Synchronized Phasor Measurements and Their Applications

Power Electronics and Power Systems

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ISBN: 978-0-387-76535-8 e-ISBN: 978-0-387-76537-2
DOI: 10.1007/978-0-387-76537-2

Library of Congress Control Number: 2008921736

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface

Synchronized phasor measurements have become the measurement technique of choice for electric power systems. They provide positive sequence voltage and current measurements synchronized to within a microsecond. This has been made possible by the availability of Global Positioning System (GPS) and the sampled data processing techniques developed for computer relaying applications. In addition to positive-sequences voltages and currents, these systems also measure local frequency and rate of change of frequency, and may be customized to measure harmonics, negative and zero sequence quantities, as well as individual phase voltages and currents. At present there are about 24 commercial manufacturers of phasor measurement units (PMUs), and industry standards developed in the Power System Relaying Committee of IEEE has made possible the interoperability of units from different manufacturers.

Recent spate of spectacular blackouts on power systems throughout the world has provided an added impetus to widescale deployment of PMUs. Positive sequence measurements provide the most direct access to the state of the power system at any given instant. Many applications of these measurements have been discussed in the technical literature, and no doubt many more applications will be developed in coming years.

The authors have been associated with this technology since its birth, and they and their colleagues and students have produced a rich body of literature on the subject of phasor measurement technology and its applications. Other researchers around the world have also made significant contributions to the field. Our aim in writing this book is to present to the interested reader a coherent account of the development of the technology, and of the emerging applications of these measurements. It is our hope that this book will help power system engineers understand the basics of "synchronized phasor measurement systems". This technology is bound to usher in an era of improved monitoring, protection, and control of power systems.

Blacksburg, Virginia — Arun G. Phadke
January, 2008 — James S. Thorp

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Part I: Phasor Measurement Techniques

Chapter 1 Introduction

1.1 Historical overview

Phase angles of voltage phasors of power network buses have always been of special interest to power system engineers. It is well-known that active (real) power flow in a power line is very nearly proportional to the sine of the angle difference between voltages at the two terminals of the line. As many of the planning and operational considerations in a power network are directly concerned with the flow of real power, measuring angle differences across transmission has been of concern for many years. The earliest modern application involving direct measurement of phase angle differences was reported in three papers in early 1980s [1,2,3]. These systems used LORAN-C, GOES satellite transmissions, and the HBG radio transmissions (in Europe) in order to obtain synchronization of reference time at different locations in a power system. The next available positive-going zero-crossing of a phase voltage was used to estimate the local phase angle with respect to the time reference. Using the difference of measured angles on a common reference at two locations, the phase angle difference between voltages at two buses was established. Measurement accuracies achieved in these systems were of the order of 40 μs. Single-phase voltage angles were measured and, of course no attempt was made to measure the prevailing voltage phasor magnitude. Neither was any account taken of the harmonics contained in the voltage waveform. These methods of measuring phase angle differences are not suitable for generalization for wide-area phasor measurement systems, and remain one-of-a-kind systems which are no longer in use.

The modern era of phasor measurement technology has its genesis in research conducted on computer relaying of transmission lines. Early work on transmission line relaying with microprocessor-based relays showed that the available computer power in those days (1970s) was barely sufficient to manage the calculations needed to perform all the transmission line relaying functions.

A significant portion of the computations was dedicated to solving six fault loop equations at each sample time in order to determine if any one of the ten

A.G. Phadke, J.S. Thorp, *Synchronized Phasor Measurements and Their Applications*,
DOI: 10.1007/978-0-387-76537-2_1,

types of faults possible on a three-phase transmission line are present. The search for methods which would eliminate the need to solve the six equations finally yielded a new relaying technique which was based on symmetrical component analysis of line voltages and currents. Using symmetrical components, and certain quantities derived from them, it was possible to perform all fault calculations with a single equation. In a paper published in 1977 [4] this new symmetrical component-based algorithm for protecting a transmission line was described. As a part of this theory, efficient algorithms for computing symmetrical components of three-phase voltages and currents were described, and the calculation of positive-sequence voltages and currents using the algorithms of that paper gave an impetus for the development of modern phasor measurement systems. It was soon recognized that the positive-sequence measurement (a part of the symmetrical component calculation) is of great value in its own right. Positive-sequence voltages of a network constitute the state vector of a power system, and it is of fundamental importance in all of power system analysis. The first paper to identify the importance of positive-sequence voltage and current phasor measurements, and some of the uses of these measurements, was published in 1983 [5], and this last paper can be viewed as the starting point of modern synchronized phasor measurement technology. The Global Positioning System (GPS) [6] was beginning to be fully deployed around that time. It became clear that this system offered the most effective way of synchronizing power system measurements over great distances. The first prototypes of the modern "phasor measurement units" (PMUs) using GPS were built at Virginia Tech in early 1980s, and two of these prototypes are shown in Figure 1.1. The prototype PMU units built at Virginia Tech were deployed at a few substations of the Bonneville Power Administration, the American Electric Power Service Corporation, and the New York Power Authority. The first commercial manufacture of PMUs with Virginia Tech collaboration was started by Macrodyne in 1991 [7]. At present, a number of manufacturers offer PMUs as a commercial product, and deployment of PMUs on power systems is being carried out in earnest in many countries around the world. IEEE published a standard in 1991 [8] governing the format of data files created and transmitted by the PMU. A revised version of the standard was issued in 2005.

Concurrently with the development of PMUs as measurement tools, research was ongoing on applications of the measurements provided by the PMUs. These applications will be discussed in greater detail in later chapters of this book. It can be said now that finally the technology of synchronized phasor measurements has come of age, and most modern power systems around the world are in the process of installing wide-area measurement systems consisting of the phasor measurement units.

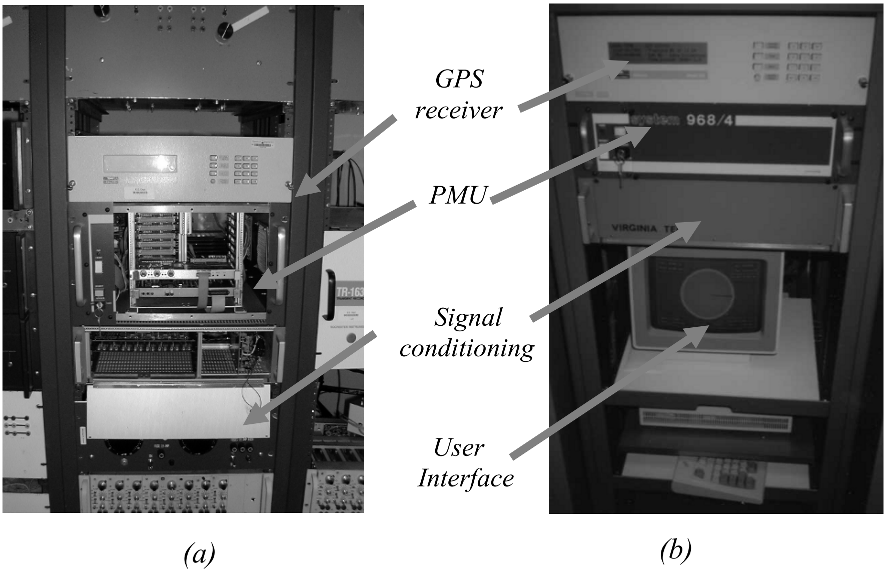

Fig. 1.1 The first phasor measurement units (PMUs) built at the Power Systems Research Laboratory at Virginia Tech. The GPS receiver clock was external to the PMU, and with the small number of GPS satellites deployed at that time, the clock had to be equipped with a precision internal oscillator which maintained accurate time in the absence of visible satellites.

1.2 Phasor representation of sinusoids

Consider a pure sinusoidal quantity given by

$$x(t) = X_{\mathrm{m}} \cos(\omega t + \phi) \tag{1.1}$$

ω being the frequency of the signal in radians per second, and ϕ being the phase angle in radians. X_{m} is the peak amplitude of the signal. The root mean square (RMS) value of the input signal is $(X_{\mathrm{m}}/\sqrt{2})$. Recall that RMS quantities are particularly useful in calculating active and reactive power in an AC circuit.

Equation (1.1) can also be written as

$$x(t) = Re\{X_{\mathrm{m}}\, e^{j(\omega t + \phi)}\} = Re[\{e^{j(\omega t)}\}\, X_{\mathrm{m}}\, e^{j\phi}].$$

It is customary to suppress the term $e^{j(\omega t)}$ in the expression above, with the understanding that the frequency is ω. The sinusoid of Eq. (1.1) is represented by a complex number X known as its phasor representation:

$$x(t) \leftrightarrow \mathrm{X} = (X_{\mathrm{m}}/\sqrt{2})\, e^{j\phi} = (X_{\mathrm{m}}/\sqrt{2})\, [\cos\phi + j\sin\phi]. \tag{1.2}$$

A sinusoid and its phasor representation are illustrated in Figure 1.2.

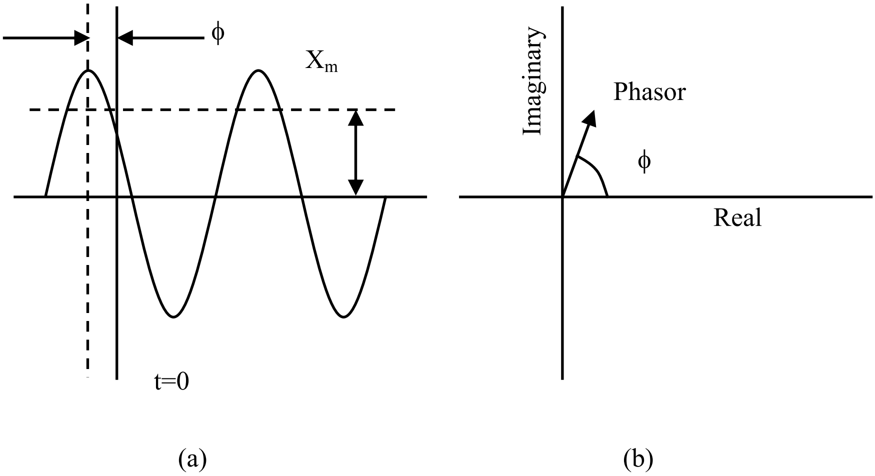

Fig. 1.2 A sinusoid (**a**) and its representation as a phasor (**b**). The phase angle of the phasor is arbitrary, as it depends upon the choice of the axis $t = 0$. Note that the length of the phasor is equal to the RMS value of the sinusoid.

It was stated earlier that the phasor representation is only possible for a pure sinusoid. In practice a waveform is often corrupted with other signals of different frequencies. It then becomes necessary to extract a single frequency component of the signal (usually the principal frequency of interest in an analysis) and then represent it by a phasor. Extracting a single frequency component is often done with a "Fourier transform" calculation. In sampled data systems, this becomes the "discrete Fourier transform" (DFT) or the "fast fourier transform" (FFT). These transforms are reviewed in the next section. The phasor definition also implies that the signal is unchanging for all time. However, in all practical cases, it is only possible to consider a portion of time span over which the phasor representation is considered. This time span, also known as the "data window", is very important in phasor estimation of practical waveforms. It will be considered in greater detail in later sections.

1.3 Fourier series and Fourier transform

1.3.1 Fourier series

Let $x(t)$ be a periodic function of t, with a period equal to T. Then $x(t+kT) = x(t)$ for all integer values of k. A periodic function can be expressed as a Fourier series:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\frac{2\pi kt}{T}) + \sum_{k=1}^{\infty} b_k \sin(\frac{2\pi kt}{T}), \qquad (1.3)$$

where the constants a_k and b_k are given by

$$a_k = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos(\frac{2\pi kt}{T})\mathrm{d}t, \quad k = 0,1,2,\cdots,$$
$$b_k = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin(\frac{2\pi kt}{T})\mathrm{d}t, \quad k = 1,2,\cdots. \qquad (1.4)$$

The Fourier series can also be written in the exponential form

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\frac{j2\pi kt}{T}} \qquad (1.5)$$

with

$$\alpha_k = \frac{1}{T}\int_{-T/2}^{+T/2} x(t) e^{-\frac{j2\pi kt}{T}} \mathrm{d}t, \quad k = 0, \pm 1, \pm 2, \cdots. \qquad (1.6)$$

Note that the summation in Eq. (1.5) goes from $-\infty$ to $+\infty$, while the summations in Eq. (1.3) go from 1 to $+\infty$. The change in summation limits is accomplished by noting that the cosine and sine functions are even and odd functions of k, and thus expanding the summation limits to ($-\infty$ to $+\infty$) and removing the factor 2 in front of the integrals for a_k and b_k leads to the desired exponential form of the Fourier series.

Example 1.1

Consider a periodic square wave signal with a period T as shown in Figure 1.3. This is an even function of time. The Fourier coefficients (in exponential form) are given by

$$\alpha_k = \frac{1}{T}\int_{-T/4}^{+T/4} e^{-\frac{j2\pi kt}{T}} \mathrm{d}t, \quad k = 0, \pm 1, \pm 2, \cdots$$
$$= \frac{1}{\pi k}\sin(\frac{k\pi}{2}).$$

Hence $\alpha_0 = 1/2$,

$\alpha_1 = 1/\pi$, $\alpha_{-1} = 1/\pi$,

$\alpha_3 = -1/3\pi$, $\alpha_{-3} = -1/3\pi$,

$\alpha_5 = 1/5\pi$, $\alpha_{-5} = 1/5\pi$, etc., and all even coefficients are zero.

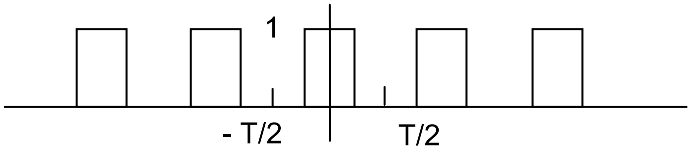

Fig. 1.3 A square wave function with a period T, with duty cycle equal to half, with the $t = 0$ axis so chosen that the function is an even function.

Thus, the Fourier series of the square wave signal is

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos(\frac{2\pi t}{T}) - \frac{1}{3}\cos(\frac{6\pi t}{T}) + \frac{1}{5}\cos(\frac{10\pi t}{T}) - \cdots \right].$$

The sum of first seven terms of the series is shown in Figure 1.4 below:

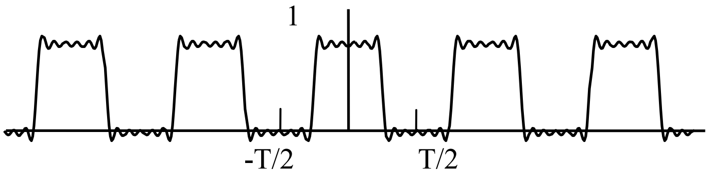

Fig. 1.4 A square wave approximated by 7 terms of the Fourier series. With more terms, the waveform approaches the square shape. The oscillations are known as the Gibbs phenomenon and are inescapable when step functions are approximated by the Fourier series.

1.3.2 Fourier transform

There are several excellent text books devoted to the subject of Fourier transforms [9,10]. The reader should consult those books for a more complete account of the Fourier transform theory. Here we present only those topics which are of direct interest for phasor estimation in power system applications.

The Fourier transform of a continuous time function $x(t)$ satisfying certain integrability conditions [9] is given by

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} \mathrm{d}t \tag{1.7}$$

and the inverse Fourier transform recovers the time function from its Fourier transform:

$$x(t)=\int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}\mathrm{d}f. \tag{1.8}$$

An important function frequently used in calculations using sampled data is the impulse function $\delta(t)$ defined by

$$x(t_0)=\int_{-\infty}^{+\infty} \delta(t-t_0)x(t)\mathrm{d}t. \tag{1.9}$$

The impulse function (also known as a distribution or a Dirac delta function) is a sampling function in the sense that when the integration in Eq. (1.9) is performed, the result is the sampled value of the function $x(t)$ at $t = t_0$. The integrals of the type shown in Eq. (1.9) are known as convolutions. Thus the sampling process at uniform intervals ΔT apart can be considered to be a convolution of the input signal and a string of impulse functions $\delta(t - k\Delta T)$, where k ranges from $-\infty$ to $+\infty$.

The convolutions of two time functions and their Fourier transforms have a convenient relationship. Consider the convolution $z(t)$ of two time functions $x(t)$ and $y(t)$:

$$z(t)=\int_{-\infty}^{+\infty} x(\tau)y(\tau-t)\mathrm{d}\tau \equiv x(t) * y(t). \tag{1.10}$$

The important result regarding convolutions is the following property:

Property 1 *The Fourier transform of a convolution is equal to the product of the Fourier transform of the functions being convolved, or:*

If $s(t) = x(t)*y(t)$, then $S(f) = X(f).Y(f)$

and similarly, the inverse Fourier transform of a convolution of two Fourier transforms is a product of the corresponding inverse Fourier transforms:

If $Z(f) = X(f)*Y(f)$, then $z(t) = x(t).y(t)$.

Next we illustrate the second of the above two statements. Consider the functions $x(t) = \cos(\omega t)$ and $y(t) = \sin(\omega t)$, with $\omega = 2\pi f_0$. The Fourier transforms of $x(t)$ and $y(t)$ are

$$X(f)=\int_{-\infty}^{+\infty} \cos(2\pi f_0 t)e^{-j2\pi ft}\mathrm{d}t = \int_{-\infty}^{+\infty} \frac{e^{-j2\pi(f-f_0)t}+e^{-j2\pi(f+f_0)t}}{2}\mathrm{d}t$$

$$=\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$$

and similarly

$$Y(f) = \frac{j}{2}[\delta(f + f_0) - \delta(f - f_0)]$$

The Fourier transforms of a pure cosine wave of unit amplitude is a pair of real impulse functions in frequency domain located at $\pm f_0$ and that of a pure sine wave of unit amplitude is a pair of imaginary impulse functions of opposite signs at $\pm f_0$.

The convolution of the two Fourier transforms determined above in the frequency domain is

$$S(f) = \int_{-\infty}^{+\infty} \frac{1}{2}[\delta(\phi - f_0) + \delta(\phi + f_0)]\frac{j}{2}[\delta(f + f_0 - \phi) - \delta(f - f_0 - \phi)]d\phi$$

$$= \frac{j}{4}\int_{-\infty}^{+\infty}[\delta(\phi - f_0)\delta(f + f_0 - \phi) + \delta(\phi + f_0)\delta(f + f_0 - \phi)$$

$$- \delta(\phi - f_0)\delta(f - f_0 - \phi) - \delta(\phi + f_0)\delta(f - f_0 - \phi)]d\phi .$$

Using the sampling property of the integrals involving impulse functions

$$S(f) = \frac{j}{4}[\delta(f + 2f_0) - \delta(f - 2f_0)]$$

the inverse Fourier transform of $S(f)$ is clearly

$$s(t) = \frac{1}{2}\sin(4\pi ft) = \sin(2\pi ft)\cos(2\pi ft) = x(t).y(t)$$

This property of convolutions will be used in discussing the sampling process and the DFT. Some other properties of the Fourier transform which are particularly useful in our development are stated next with accompanying examples.

Property 2 *The Fourier transform of an even function is an even function of frequency. If the even function is real, the Fourier transform is also real and even.*

Consider an even function of time, $x(t)$, so that $x(-t) = x(t)$. Let $x(t)$ be complex, $x(t) = r(t) + js(t)$. The Fourier transform $X(f)$ of this function is given by

$$X(f)=\int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}\,\mathrm{d}t=\int_{-\infty}^{+\infty} r(t)e^{-j2\pi ft}\,\mathrm{d}t+j\int_{-\infty}^{+\infty} s(t)e^{-j2\pi ft}\,\mathrm{d}t=$$

$$=\int_{-\infty}^{+\infty} r(t)\cos(2\pi ft)\mathrm{d}t+j\int_{-\infty}^{+\infty} r(t)\sin(2\pi ft)\mathrm{d}t+j\int_{-\infty}^{+\infty} s(t)\cos(2\pi ft)\mathrm{d}t-\int_{-\infty}^{+\infty} s(t)\sin(2\pi ft)\mathrm{d}t$$

The second and fourth integrals are zero, since the integrands are odd functions of time. Thus

$$X(f)=\int_{-\infty}^{+\infty} r(t)\cos(2\pi ft)\mathrm{d}t+j\int_{-\infty}^{+\infty} s(t)\cos(2\pi ft)\mathrm{d}t$$

Since $\cos(2\pi ft) = \cos(-2\pi ft)$, it follows that $X(f) = X(-f)$.

Also, if $x(t)$ is real $= r(t)$, the Fourier transform of $x(t)$ is $R(f)$, which is real and even.

Property 3 *The Fourier transform of an odd function is an odd function of frequency. If the odd function is real, the Fourier transform is imaginary and odd.*

Consider an odd function of time, $x(t)$, so that $x(-t) = -x(t)$. Let $x(t)$ be complex, $x(t) = r(t) + js(t)$. The Fourier transform $X(f)$ of this function is given by

$$X(f)=\int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}\,\mathrm{d}t=\int_{-\infty}^{+\infty} r(t)e^{-j2\pi ft}\,\mathrm{d}t+j\int_{-\infty}^{+\infty} s(t)e^{-j2\pi ft}\,\mathrm{d}t=$$

$$=\int_{-\infty}^{+\infty} r(t)\cos(2\pi ft)\mathrm{d}t+j\int_{-\infty}^{+\infty} r(t)\sin(2\pi ft)\mathrm{d}t+j\int_{-\infty}^{+\infty} s(t)\cos(2\pi ft)\mathrm{d}t-\int_{-\infty}^{+\infty} s(t)\sin(2\pi ft)\mathrm{d}t\,.$$

The first and third integrals are zero, since the integrands are odd functions of time. Thus

$$X(f)=j\int_{-\infty}^{+\infty} r(t)\sin(2\pi ft)\mathrm{d}t-\int_{-\infty}^{+\infty} s(t)\sin(2\pi ft)\mathrm{d}t\,.$$

Since $\sin(2\pi ft) = -\sin(-2\pi ft)$, it follows that $X(f) = -X(-f)$.

Also, if $x(t)$ is real $= r(t)$, the Fourier transform of $x(t)$ is $jR(f)$, which is imaginary and odd.

Property 4 *The Fourier transform of a real function has an even real part and an odd imaginary part.*

Consider a real function of time $x(t) = r(t) + j_0$. The Fourier transform is given by

$$X(f) = \int_{-\infty}^{+\infty} r(t)\cos(2\pi ft)\mathrm{d}t + j\int_{-\infty}^{+\infty} r(t)\sin(2\pi ft)\mathrm{d}t = R_1(f) + jR_2(f)$$

Since the cosine and sine functions are respectively even and odd functions of frequency, it is clear that $R_1(f)$ is an even function, and $R_2(f)$ is an odd function of frequency.

Property 5 *The Fourier transform of a periodic function is a series of impulse functions of frequency.*

If $x(t)$ is a periodic function of t, with a period equal to T, it can be expressed as an exponential Fourier series given by Eqs. (1.5) and (1.6):

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\frac{j2\pi kt}{T}}$$

with

$$\alpha_k = \frac{1}{T}\int_{-T/2}^{+T/2} x(t) e^{-\frac{j2\pi kt}{T}} \mathrm{d}t, \quad k = 0, \pm1, \pm2, \ldots$$

The Fourier transform of $x(t)$ expressed in the exponential form is given by

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft}\mathrm{d}t = \int_{-\infty}^{+\infty} [\sum_{k=-\infty}^{\infty} \alpha_k e^{\frac{j2\pi kt}{T}}] e^{-j2\pi ft}\mathrm{d}t = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} \alpha_k \mathrm{e}^{\frac{j2\pi kt}{T}} e^{-j2\pi ft}\mathrm{d}t$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} \alpha_k e^{j2\pi t\left\{\frac{k}{T} - f\right\}} \mathrm{d}t$$

where the order of the summation and integration has been reversed (assuming that this is permissible). Setting $f_0 = 1/T$, the fundamental frequency of the periodic signal, the integral of the exponential term in the last form is the impulse function $\delta(kf_0 - \mathrm{f})$, and thus the Fourier transform of periodic $x(t)$ is

$$X(f) = \sum_{k=-\infty}^{\infty} \alpha_k \, \delta(f - \frac{k}{T}), \text{ with}$$

$$\alpha_k = \frac{1}{T} \int_{-T/2}^{+T/2} x(t) e^{-\frac{j2\pi kt}{T}} \mathrm{d}t, \quad k = 0, \pm 1, \pm 2, \ldots$$

These are a series of impulses located at multiples of the fundamental frequency f_0 of the periodic signal with impulse magnitudes being equal to amplitude of each frequency component in the input signal.

Property 6 *The Fourier transform of a series of impulses is a series of impulse functions in the frequency domain.*

Consider the function

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

This is a periodic function with period T. Hence its Fourier transform (by Property 5 above) is

$$X(f) = \sum_{k=-\infty}^{\infty} \alpha_k \, \delta(f - \frac{k}{T}), \text{ with}$$

$$\alpha_k = \frac{1}{T} \int_{-T/2}^{+T/2} \delta(t) e^{-\frac{j2\pi kt}{T}} \mathrm{d}t, \quad k = 0, \pm 1, \pm 2, \cdots$$

Since the delta function in the integrand produces a sample of the exponent at $t = 0$, a_k is equal to $1/T$ for all k, and the Fourier transform of $x(t)$ becomes

$$X(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T}) \quad ,$$

which is a pulse train in the frequency domain at intervals kf_0 and a magnitude of $1/T$.

Example 1.2
Consider a rectangular input signal as shown in Figure 1.5. This is an even function of time.

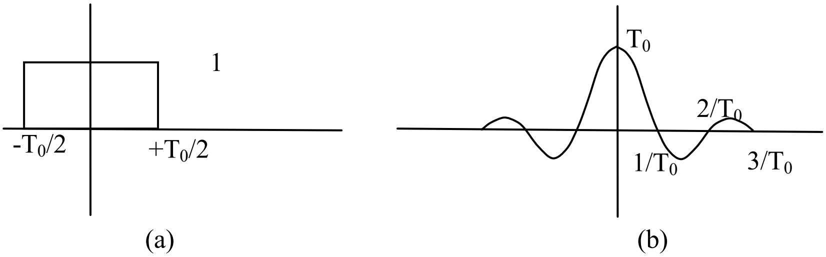

Fig. 1.5 (**a**) A rectangular function of time with the $t = 0$ axis so chosen that the function is an even function. The duration of the signal is $2T_0$. (**b**) Fourier transform of the function.

The Fourier transform of this time function is given by

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}\mathrm{d}t = \int_{t_1}^{t_1+T_0} e^{-j2\pi ft}\mathrm{d}t = e^{j2\pi f(t_1+\frac{T_0}{2})}T_0\frac{\sin(2\pi f\frac{T_0}{2})}{(2\pi f\frac{T_0}{2})}.$$

The first term in the Fourier transform is a phase shift factor and has been omitted from the plot in Figure 1.5b for convenience. If the rectangular wave is centered at the origin, $t_1 = -T_0/2$, and the phase shift factor vanishes. This is also in keeping with Property 2 of the Fourier transform given above, which states that the Fourier transform of a real even function must be real and even function of frequency.

1.4 Sampled data and aliasing

Sampled data from input signals are the starting point of digital signal processing. The computation of phasors of voltages and currents begins with samples of the waveform taken at uniform intervals $k\Delta T$, ($k = 0, \pm1, \pm2, \pm3, \pm4, \cdots\cdots$ }. Consider an input signal $x(t)$ which is being sampled, yielding sampled data $x(k\Delta T)$. We may view the sampled data as a time function $x'(t)$ consisting of uniformly spaced impulses, each with a magnitude $x(k\Delta T)$:

$$x'(t) = \sum_{k=-\infty}^{\infty} x(k\Delta t)\,\delta(t - k\Delta T) \tag{1.11}$$

It is interesting to determine the Fourier transform of the sampled data function given by Eq. (1.11). Note that the sampled data function is a product of the function $x(t)$ and the sampling function $\delta(t - k\Delta T)$, the product being interpreted in the sense of Eq. (1.9). Hence the Fourier transform $X'(f)$ of $x'(t)$ is the convolution of the Fourier transforms of $x(t)$ and of the unit impulse train. By Property 6 of Section 1.3, the Fourier transform of the impulse train is

$$\Delta(f) = \frac{1}{\Delta T} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{\Delta T}) \quad . \tag{1.12}$$

Hence the Fourier transform of the sampled data function is the convolution of $\Delta(f)$ and $X(f)$

$$\begin{aligned} X'(f) &= \frac{1}{\Delta T} \int_{-\infty}^{+\infty} X(\phi) \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{\Delta T} - \phi)\, \mathrm{d}\phi \\ &= \frac{1}{\Delta T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{+\infty} X(\phi)\delta(f - \frac{k}{\Delta T} - \phi)\, \mathrm{d}\phi \\ &= \frac{1}{\Delta T} \sum_{k=-\infty}^{\infty} X(f - \frac{k}{\Delta T}) \end{aligned} \tag{1.13}$$

Once again the order of summation and integration has been reversed (it being assumed that this is permissible), and the integral is evaluated by the use of the sampling property of the impulse function.

The relationship between the Fourier transforms of $x(t)$ and $x'(t)$ are as shown in Figure 1.6. The Fourier transform of $x(t)$ is shown to be band-limited, meaning that it has no components beyond a cut-off frequency f_c. The sampled data has a Fourier transform which consists of an infinite train of the Fourier transforms of $x(t)$ centered at frequency intervals of $(k/\Delta T)$ for all k. Recall that the sampling interval is ΔT, so that the sampling frequency $f_s = (1/\Delta T)$.

If the cut-off frequency f_c is greater than one-half of the sampling frequency f_s, the Fourier transform of the sampled data will be as shown in Figure 1.7. In this case, the spectrum of the sampled data is different from that of the input signal in the region where the neighboring spectra overlap as shown by the shaded region in Figure 1.7. This implies that frequency components estimated from the sampled data in this region will be in error, due to a phenomenon known as "aliasing".

It is clear from the above discussion that in order to avoid errors due to aliasing, the bandwidth of the input signal must be less than half the sampling frequency utilized in obtaining the sampled data. This requirement is known as the "Nyquist criterion".

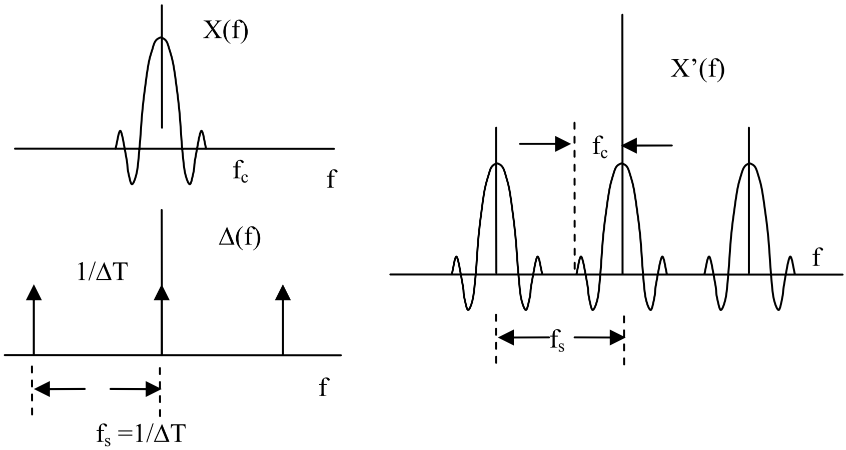

Fig. 1.6 Fourier transform of the sampled data function as a convolution of the Transforms $X(f)$ and $\Delta(f)$. The sampling frequency is f_s, and $X(f)$ is band-limited between $\pm f_c$.

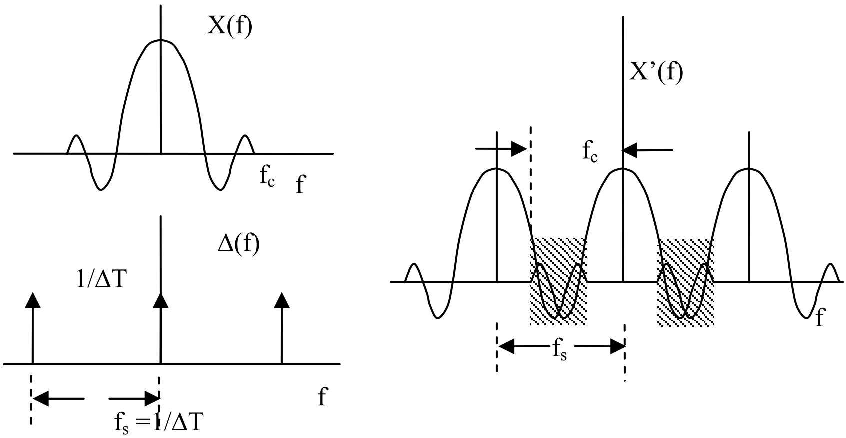

Fig. 1.7 Fourier transform of the sampled data function when the input signal is band-limited to a frequency greater than half the sampling frequency. The estimate of frequencies from sampled data in the shaded region will be in error because of aliasing.

In order to avoid aliasing errors, it is customary in all sampled data systems used in phasor estimation to use anti-aliasing filters which band-limit the input signals to below half the sampling frequency chosen. Note that the signal input cut-off frequency must be *less than* one half the sampling frequency. In practice, the signal is usually band-limited to a value much smaller than the one required for meeting the Nyquist criterion. Anti-aliasing filters are generally passive low-pass R-C filters [11], although active filters may also be used for obtaining a sharp cut-off characteristic. In addition to passive anti-aliasing filters, digital filters may also be used in special cases (e.g., with oversampling and decimation). All anti-aliasing filters introduce frequency-dependent phase shift in the input signal which must be compensated for in determining the phasor representation of the input signal. This will be discussed further in Chapter 5 where the 'Synchrophasor' standard is described.

1.5 Discrete Fourier transform (DFT)

DFT is a method of calculating the Fourier transform of a small number of samples taken from an input signal $x(t)$. The Fourier transform is calculated at discrete steps in the frequency domain, just as the input signal is sampled at discrete instants in the time domain. Consider the process of selecting N samples: $x(k\Delta T)$ with $\{k = 0, 1, 2, \cdots, N-1\}$, ΔT being the sampling interval. This is equivalent to multiplying the sampled data train by a "windowing function" $w(t)$, which is a rectangular function of time with unit magnitude and a span of $N\Delta T$. With the choice of samples ranging from 0 to $N-1$, it is clear that the windowing function can be viewed as starting at $-\Delta T/2$ and ending at $(N-1/2)\Delta T$. The function $x(t)$, the sampling function $\Delta(t)$, and the windowing function $w(t)$ along with their Fourier transforms are shown in Figure 1.8.

Consider the collection of signal samples which fall in the data window: $x(k\Delta T)$ with $\{k = 0,1,2, \cdots, N-1\}$. These samples can be viewed as being obtained by the multiplication of the signal $x(t)$, the sampling function $\delta(t)$, and the windowing function $\omega(t)$:

$$y(t) = x(t)\delta(t)w(t) = \sum_{k=0}^{N-1} x(k\Delta T)\,\delta(t - k\Delta T), \tag{1.14}$$

where once again the multiplication with the delta function is to be understood in the sense of the integral of Eq. (1.9). The Fourier transform of the sampled windowed function $y(t)$ is then the convolution of Fourier transforms of the three functions.

The Fourier transform of $y(t)$ is to be sampled in the frequency domain in order to obtain the DFT of $y(t)$. The discrete steps in the frequency domain are multiples of $1/T_0$, where T_0 is the span of the windowing function. The frequency sampling function $\Phi(f)$ is given by