0,\ (R^2 - p^2a^2)^1,\ (R^2 - p^2a^2)^2,\ (R^2 - p^2a^2)^3,\ \ldots \tag{11}$$

which have for their corresponding ratios

$$1,\ \frac{2}{3},\ \frac{8}{15},\ \frac{48}{105},\ \ldots.$$

If the series $\zeta_p = \sqrt{R^2 - p^2a^2}$ is interpolated between the first and second terms of (11), the corresponding ratio of

$$\zeta_p = \sqrt{R^2 - p^2a^2}$$

[23] See also Ferraro [1998, 291–293].
[24] See Wallis [1656, 89–182].

is given by the interpolated value of 1, $\frac{2}{3}$, $\frac{8}{15}$, $\frac{48}{105}$, ... between 1 and $\frac{2}{3}$. Wallis introduced the symbol □ to denote the sought-after number and constructed several numerical tables such as

1		1		1		1	
	□						
1		2		3		4	
1		3		6		10	
1		4		10		20	

where

- the numbers in the first row and column are 1,
- those in the second row and column are the natural numbers $\{n\}_{n=1,2,\ldots,\infty}$,
- those in the third row and column are $\left\{\frac{n(n+1)}{1\cdot 2}\right\}_{n=1,2,\ldots,\infty}$ (triangular numbers),
- those in the fourth row and column are $\left\{\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\right\}_{n=1,2,\ldots,\infty}$ (triangular pyramidal number),
- ...

After a long sequence of calculations, he succeeded in expressing □ as the infinite product

$$\square = \left(\frac{4}{\pi}\right) = \frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\ldots}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\ldots}. \tag{12}$$

* * *

Formula (12) was not the first infinite product to be found in the history of mathematics. In his *Variorum de rebus mathematicis responsorum* [1593], Viète had already squared the circle by means of an infinite product. He assumed the circle to be a polygon with infinite sides and considered regular inscribed polygons of 4, 8, 16, ... sides. By using geometric properties of these polygons he represented π in the form[25]

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots. \tag{13}$$

[25] See Viète [1593, 400].

In the *Arithmetica infinitorum,*[26] Wallis also published an expansion of $\frac{4}{\pi}$ into continued fractions. He had submitted (12) to Lord Brouncker, who expressed $\frac{4}{\pi}$ in the form

$$\frac{4}{\pi} = 1 + \frac{1}{2+}\frac{9}{2+}\frac{25}{2+}\frac{49}{2+}. \tag{14}$$

This formula was published by Wallis in the *Arithmetica Infinitorum*, Proposition 191. On this occasion Wallis introduced the term "continued fraction". However, he did not expound the procedure used by Lord Brouncker to derive (14).

It should be emphasised that when Brouncker obtained (14), continued fractions were already known, at least since 1613 when Cataldi had shown how a root $\sqrt{p}$ could be expanded into a continued fraction. Earlier, in his *Algebra* [1572, 37–38], Bombelli had published a procedure for calculating the approximate value of a root which can be interpreted *a posteriori* as a procedure for developing numbers into continued fractions.

To compute the value of $\sqrt{13}$, Bombelli first observed that 3 is the greatest integer less than $\sqrt{13}$. Then he considered the difference $\sqrt{13} - 3 = x$ (for the sake of simplicity, I use the letter x to denote this difference, though Bombelli did not use symbols of this kind). The first approximation of x (say x_1) is given by $\frac{2}{3}$ because $13 - 3^2 = 4$ and

$$x_1 = \frac{4}{2 \cdot 3} = \frac{4}{6} = \frac{2}{3}.$$

To find a second approximation x_2, he set

$$x_2 = \frac{4}{6 + \frac{2}{3}} = \frac{3}{5}.$$

The third approximation is

$$x_3 = \frac{4}{6 + \frac{3}{5}} = \frac{20}{33}.$$

Similarly, he found

$$x_4 = \frac{66}{109}, \quad x_5 = \frac{109}{180}, \quad x_6 = \frac{720}{1189}.$$

The approximation can be improved as desired.

In modern terms, Bombelli's procedure can be described as follows. If one sets

$$\sqrt{p} = n + x,$$

[26]See Wallis [1665, 181–193].