Sign-Changing Critical Point Theory

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I dedicate this book to my wife and son

Qinying Wang and Yuezhang Zou

for their love and support

Contents

Preface

There has been increasing interest in recent years to develop a critical point theory by which one can obtain additional information on the critical points of a differentiable functional. What I mean by additional information is the locations of the critical points related to closed convex subsets in Banach spaces. This is the theme of the current book.

This book mainly reflects a significant part of my research activity during recent years. Except for the last chapter, it is constructed based on the results obtained myself or through direct cooperation with other mathematicians. On the whole, the readers will observe that the main abstract existence theorems of critical points in classical minimax theory are generalized to the cases of sign-changing critical points. Hence, a new theory is built. To the best of my knowledge, no book on sign-changing critical point theory has ever been published.

The material covered in this book is for advanced graduate and PhD students or anyone who wishes to seek an introduction into sign-changing critical point theory. The chapters are designed to be as self-contained as possible.

I have had the good fortune to teach at the University of California at Irvine and to work with Martin Schechter for the years 2001 to 2004. During that period, some results of the current book were obtained. M. Schechter has had a profound influence on me not only by his research, but also by his writing and his generosity. I am grateful to T. Bartsch and Z. Q. Wang for sending me their interesting papers and enlightening discussions with Wang when I visited Utah. Thanks also go to A. Szulkin and M. Willem for inviting me to visit their prestigious departments years ago. Special thanks are also given to S. Li who first introduced me into the variational and topological methods ten years ago. I wish to thank the University of California at Irvine for providing me a favorable environment during the period 2001 to 2004. This book is supported by the NSFC (No. 10001019 & 10571096), the SRF-ROCS-SEM, the Program of the Education Ministry in P. R. China for New Century Excellent Talents in Universities of China.

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Tsinghua University, Beijing Wenming Zou

Introduction

A theory is the more impressive,
the simpler are its premises,
the more distinct are the things it connects,
and the broader is its range of applicability.

Albert Einstein

Many nonlinear problems in physics, engineering, biology, and social sciences can be reduced to finding critical points (minima, maxima, and minimax points) of real-valued functions on various spaces. The first class of critical points to be studied were minima and maxima and much of the activity in the calculus of variations has been devoted to finding such points. A more difficult problem is to find critical points that are neither maxima nor minima. So far we may say, to some extent, that there is an organized procedure for producing such critical points and these methods are called global variational and topological methods. Roughly speaking, the modern variational and topological methods consist of the following two parts.

Minimax Methods. Ljusternik and Schnirelman [214] in 1929 mark the beginning of global analysis, by which some earlier mathematicians no longer consider only the minima or maxima of variational integrals. In 1934, Ljusternik and Schnirelman [215] developed a method that seeks to get information concerning the number of critical points of a functional from topological data. These ideas are referred to as the Ljusternik–Schnirelman theory. One celebrated and important result in the last 30 years has been the mountain pass theorem due to Ambrosetti and Rabinowitz [15] in 1973. Since then, a series of new theorems in the form of minimax have appeared via various linking, category, and index theories. Now these results in fact become a wonderful tool in studying the existence of solutions to differential equations with variational structures. We refer readers to the books (or surveys) due to Brézis and Nirenberg [71], Nirenberg [232, 233, 235], Rabinowitz [255], Schechter [275], Struwe [313], Willem [335], Mawhin and Willem [225], and Zou and Schechter [351], among others.

Morse Theory. This approach towards a global theory of critical points was pursued by Morse [229] in 1934. It reveals a deep relation between the topology of spaces and the number and types of critical points of any function defined on it. This theory was highly successful in topology in the 1950s due to the efforts of Milnor [226] and Smale [303]. In the works of Palais [239], Smale [304], and Rothe [264, 263], Morse theory was generalized to infinite-dimensional spaces. By then it was recognized as a useful approach in dealing with differential equations and in particular, in finding the existence of multiple solutions (see Chang [92, 94]). The critical group and Morse index also can be derived in some cases. Although there are some profound works on Morse theory and related topics, the applications are somewhat limited by the smoothness and nondegeneracy assumptions on the functionals. Readers may consult Mawhin and Willem [225], Conley [106], and Benci [51], among others.

However, both minimax theory and Morse theory essentially give answers on the existence of (multiple) critical points of a functional. They usually cannot provide many more additional properties of the critical points except some special profiles such as the Morse index, critical groups, and so on. I make no attempt here to give an exhaustive account of the field or a complete survey of the literature.

There has been increasing interest in recent years to develop a theory by which one can obtain much more information on critical points. The central theme of the current volume is the theory of finding sign-changing critical points. The book is organized as follows.

In Chapter 1, we provide some prerequisites for this book such as degree theory, Sobolev space, and so on. Basically, these theories are relatively mature and readily available in many existing books. However, we still spend some pages on the flows of the ODEs in Banach spaces which play important roles in this book. Well-trained readers may skip over this chapter to the next parts.

In Chapter 2, we establish the relation between linking and the sign-changing critical point. The linking introduced by Schechter is more general and realistic. We say that a set A links another set B if they do not intersect and A cannot be continuously shrunk to a point without intersecting B. This kind of linking includes the original ones. But more examples can be found. We show how the new linking produces sign-changing critical points.

We devote Chapter 3 to the sign-changing saddle point theory. The saddle point theory can be traced back to Rabinowitz's theory 30 years ago, which gives the sufficient conditions on the existence of a saddle point. But it never excludes the triviality of that point, nor the sign-changingness of it. We solve this question.

Essentially, in Chapter 4, we generalize the Brezis–Nirenberg critical point theorem obtained in 1991 by judging the location and nodal structure of the (PS) sequences and critical points.

Chapter 5 is about the even functionals. We obtain the relationship between the classical symmetric mountain pass theorem and the sign-changing critical points.

Chapter 6 discusses the parameter dependence of sign-changing critical points. This theory is independent of the (PS) compactness condition.

In Chapter 7 we provide sign-changing critical point theories due Bartsch, Chang and Wang, and Bartsch and Weth. The Morse index and the number of nodal domains are included.

In each chapter, based on the new abstract sign-changing critical point theory, applications are considered mainly on Schrödinger equations or Dirichlet boundary value problems.

This book mainly consists of the results of my recent research. It is not intended and nor is it possible to be complete. In fact, many other results on sign-changing solutions of elliptic equations in recently years are not in this book. I just cite them in the bibliography or quote some lemmas from them. We refer the readers to the references in the bibliography written by T. Bartsch, A. Castro, G. Cerami, K. C. Chang, M. Clapp, V. Coti-Zelati, E. N. Dancer, Y. Du, N. Ghoussoub, F. A. van Heerden, N. Hirano, S. Li, J. Q. Liu, Z. Liu, P. H. Rabinowitz, S. Solimini, M. Struwe, Z. Q. Wang, T. Weth, C. Yuan, et al. for other interesting results on concrete elliptic equations. Finally, although Chapter 7 involves some theories due to Bartsch and others, I would like to mention the following additional topics due to them: symmetry results for sign-changing solutions, in particular for the least energy nodal solution; upper estimates on the number of nodal domains; and some discussions of singularly perturbed equations and multiple nodal solutions without oddness of the nonlinearity.

Chapter 1
Preliminaries

For readers' convenience, we collect in this chapter some classical results on nonlinear functional analysis and the elementary theory of partial differential equations. Some of them are well known and their proofs are omitted. For others, although their proofs may be found in many existing books, we make no apology for repeating them.

1.1 Partition of Unity

Let E be a metric space with a distance function $\mathrm{dist}(\cdot,\cdot)$ on it. Let $A \subset E$ and $\mathcal{O}$ be a family of open subsets of E. If each point of A belongs to at least one member of $\mathcal{O}$, then $\mathcal{O}$ is called an open covering of A.

Definition 1.1. Let $\mathcal{O}$ be an open covering of a subset A of E. $\mathcal{O}$ is called locally finite if for any $u \in A$, there is an open neighborhood U such that $u \in U$ and that U intersects only finitely many elements of $\mathcal{O}$.

A well-known result on this line is the underlying proposition due to Stone [308].

Proposition 1.2. *Any metric space E is paracompact; that is, every open covering $\mathcal{O}$ of E has an open, locally finite refinement Θ. That is, Θ is a locally finite covering of E and for any V_i in Θ, we can find a U_i in $\mathcal{O}$ such that $V_i \subset U_i$.*

Proposition 1.3. *Assume that E is a metric space with the distance function* $\mathrm{dist}(\cdot,\cdot)$. *Let $\mathcal{O}$ be an open covering of E. Then $\mathcal{O}$ admits a locally finite partition of unity $\{\lambda_i\}_{i\in J}$ subordinate to it satisfying*

(1) $\lambda_i : E \to [0,1]$ is Lipschitz continuous.
(2) $\{u \in E : \lambda_i(u) \neq 0\}_{i\in J}$ is a locally finite covering of E.

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(3) For each V_i*, there is a* $U_i \in \mathcal{O}$ *such that* $V_i \subset U_i$*.*
(4) $\sum_{i\in J} \lambda_i(u) = 1, \forall u \in E,$

where J *is the index set.*

Proof. Because (E, dist) is a metric space with an open covering $\mathcal{O}$, by Proposition 1.2, there is an open, locally finite refinement Θ; that is, Θ is locally finite and for any V_i of Θ, we can find a U_i of $\mathcal{O}$ such that $V_i \subset U_i$. Define

$$\rho_i(u) = \text{dist}(u, E\backslash V_i), \qquad i \in J.$$

Then ρ_i is locally Lipschitz. Let

$$\lambda_i(u) = \frac{\rho_i(u)}{\sum_{j\in J} \rho_j(u)}, \qquad i \in J.$$

Then $\{\lambda_i\}_{i\in J}$ is what we want. □

1.2 Ekeland's Variational Principle

We recall Ekeland's variational principle (see Ekeland [137]).

Lemma 1.4. *Let* E *be a complete metric space with a metric* dist *and* $I : E \to \mathbf{R}$ *be a lower semicontinuous functional that is bounded below. For any* $T > 0, \varepsilon > 0$*, let* $u_1 \in E$ *be such that* $I(u_1) \le \inf_E I + \varepsilon$*. Then there exists a* $v_1 \in E$ *such that*

$$I(v_1) \le I(u_1), \tag{1.1}$$

$$\text{dist}(u_1, v_1) \le 1/T, \tag{1.2}$$

$$I(v_1) < I(w) + \varepsilon T \text{dist}(v_1, w), \qquad \textit{for all } w \ne v_1. \tag{1.3}$$

Proof. Define a partial order $\preceq$ in E as the following.

$$u \preceq v \Leftrightarrow I(u) \le I(v) - \varepsilon T \text{dist}(v, u).$$

Then obviously,

$$\begin{aligned} &u \preceq u, && \text{for all } u \in E,\\ &u \preceq v, v \preceq u \Rightarrow u = v, && \text{for all } u, v \in E,\\ &u \preceq v, v \preceq w \Rightarrow u \preceq w, && \text{for all } u, v, w \in E. \end{aligned}$$

Let $C_1 := \{u \in E : u \preceq u_1\}$ and let $u_2 \in C_1$ be such that

$$I(u_2) \le \inf_{C_1} I + \frac{\varepsilon}{2^2}.$$

Then, let $C_2 := \{u \in E : u \preceq u_2\}$. Inductively,

$$u_{n+1} \in C_n := \{u \in E : u \preceq u_n\}, \qquad I(u_{n+1}) \le \inf_{C_n} I + \frac{\varepsilon}{2^{n+1}}.$$

By the lower semicontinuity of I and the continuity of $\mathrm{dist}(\cdot,\cdot)$, we see that C_n is closed. Moreover,

$$C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots,$$

$$\cdots \preceq u_n \preceq \cdots \preceq u_2 \preceq u_1.$$

For any $v \in C_n$, then

$$I(v) \le I(u_n) - \varepsilon T \mathrm{dist}(v, u_n). \tag{1.4}$$

Note that $v \in C_{n-1}$; we have

$$I(u_n) \le \inf_{C_{n-1}} I + \frac{\varepsilon}{2^n} \le I(v) + \frac{\varepsilon}{2^n}. \tag{1.5}$$

Combine Equations (1.4) and (1.5); we have that $\mathrm{dist}(v, u_n) \le (1/T2^n)$. Because $v \in C_n$ is arbitrary, we know that the diameter of C_n is less than or equal to $(1/T2^{n-1})$, hence, approaches zero. Therefore,

$$\bigcap_{n=1}^{\infty} C_n = \{v_1\}.$$

We claim that v_1 is what we want. Indeed, $v_1 \in C_1$ implies that

$$I(v_1) \le I(u_1) - \varepsilon T \mathrm{dist}(u_1, v_1) \le I(u_1).$$

For any $w \ne v_1$, we observe that we cannot have $w \preceq v_1$, otherwise $w \in \bigcap_{n=1}^{\infty} C_n$ hence $w = v_1$. That is, we must have

$$I(w) > I(v_1) - \varepsilon T \mathrm{dist}(w, v_1).$$

Finally, noting that

$$\mathrm{dist}(u_1, u_n) \le \sum_{i=1}^{n-1} \mathrm{dist}(u_i, u_{i+1}) \le \sum_{i=1}^{n-1} \frac{1}{T2^i} \le \frac{1}{T}$$

and that $\lim_{n\to\infty} u_n = v_1$, then we get that $\mathrm{dist}(u_1, v_1) \le 1/T$. Thus, v_1 satisfies Equations (1.1) to (1.3). This completes the proof. □

Notes and Comments. Readers may consult Ekeland [138], de Figueiredo [147], Ghoussoub [156], Grossinho and Tersian [162], Mawhin and Willem [225], Struwe [313], and Willem [335] for the variants of Ekeland's variational principle and their applications. Ghoussoub [156] contains the Borwein and Preiss principle and also the mountain pass principle which is presented as a "multidimensional extension" of the Ekeland variational principle. A simple and elegant generalization of Ekeland's variational principle to a general form on ordered sets was obtained in Brézis and Browder [66]. It was applied to nonlinear semigroups and to derive diverse results from nonlinear analysis including the variational principle and one of its equivalent forms, the Bishop–Phelps theorem. Some other generalizations of Ekeland's variational principle can also be found in Li and Shi [203] and Zhong [339, 340].

1.3 Sobolev Spaces and Embedding Theorems

Let Ω be an open subset of $\mathbf{R}^N, N \in \mathbf{N}$. Denote

$$L^p(\Omega) := \{u : \Omega \to \mathbf{R} \text{ is Lebesgue measurable, } \|u\|_{L^p(\Omega)} < \infty\},$$

where

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{1/p}, \qquad 1 \le p < +\infty.$$

If $p = +\infty$,

$$\|u\|_{L^\infty(\Omega)} = ess \sup_\Omega |u| := \inf_{A \subset \Omega, \mathrm{meas}(A)=0} \sup_{\Omega \setminus A} |u|,$$

where meas denotes the Lebesgue measure. If $\|u\|_{L^\infty(\Omega)} < \infty$, we say that u is essentially bounded on Ω. Let

$$L^p_{loc}(\Omega) := \{u : \Omega \to \mathbf{R}, u \in L^p(V) \text{ for each } V \subset\subset \Omega\},$$

where $V \subset\subset \Omega \Leftrightarrow V \subset \bar{V} \subset \Omega$ and $\bar{V}$ is compact. Sometimes in this book we denote $\|u\|_{L^p(\Omega)}$ by $\|u\|_p$ or $|u|_p$.

We denote by $\mathrm{supp}(u) := \overline{\{x \in \Omega : u(x) \neq 0\}}$ the support of $u : \Omega \to \mathbf{R}$. Let $\mathbf{C}_c^\infty(\Omega)$ denote the space of infinitely differentiable functions $\phi : \Omega \to \mathbf{R}$ with compact support in Ω. For each $\phi \in \mathbf{C}_c^\infty(\Omega)$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$ with order $|\alpha| := \alpha_1 + \cdots + \alpha_N$, we denote

$$D^\alpha \phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} \phi.$$

Definition 1.5. Suppose $u, v \in L^1_{loc}(\Omega)$. We say that v is the αth-weak partial derivative of u, written $D^\alpha u = v$ provided

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx$$

for all $\phi \in \mathbf{C}_c^\infty(\Omega)$.

It is easy to check that the αth-weak partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.

Let $\mathbf{C}^m(\Omega)$ be the set of functions having derivatives of order $\leq m$ being continuous in Ω ($m = integer \geq 0$ or $m = \infty$). Let $\mathbf{C}^m(\bar{\Omega})$ be the set of functions in $\mathbf{C}^m(\Omega)$ all of whose derivatives of order $\leq m$ have continuous extension to $\bar{\Omega}$.

Definition 1.6. Fix $p \in [1, +\infty]$ and $k \in \mathbf{N} \cup \{0\}$. The Sobolev space

$$W^{k,p}(\Omega)$$

consists of all $u : \Omega \to \mathbf{R}$ which has αth-weak partial derivative $D^\alpha u$ for each multi-index α with $|\alpha| \leq k$ and $D^\alpha u \in L^p(\Omega)$.

If $p = 2$, we usually write

$$H^k(\Omega) = W^{k,2}(\Omega), \qquad k = 0, 1, 2, \ldots .$$

Note that $H^0(\Omega) = L^2(\Omega)$. We henceforth identify functions in $W^{k,p}(\Omega)$ which agree a.e

Definition 1.7. If $u \in W^{k,p}(\Omega)$, we define its norm to be

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx\right)^{1/p}, & p \in [1, +\infty), \\ \sum_{|\alpha| \leq k} ess \sup_\Omega |D^\alpha u|, & p = +\infty. \end{cases}$$

Definition 1.8. We denote $W_0^{k,p}(\Omega)$ the closure of $\mathbf{C}_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$ with respect to its norm defined in Definition 1.7. It is customary to write

$$H_0^k(\Omega) = W_0^{k,2}(\Omega)$$

and denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$.

The following results can be found in Evans [141] and Adams and Fournier [2].

Proposition 1.9. *For each $k = 1, 2, \ldots$ and $1 \leq p \leq +\infty$, the Sobolev space*

$$(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$$

is a Banach space and so is $W_0^{k,p}(\Omega)$. In particular, $H^k(\Omega), H_0^k(\Omega)$ are Hilbert spaces; $W_0^{k,p}(\mathbf{R}^N) = W^{k,p}(\mathbf{R}^N)$.

Definition 1.10. Let $(E, \|\cdot\|_E)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces, $E \subset Y$. We say that E is continuously embedded in Y (denoted by $E \hookrightarrow Y$) if the identity $\mathbf{id} : E \to Y$ is a linear bounded operator; that is, there is a constant $C > 0$ such that $\|u\|_Y \leq C\|u\|_E$ for all $u \in E$. In this case, the constant $C > 0$

is called the embedding constant. If moreover, each bounded sequence in E is precompact in Y, we say the embedding is compact, written $E \hookrightarrow\hookrightarrow Y$.

Definition 1.11. A function $u : \Omega \subset \mathbf{R}^N \to \mathbf{R}$ is Hölder continuous with exponent $\gamma > 0$ if

$$[u]^{(\gamma)} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\gamma} < \infty.$$

Definition 1.12. The Hölder space $\mathbf{C}^{k,\gamma}(\bar{\Omega})$ consists of all functions $u \in \mathbf{C}^k(\bar{\Omega})$ for which the norm

$$\|u\|_{\mathbf{C}^{k,\gamma}}(\bar{\Omega}) := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{\mathbf{C}(\bar{\Omega})} + \sum_{\||\alpha|=k} [D^\alpha u]^{(\gamma)}$$

is finite. It is a Banach space. We set $\mathbf{C}^{k,0}(\bar{\Omega}) = \mathbf{C}^k(\bar{\Omega})$.

We have the following embedding results; see Adams [1], Adams and Fournier [2], Evans [141], and Gilbarg and Trudinger [160].

Proposition 1.13. *If Ω is a bounded domain in $\mathbf{R}^N$, then*

$$W_0^{k,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega), & kp < N, 1 \leq q \leq Np/(N - kp); \\ \mathbf{C}^{m,\alpha}(\bar{\Omega}), & 0 \leq \alpha \leq k - m - N/p\,, \\ & 0 \leq m < k - N/p < m + 1. \end{cases}$$

Proposition 1.14. *If Ω is a bounded domain in $\mathbf{R}^N$, then*

$$W_0^{k,p}(\Omega) \hookrightarrow\hookrightarrow \begin{cases} L^q(\Omega), & kp < N, 1 \leq q < Np/(N - kp); \\ \mathbf{C}^{m,\alpha}(\bar{\Omega}), & 0 \leq \alpha < k - m - N/p\,, \\ & 0 \leq m < k - N/p < m + 1. \end{cases}$$

In general, $W_0^{k,p}(\Omega)$ cannot be replaced by $W^{k,p}(\Omega)$ in Proposition 1.13. However, this replacement can be made for a large class of domains, which includes, for example, domains with a smooth boundary.

Definition 1.15. A bounded domain $\Omega \subset \mathbf{R}^N$ with boundary $\partial\Omega$. Let k be a nonnegative integer and $\alpha \in [0, 1]$. Ω is called $\mathbf{C}^{k,\alpha}$ if at each point $x_0 \in \partial\Omega$ there is a ball $B = B(x_0)$ and one-to-one mapping φ from B onto $D \subset \mathbf{R}^N$ such that

(1) $\varphi(B \cap \Omega) \subset \mathbf{R}_+^N := \{x = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N : x_N > 0\}$.
(2) $\varphi(B \cap \partial\Omega) \subset \partial\mathbf{R}_+^N := \{x = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N : x_N = 0\}$.
(3) $\varphi \in \mathbf{C}^{k,\alpha}(B), \varphi^{-1} \in \mathbf{C}^{k,\alpha}(D)$.

The following proposition is due to Gilbarg and Trudinger [160, Theorem 7.26].

Proposition 1.16. *Let Ω be a $\mathbf{C}^{0,1}$ domain in $\mathbf{R}^N$. Then*

(1) If $kp < N$, then $W^{k,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, where $p^* = Np/(N-kp)$; and $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q < p^*$.
(2) If $0 \leq m < k - N/p < m+1$, then $W^{k,p}(\Omega) \hookrightarrow \mathbf{C}^{m,\alpha}(\bar{\Omega}), \alpha = k - N/p - m$; and $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow \mathbf{C}^{m,\beta}(\bar{\Omega})$ for any $\beta < \alpha$.

The following proposition can be found in Brezis [64] and Willem [335].

Proposition 1.17. *The following embeddings are continuous.*

$$H^1(\mathbf{R}^N) \hookrightarrow L^p(\mathbf{R}^N), \qquad 2 \leq p < \infty, N = 1, 2,$$

$$H^1(\mathbf{R}^N) \hookrightarrow L^p(\mathbf{R}^N), \qquad 2 \leq p \leq 2^*, N \geq 3,$$

where $2^* := 2N/(N-2)$ if $N \geq 3$; $2^* = +\infty$ if $N = 1, 2$, is called a critical exponent.

For $N \geq 3$, let

$$S := \inf_{u \in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}$$

be the best Sobolev constant. Then, by Talenti's [321] result,

$$S = \frac{\|\nabla U\|_2^2}{\|U\|_{2^*}^2},$$

where

$$U^*(x) = \frac{(N(N-2))^{(N-2)/4}}{(1+|x|^2)^{(N-2)/2}}.$$

Note that if $\mathbf{R}^N$ is replaced by a bounded domain, S is never achieved. We frequently use the following Gagliardo–Nirenberg inequality, see Chabrowski [88], Evans [141], and Nirenberg [231].

Proposition 1.18. *For every $v \in H^1(\mathbf{R}^N)$,*

$$\|v\|_p \leq c\|\nabla v\|_2^{\gamma}\|v\|_q^{1-\gamma}$$

with

$$\frac{N}{p} = \gamma\frac{N-2}{2} + (1-\gamma)\frac{N}{q}, \qquad q \geq 1, \gamma \in [0,1],$$

where c is a constant depending on p, γ, q, N.

Note. In this book, from time to time the letter c is indiscriminately used to denote various constants when the exact values are irrelevant.

The following concentration-compactness lemma due to Lions [196] is also a powerful tool in dealing with Schrödinger equations.

Lemma 1.19. *Let $r > 0$ and $q \in [2, 2^*)$. For any bounded sequence $\{w_n\}$ of $E := H^1(\mathbf{R}^N)$, if*

$$\sup_{y\in\mathbf{R}^N} \int_{B(y,r)} |w_n|^q dx \to 0, \qquad n \to \infty,$$

where $B(y,r) := \{u \in E : \|u - y\| \leq r\}$; then $w_n \to 0$ in $L^p(\mathbf{R}^N)$ for $q < p < 2^*$.

Proof. We only consider $N \geq 3$. Choose $p_1, p_2, t > 1, t' > 1$ such that

$$p_1 t = q, \qquad p_2 t' = 2^*, \qquad 1/t + 1/t' = 1, \qquad p_1 + p_2 = p.$$

By the Hölder inequality and Proposition 1.14, we have

$$\begin{aligned}
&\int_{B(y,r)} |w_n|^p dx \\
&\quad \leq \left(\int_{B(y,r)} |w_n|^{p_1 t} dx\right)^{1/t} \left(\int_{B(y,r)} |w_n|^{p_2 t'} dx\right)^{1/t'} \\
&\quad \leq c\left(\int_{B(y,r)} |w_n|^{p_1 t} dx\right)^{1/t} \|w_n\|_{2^*}^{p_2} \\
&\quad \leq c\left(\int_{B(y,r)} |w_n|^{p_1 t} dx\right)^{1/t} \left(\int_{B(y,r)} (w_n^2 + |\nabla w_n|^2) dx\right)^{p_2/2} \\
&\quad \leq c\left(\int_{B(y,r)} |w_n|^{p_1 t} dx\right)^{1/t} \left(\int_{B(y,r)} (w_n^2 + |\nabla w_n|^2) dx\right)^{p_2/2}.
\end{aligned}$$

Covering $\mathbf{R}^N$ by balls of radius r in such a way that each point of $\mathbf{R}^N$ is contained in at most $N + 1$ balls, we have

$$\int_{\mathbf{R}^N} |w_n|^p dx \leq (N+1)c \sup_{y\in\mathbf{R}^N} \left(\int_{B(y,r)} |w_n|^q dx\right)^{1/t},$$

which implies the conclusion. □

1.4 Differentiable Functionals

Let E be a Banach space with the norm $\|\cdot\|$. Let $U \subset E$ be an open set of E. The conjugate (or dual) space of E is denoted by E'; that is, E' denotes the set of all bounded linear operators on E. Consider a functional $G : U \to \mathbf{R}$.

Definition 1.20. The functional G has a Fréchet derivative $F \in E'$ at $u \in U$ if

$$\lim_{h\in E, h\to 0} \frac{G(u+h) - G(u) - F(h)}{\|h\|} = 0.$$

We denote $G'(u) = F$ or $\nabla G(u) = F$ and sometimes say the gradient of G at u. Usually, $G'(\cdot)$ is a nonlinear operator. We use $\mathbf{C}^1(U, \mathbf{R})$ to denote the set of all functionals G that have a continuous Fréchet derivative on U. A point $u \in U$ is called a critical point of a functional $G \in \mathbf{C}^1(U, \mathbf{R})$ if $G'(u) = 0$.

Definition 1.21. The functional G has a Gateaux derivative $I \in E'$ at $u \in U$ if, for every $h \in E$,

$$\lim_{t\to 0} \frac{G(u+th) - G(u)}{t} = I(h).$$

The Gateaux derivative at $u \in U$ is denoted by $DG(u)$. Obviously, if G has a Fréchet derivative $F \in E'$ at $u \in U$, then G has a Gateaux derivative $I \in E'$ at $u \in U$ and $G'(u) = DG(u)$. Unfortunately, the converse is not true. However, if G has Gateaux derivatives at every point of some neighborhood of $u \in U$ such that $DG(u)$ is continuous at u, then G has a Fréchet derivative and $G'(u) = DG(u)$. This is a straightforward consequence of the mean value theorem.

Sometimes, we use the concepts of the second-order Fréchet and Gateaux derivatives.

Definition 1.22. The functional $G \in \mathbf{C}^1(U, \mathbf{R})$ has a second-order Fréchet derivative at $u \in U$ if there is an L, which is a linear bounded operator from E to E', such that

$$\lim_{h\in E, h\to 0} \frac{G'(u+h) - G'(u) - Lh}{\|h\|} = 0;$$

we denote $G''(u) = L$.

We say that $G \in \mathbf{C}^2(U, \mathbf{R})$ if the second-order Fréchet derivative of G exists and is continuous on U.

Definition 1.23. The functional $G \in \mathbf{C}^1(U, \mathbf{R})$ has a second-order Gateaux derivative at $u \in U$ if there is an L, which is a linear bounded operator from E to E', such that

$$\lim_{t\to 0} \frac{(G'(u+th) - G'(u) - Lth)v}{t} = 0, \qquad \forall h, v \in E.$$

We denote $D^2G(u) = L$.

Evidently, any second-order Fréchet derivative of G is a second-order Gateaux derivative. Using the mean value theorem, if G has a continuous second-order Gateaux derivative on U, then $G \in \mathbf{C}^2(U, \mathbf{R})$.

Definition 1.24. Let $f(x,t)$ be a function on $\Omega \times \mathbf{R}$, where Ω is either bounded or unbounded. We say that f is a Carathéodory function if $f(x,t)$ is continuous in t for a.e. $x \in \Omega$ and measurable in x for every $t \in \mathbf{R}$.

Lemma 1.25. *Assume $p \geq 1, q \geq 1$. Let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbf{R}$ and satisfy*

$$|f(x,t)| \leq a + b|t|^{p/q}, \qquad \forall (x,t) \in \Omega \times \mathbf{R},$$

where $a, b > 0$ and Ω is either bounded or unbounded. Define a Carathéodory operator by

$$Ou := f(x, u(x)), \qquad u \in L^p(\Omega).$$

Let $\{w_k\}_{k=0}^{\infty} \subset L^p(\Omega)$. If $\|w_k - w_0\|_p \to 0$ as $k \to +\infty$, then

$$\|Ow_k - Ow_0\|_q \to 0$$

as $k \to \infty$. In particular, if Ω is bounded, then O is a continuous and bounded mapping from $L^p(\Omega)$ to $L^q(\Omega)$ and the same conclusion is true if Ω is unbounded and $a = 0$.

Proof. Note that

$$w_k(x) \to w_0(x), \qquad \text{a.e. } x \in \Omega. \tag{1.6}$$

Because f is a Carathéodory function,

$$Ow_k(x) \to Ow_0(x), \qquad \text{a.e. } x \in \Omega. \tag{1.7}$$

Let

$$v_k(x) := a + b|w_k(x)|^{p/q}, \qquad k = 0, 1, 2, \ldots. \tag{1.8}$$

Then by (1.6)–(1.8),

$$|Ow_k(x)| \leq v_k(x) \quad \text{for all } x \in \Omega; \qquad v_k(x) \to v_0(x) \ \text{ a.e. } x \in \Omega. \tag{1.9}$$

Because

$$|w_k|^p + |w_0|^p - ||w_k|^p - |w_0|^p| \geq 0,$$

by Fatou's theorem, we have

$$\begin{aligned} &\int_\Omega \liminf_{k \to +\infty} (|w_k|^p + |w_0|^p - ||w_k|^p - |w_0|^p|) dx \\ &\leq \liminf_{k \to +\infty} \int_\Omega (|w_k|^p + |w_0|^p - ||w_k|^p - |w_0|^p|) dx. \end{aligned} \tag{1.10}$$

Combining (1.6)–(1.10), thus we see that

$$\lim_{k\to+\infty}\int_\Omega ||w_k|^p-|w_0|^p|dx=0. \tag{1.11}$$

It follows that

$$\int_\Omega |v_k-v_0|^q dx \le b^q \int_\Omega ||w_k|^p-|w_0|^p|dx \to 0 \tag{1.12}$$

as $k\to\infty$. Because there is a constant $C>0, C_1>0$ such that

$$\begin{aligned}&|Ow_k-Ow_0|^q\\ &\quad\le C(|Ow_k|^q+|Ow_0|^q)\\ &\quad\le C(|v_k|^q+|v_0|^q)\\ &\quad\le C_1(|v_k-v_0|^q+|v_0|^q)\end{aligned}$$

a.e. $x\in\Omega$, then by Fatou's theorem,

$$\begin{aligned}&\int_\Omega \liminf_{k\to+\infty}(C_1(|v_k-v_0|^q+|v_0|^q)-|Ow_k-Ow_0|^q|)dx\\ &\quad\le \liminf_{k\to+\infty}\int_\Omega (C_1(|v_k-v_0|^q+|v_0|^q)-|Ow_k-Ow_0|^q)dx.\end{aligned} \tag{1.13}$$

By (1.7), (1.8), (1.12), and (1.13), we have

$$\|Ow_k-Ou_0\|_q\to 0.$$

Finally, if Ω is bounded, then for any $u\in L^p(\Omega)$, evidently we have

$$\|Ou\|_q\le c+c\|u\|_p^{p/q}, \tag{1.14}$$

where $c>0$ is a constant. Equation (1.14) remains true if Ω is unbounded and $a=0$. Therefore, O is a continuous and bounded mapping from $L^p(\Omega)$ to $L^q(\Omega)$ and the same conclusion is true if Ω is unbounded and $a=0$. □

The following lemma comes from Willem [335].

Lemma 1.26. *Assume $p_1,p_2,q_1,q_2\ge 1$. Let $f(x,t)$ be a Carathéodory function on $\Omega\times\mathbf{R}$ and satisfy*

$$|f(x,t)|\le a|t|^{p_1/q_1}+b|t|^{p_2/q_2},\qquad \forall(x,t)\in\Omega\times\mathbf{R},$$

where $a,b\ge 0$ and Ω is either bounded or unbounded. Define a Carathéodory operator by

$$Ou:=f(x,u(x)),\qquad u\in\mathcal{H}:=L^{p_1}(\Omega)\cap L^{p_2}(\Omega).$$

Define the space

$$\mathcal{E}_0 := L^{q_1}(\Omega) + L^{q_2}(\Omega)$$

with a norm

$$\|u\|_{\mathcal{E}_0} = \inf\{\|v\|_{L^{q_1}(\Omega)} + \|w\|_{L^{q_2}(\Omega)} : u = v + w \in \mathcal{E}_0, v \in L^{q_1}(\Omega), w \in L^{q_2}(\Omega)\}.$$

Then $O = O_1 + O_2$, where O_i is bounded continuous from $L^{p_i}(\Omega)$ to $L^{q_i}(\Omega), i = 1, 2$. In particular, O is a bounded continuous mapping from $\mathcal{H}$ to $\mathcal{E}_0$.

Proof. Let $\xi : \mathbf{R} \to [0, 1]$ be a smooth function such that $\xi(t) = 1$ for $t \in (-1, 1)$; $\xi(t) = 0$ for $t \notin (-2, 2)$. Let

$$\phi(x, t) = \xi(t)f(x, t), \qquad \psi(x, t) = (1 - \xi(t))f(x, t).$$

We may assume that $p_1/q_1 \leq p_2/q_2$. Then there are two constants $d > 0$, $m > 0$ such that

$$|\phi(x, t)| \leq d|t|^{p_1/q_1}, \qquad |\psi(x, t)| \leq m|t|^{p_2/q_2}.$$

Define

$$O_1 u = \phi(x, u), \qquad u \in L^{p_1}(\Omega);$$
$$O_2 u = \psi(x, u), \qquad u \in L^{p_2}(\Omega).$$

Then by Lemma 1.25, O_i is bounded continuous from $L^{p_i}(\Omega)$ to $L^{q_i}(\Omega), i = 1, 2$. It is readily seen that $O = O_1 + O_2$ is a bounded continuous mapping from $\mathcal{H}$ to $\mathcal{E}_0$. □

The following theorem and its idea of proof are enough for us to see those functionals encountered in this book are of $\mathbf{C}^1$.

Theorem 1.27. *Assume $\kappa \geq 0, p \geq 0$. Let $f(x, t)$ be a Carathéodory function on $\Omega \times \mathbf{R}$ and satisfy*

$$|f(x, t)| \leq a|t|^\kappa + b|t|^p, \qquad \forall (x, t) \in \Omega \times \mathbf{R}, \tag{1.15}$$

where $a, b > 0$ and Ω is either bounded or unbounded. Define a functional

$$J(u) := \int_\Omega F(x, u)dx, \qquad \text{where } F(x, u) = \int_0^u f(x, s)ds.$$

Assume $(E, \|\cdot\|)$ is a Sobolev Banach space such that $E \hookrightarrow L^{p+1}(\Omega)$ and $E \hookrightarrow L^{\kappa+1}(\Omega)$; then $J \in \mathbf{C}^1(E, \mathbf{R})$ and

$$J'(u)h := \int_\Omega f(x, u)hdx, \qquad \forall\, h \in E.$$

Moreover, if $E \hookrightarrow\hookrightarrow L^{\kappa+1}, E \hookrightarrow\hookrightarrow L^{p+1}$*, then* $J' : E \to E'$ *is compact.*

Proof. Because $E \hookrightarrow L^{\kappa+1}(\Omega)$ and $E \hookrightarrow L^{p+1}(\Omega)$, we may find a constant $C_0 > 0$ such that

$$\|w\|_{\kappa+1} \leq C_0\|w\|, \qquad \|w\|_{p+1} \leq C_0\|w\|, \qquad \forall w \in E. \tag{1.16}$$

Recall the Young inequality and

$$(|s| + |t|)^\tau \leq 2^{\tau-1}(|s|^\tau + |t|^\tau), \qquad \tau \geq 1, s, t \in \mathbf{R}.$$

Combining the assumptions on f, for any $\gamma \in [0, 1]$, it is easy to check that

$$|f(x, u + \gamma h)h| \leq C_1(|u|^{(p+1)} + |h|^{(p+1)} + |u|^{\kappa+1} + |h|^{\kappa+1}),$$

where C_1 is a constant independent of γ. Therefore, for any $u, h \in E$, by the mean value theorem and Lebesgue theorem,

$$\begin{aligned} &\lim_{t\to 0} \frac{J(u + th) - J(u)}{t} \\ &= \lim_{t\to 0} \int_\Omega f(x, u + \theta th)hdx \\ &= \int_\Omega f(x, u)hdx \\ &=: T_0(u, h), \end{aligned} \tag{1.17}$$

where $\theta \in [0, 1]$ depending on u, h, t. Obviously, $T_0(u, h)$ is linear in h. Furthermore, by (1.16),

$$\begin{aligned} &|T_0(u, h)| \\ &\leq \int_\Omega |f(x, u)h|dx \\ &\leq c(\|u\|_{\kappa+1}^\kappa \|h\|_{\kappa+1} + \|u\|_{p+1}^p \|h\|_{p+1}) \\ &\leq c(\|u\|^\kappa + \|u\|^p)\|h\|. \end{aligned}$$

It follows that $T_0(u, h)$ is linear bounded in h. Therefore, $DJ(u) = T_0(u, \cdot) \in E'$ is the Gateaux derivative of J at u. Next, we show that $DJ(u)$ is continuous in u. Let $Ou := f(x, u), u \in E$. By Lemma 1.26, $O = O_1 + O_2$, where O_1 is bounded continuous from $L^{\kappa+1}(\Omega)$ to $L^{(\kappa+1)/\kappa}(\Omega)$ and O_2 is bounded continuous from $L^{p+1}(\Omega)$ to $L^{(p+1)/\kappa}(\Omega)$. For any $v, h \in E$,

$$\begin{aligned} &|(DJ(u) - DJ(v))h| \\ &= \left|\int_\Omega (f(x, u) - f(x, v))hdx\right| \end{aligned}$$

$$= \left| \int_\Omega (Ou - Ov) h dx \right|$$

$$= \left| \int_\Omega (O_1 u + O_2 u - O_1 v - O_2 v) h dx \right|$$

$$\leq \int_\Omega |O_1 u - O_1 v| |h| dx + \int_\Omega |O_2 u - O_2 v| |h| dx$$

$$\leq C_0 \|O_1 u - O_1 v\|_{(\kappa+1)/\kappa} \|h\| + C_0 \|O_2 u - O_2 v\|_{(p+1)/p} \|h\|.$$

It implies that

$$(1.18) \qquad \|DJ(u) - DJ(v)\|_{E'} \leq C_0 (\|O_1 u - O_1 v\|_{(\kappa+1)/\kappa} + \|O_2 u - O_2 v\|_{(p+1)/p}),$$

where $\|\cdot\|_{E'}$ is the norm in E'. If $v_k \to u$ in $E \subset L^{\kappa+1}(\Omega) \cap L^{p+1}(\Omega)$, then

$$\|O_1 v_k - O_1 u\|_{(\kappa+1)/\kappa} \to 0,$$

$$\|O_2 v_k - O_2 u\|_{(p+1)/p} \to 0.$$

Therefore, $DJ(v_k) \to DJ(u)$. This means $DJ(u)$ is continuous in u. Hence, $J'(u) = DJ(u)$; that is, $J \in \mathbf{C}^1(E, \mathbf{R})$. Furthermore, if $E \hookrightarrow\hookrightarrow L^{p+1}$, $E \hookrightarrow\hookrightarrow L^{\kappa+1}$, then any bounded sequence $\{u_k\}$ in E has a subsequence denoted by $\{u_k\}$ that converges to u_0 in $L^{p+1}(\Omega)$ and in $L^{\kappa+1}(\Omega)$. Hence, $O_1(u_k) \to O_1(u_0)$ in $L^{(\kappa+1)/\kappa}(\Omega)$; $O_2(u_k) \to O_2(u_0)$ in $L^{(p+1)/p}(\Omega)$. Finally, $DJ(u_k) \to DJ(u_0)$ in E'; that is, J' is compact in E. □

1.5 The Topological Degree

Since the invention of Brouwer's degree in 1912, topological degree has become an eternal topic of every book on nonlinear functional analysis. Therefore, we just outline the main ideas and results and omit the proofs. Readers may consult the books of Berger [57], Chang [91], Deimling [134], Mawhin [224], Nirenberg [234], and Zeidler [337] (also Brézis and Nirenberg [72] for applications).

Definition 1.28. Let $W \subset X := \mathbf{R}^N$ $(N \geq 1)$ be an open subset and a mapping $J \in \mathbf{C}^1(W, X)$. A point $u \in W$ is called a regular point and $J(u)$ is a regular value if $J'(u) : X \to X$ is surjective. Otherwise, u is called a critical point and $J(u)$ is the critical value.

To construct the degree theory, we need a simplified Sard's theorem. Refer to Sard [266].

Theorem 1.29. *Let $W \subset X := \mathbf{R}^N$ $(N \geq 1)$ be an open subset and $J \in \mathbf{C}^1(W, X)$. Then the set of all critical values of J has zero Lebesgue measure in X.*

Definition 1.30 (Brouwer's degree). Let $W \subset X := \mathbf{R}^N$ $(N \geq 1)$ be a bounded open subset, $J \in \mathbf{C}^2(\bar{W}, X)$, $p \in X \backslash J(\partial W)$.

(1) If p is a regular value of J, define the Brouwer degree by

$$\deg(J, W, p) := \sum_{v \in J^{-1}(p)} \text{sign } \det J'(v),$$

where det denotes the determinant.

(2) If p is a critical value of J, choose p_1 to be a regular value (by Sard's theorem) such that $\|p - p_1\| < \text{dist}(p, J(\partial W))$ and define the Brouwer degree by

$$\deg(J, W, p) := \deg(J, W, p_1).$$

In item (1), $J^{-1}(p)$ is a finite set when p is a regular value. In item (2), the degree is independent of the choice of p_1.

If $J \in \mathbf{C}(\bar{W}, X)$, we may find by Weierstrass's theorem an approximation of J via a smooth function.

Definition 1.31 (Brouwer's degree). Let $W \subset X := \mathbf{R}^N$ $(N \geq 1)$ be a bounded open subset, $J \in \mathbf{C}(\bar{W}, X)$, $p \in X \backslash J(\partial W)$. Choose $\tilde{J} \in \mathbf{C}^2(\bar{W}, X)$ such that

$$\sup_{u \in W} \|J(u) - \tilde{J}(u)\| < \text{dist}(p, J(\partial W))$$

and define Brouwer's degree by

$$\deg(J, W, p) := \deg(\tilde{J}, W, p),$$

which is independent of the choice of $\tilde{J}$.

Proposition 1.32. *Let $W \subset X := \mathbf{R}^N$ $(N \geq 1)$ be a bounded open subset, $J \in \mathbf{C}(\bar{W}, X)$, $p \in X \backslash J(\partial W)$.*

(1)

$$\deg(\mathbf{id}, W, p) = \begin{cases} 1, & p \in W, \\ 0, & p \notin \bar{\Omega}, \end{cases}$$

where **id** *is the identity.*

(2) Let W_1, W_2 be two disjoint open subsets of W, $p \notin J(\bar{W} \backslash (W_1 \cup W_2))$; then

$$\deg(J, W, p) = \deg(J, W_1, p) + \deg(J, W_2, p).$$

(3) Let $H \in \mathbf{C}([0,1] \times \bar{W}, \mathbf{R}^N)$, $p \in \mathbf{C}([0,1], \mathbf{R}^N)$ and $p(t) \notin H(t, \partial W)$. Then $\deg(H(t, \cdot), W, p(t))$ is independent of $t \in [0,1]$.

(4) (Kronecker's theorem) If $\deg(J, W, p) \neq 0$, *then there exists a* $u \in W$ *such that* $J(u) = p$.

Theorem 1.33 (Borsuk–Ulam theorem). *Let* W *be an open bounded symmetric neighborhood of* 0 *in* $\mathbf{R}^N$. *Every continuous odd map* $f : \partial W \to \mathbf{R}^{N-1}$ *has a zero.*

Brouwer's degree can be extended to infinite-dimensional spaces. This is the Leray–Schauder degree for a compact perturbation of the identity.

Definition 1.34. Let E be a Banach space; $M \subset E$. A mapping $J : M \to E$ is called compact if $\overline{J(S)}$ is compact for any bounded subset S of E. Furthermore, if J is continuous, we say that J is completely continuous. In this case, $\mathbf{id} - J$ is called a completely continuous field.

Theorem 1.35. *Let* E *be a Banach space and* $M \subset E$ *be a bounded closed subset. Let* $J : M \to E$ *be a continuous mapping. Then* J *is completely continuous if and only if, for any* $\varepsilon > 0$, *there exists a finite-dimensional subspace* E_n *of* E *and a bounded continuous mapping* $J_n : M \to E_n$ *such that*

$$\sup_{u \in D} \|J(u) - J_n(u)\| < \varepsilon.$$

Let E be a Banach space and $W \subset E$ be a bounded open subset. Let $J : \bar{W} \to E$ be completely continuous and $f = \mathbf{id} - J$. If $p \in E \backslash f(\partial W)$, then by Theorem 1.35, there exists a finite-dimensional subspace E_n of E and a bounded continuous mapping $J_n : \bar{W} \to E_n$ such that

$$\sup_{u \in W} \|J(u) - J_n(u)\| < \operatorname{dist}(p, f(\partial W)).$$

Denote $W_n = E_n \cap W$; $f_n(u) = u - J_n(u)$; then $f_n \in \mathbf{C}(\bar{W}_n, E_n), p \in E_n \backslash f_n(\partial W_n)$. Hence, $\deg(f_n, W_n, p_n)$ is well defined.

Definition 1.36 (Leray–Schauder degree). Let f be the completely continuous field defined as above. Define the Leray–Schauder degree of f at $p \in E \backslash f(\partial W)$ by

$$\deg(f, W, p) = \deg(f_n, W_n, p),$$

which is independent of the choice of E_n, p, J_n.

Proposition 1.37. *Let* $W \subset E$ *be a bounded open subset of the Banach space* E; $f = \mathbf{id} - J$ *is a completely continuous field,* $p \in E \backslash f(\partial W)$.

(1)

$$\deg(\mathbf{id}, W, p) = \begin{cases} 1, & p \in W, \\ 0, & p \notin \bar{W}. \end{cases}$$

(2) Let W_1, W_2 *be two disjoint open subsets of* W, $p \notin f(\bar{W} \backslash (W_1 \cup W_2))$; *then*

$$\deg(f, W, p) = \deg(f, W_1, p) + \deg(f, W_2, p).$$

(3) Let $H \in \mathbf{C}([0,1]\times\bar{W}, E)$ *be completely continuous,* $h_t(u) = u - H(t,u)$, $p \in \mathbf{C}([0,1], E)$, *and* $p(t) \notin h_t(\partial W)$ *for each* $t \in [0,1]$. *Then*

$$\deg(h_t(\cdot), W, p(t))$$

is independent of $t \in [0,1]$.

(4) (Kronecker's theorem) If $\deg(f, W, p) \neq 0$, *then there exists a* $u \in W$ *such that* $f(u) = p$.

Theorem 1.38 (Borsuk–Ulam theorem). *Let* W *be an open bounded symmetric neighborhood of* 0 *in a Banach space* E. *A completely continuous field* $f = \mathbf{id} - J : \bar{W} \to E$, *where* J *is odd on* ∂W; $p \in E \backslash f(\partial W)$; *then* $\deg(f, W, p)$ *is an odd number.*

1.6 The Global Flow

Let $(E, \|\cdot\|)$ be a Banach space. Consider the following Cauchy initial value problem of the ordinary differential equation.

$$\begin{cases} \dfrac{d\sigma}{dt} = W(\sigma(t, u_0)), \\ \sigma(0, u_0) = u_0 \in E, \end{cases} \tag{1.19}$$

where W is a potential function. We are interested in the existence of a solution to (1.19), which plays an important role in the following chapters. First, we prepare two auxiliary results.

Lemma 1.39 (Gronwall's inequality). *If* $\kappa \geq 0, \gamma > 0$ *and* $f \in \mathbf{C}([0,T], \mathbf{R}^+)$ *satisfies*

$$f(t) \leq \kappa + \gamma \int_0^t f(s)ds, \qquad \forall t \in [0,T], \tag{1.20}$$

then $f(t) \leq \kappa e^{\gamma t}$ *for all* $t \in [0,T]$.

Proof. By (1.20), we observe that $(d/dt)\big(e^{-\gamma t}\int_0^t f(s)ds\big) \leq \kappa e^{-\gamma t}$. Integrating both sides on $[0,t]$, we get the conclusion. □

Lemma 1.40 (Banach's fixed point theorem). *Let* $D \subset E$ *be closed. Let* $H : D \to D$ *satisfy*

$$\|Hu - Hv\| \leq k\|u - v\| \qquad \text{for some } k \in (0,1) \text{ and all } u, v \in D. \tag{1.21}$$

Then there exists a unique u^* *such that* $Hu^* = u^*$.

Proof. Let $u_{n+1} = Hu_n \quad (n = 0, 1, 2, \ldots)$ with $u_0 \in D$. Using (1.21) repeatedly, we have $\|u_{n+m+1} - u_n\| \le (1-k)^{-1} k^n \|u_1 - u_0\| \to 0$ as $n \to +\infty$. Therefore, $\{u_n\}$ is a Cauchy sequence. The conclusion follows from the continuity of H. □

We assume

(O) $W : E \to E$ is a locally Lipschitz continuous mapping; that is, for any $u \in E$, there exists a ball $B(u,r) := \{w \in E : \|w - u\| < r\}$ with radius r and a constant $\rho > 0$ depending on r and u such that

$$\|W(w) - W(v)\| \le \rho \|w - v\|, \qquad \forall w, v \in B(u,r).$$

Moreover, $\|W(u)\| \le a + b\|u\|$ for all $u \in E$, where $a, b > 0$ are constants.

Theorem 1.41. *Assume* (O)*. Then for any* $u \in E$*, Cauchy problem (1.19) has a unique solution* $\sigma(t,u)$ *(called the flow or trajectory) defined in a maximal interval* $[0, +\infty)$ *of* t*.*

Proof. For any fixed $u_0 \in E$, by condition (O), we find a ball $B(u_0, r) := \{w \in E : \|w - u_0\| < r\}$ with radius r and a constant $\rho > 0$ depending on r and u_0 such that

$$\|W(w) - W(u_0)\| \le \rho \|w - u_0\|, \qquad \forall w \in B(u_0, r).$$

Let $\Lambda := \sup_{B(u_0,r)} \|W\|$. Then $\Lambda < +\infty$. Choose $\varepsilon > 0$ such that $\varepsilon\rho < 1$, $\varepsilon\Lambda \le r$. Consider the Banach space

$$\hat{E} := \mathbf{C}([0,\varepsilon], E) := \{u : [0,\varepsilon] \to E \text{ is a continuous function}\}$$

with the norm $\|u\|_{\hat{E}} := \max_{t\in[0,\varepsilon]} \|u(t)\|$ for each $u \in \hat{E}$. Let $D := \{u \in \hat{E} : \|u - u_0\|_{\hat{E}} \le r\}$. Define a mapping $H : \hat{E} \to \hat{E}$ by

$$Hu := u_0 + \int_0^t W(u(s))ds, \qquad u \in \hat{E}.$$

For any $u, w \in D$ we have

$$\|Hu - u_0\|_{\hat{E}} \le \int_0^t \|W(u(s))\|_{\hat{E}} ds \le \Lambda\varepsilon \le r$$

and

$$\|Hu - Hw\|_{\hat{E}} \le \max_{t\in[0,\varepsilon]} \int_0^t \|W(u) - W(w)\|_{\hat{E}} ds \le \rho\varepsilon \|u - w\|_{\hat{E}}.$$

Therefore, $H : D \to D$ satisfies all conditions of Lemma 1.40. Hence, H has a unique fixed point $u^* \in D$, which is a solution of Cauchy problem (1.19).