# VARIATIONAL INEQUALITIES WITH APPLICATIONS

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# VARIATIONAL INEQUALITIES WITH APPLICATIONS

## A Study of Antiplane Frictional Contact Problems

By

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*To my teachers Nicolae Cristescu, Caius Iacob,*
*and Gheorghe Popescu, with gratitude*
*(Mircea Sofonea)*

*To Claudiu, Alexandru, and Robert, with love*
*(Andaluzia Matei)*

# Series Preface

*As any human activity needs goals, mathematical research needs problems.*
—David Hilbert

*Mechanics is the paradise of mathematical sciences.*
—Leonardo da Vinci

Mechanics and mathematics have been complementary partners since Newton's time, and the history of science shows much evidence of the beneficial influence of these disciplines on each other. Driven by increasingly elaborate modern technological applications, the symbiotic relationship between mathematics and mechanics is continually growing. However, the increasingly large number of specialist journals has generated a duality gap between the partners, and this gap is growing wider.

*Advances in Mechanics and Mathematics* (AMMA) is intended to bridge the gap by providing multidisciplinary publications that fall into the two following complementary categories:

1. An annual book dedicated to the latest developments in mechanics and mathematics;
2. Monographs, advanced textbooks, handbooks, edited volumes, and selected conference proceedings.

The AMMA annual book publishes invited and contributed comprehensive research and survey articles within the broad area of modern mechanics and applied mathematics. The discipline of mechanics, for this series, includes relevant physical and biological phenomena such as: electromagnetic, thermal, and quantum effects, biomechanics, nanomechanics, multiscale modeling, dynamical systems, optimization and control, and computation methods. Especially encouraged are articles on mathematical and computational models and methods based on mechanics and their interactions with other fields. All contributions will be reviewed so as to guarantee the highest possible scientific standards. Each chapter will reflect the most recent achievements in the area. The coverage should be conceptual, concentrating on the methodological thinking that will allow the nonspecialist reader to understand it. Discussion of possible future research directions in the area is welcome.

Thus, the annual volumes will provide a continuous documentation of the most recent developments in these active and important interdisciplinary fields. Chapters published in this series could form bases from which possible AMMA monographs or advanced textbooks could be developed.

Volumes published in the second category contain review/research contributions covering various aspects of the topic. Together these will provide an overview of the state-of-the-art in the respective field, extending from an introduction to the subject right up to the frontiers of contemporary research. Certain multidisciplinary topics, such as duality, complementarity, and symmetry in mechanics, mathematics, and physics are of particular interest.

The *Advances in Mechanics and Mathematics* series is directed to all scientists and mathematicians, including advanced students (at the doctoral and postdoctoral levels) at universities and in industry who are interested in mechanics and applied mathematics.

*David Y. Gao*
*Ray W. Ogden*

# Preface

The theory of variational inequalities plays an important role in the study of both the qualitative and numerical analysis of nonlinear boundary value problems arising in mechanics, physics, and engineering science. For this reason, the mathematical literature dedicated to this field is extensive, and the progress made in the past four decades is impressive. A part of this progress was motivated by new models arising in contact mechanics. At the heart of this theory is the intrinsic inclusion of free boundaries in an elegant mathematical formulation.

Contact between deformable bodies abounds in industry and everyday life. Because of the industrial importance of the physical processes that take place during contact, a considerable effort has been made in their modeling, analysis, numerical analysis and numerical simulations, and, as a result, the mathematical theory of contact mechanics has made impressive progress recently. Owing to their inherent complexity, contact phenomena lead to mathematical models expressed in terms of strongly nonlinear evolutionary problems.

Antiplane shear deformations are one of the simplest classes of deformations that solids can undergo: in antiplane shear (or longitudinal shear) of a cylindrical body, the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. For this reason, the antiplane problems play a useful role as pilot problems, allowing for various aspects of solutions in solid mechanics to be examined in a particularly simple setting. In recent years, considerable attention has been paid to the analysis of such kinds of problems.

The purpose of this book is to introduce to the reader the theory of variational inequalities with emphasis on the study of contact mechanics and, more specifically, with emphasis on the study of antiplane frictional contact problems. The contents cover both abstract results in the study of variational inequalities as well as the study of specific antiplane frictional contact problems. This includes their modeling and variational analysis. Our intention is to illustrate the cross-fertilization between modeling and applications on the one hand, and nonlinear mathematical analysis on the other hand.

Thus, within the particular setting of antiplane shear, we show how new and nonstandard models in contact mechanics lead to new types of variational inequalities and, conversely, we show how the abstract results on variational inequalities can be applied to prove the unique solvability of the corresponding contact problems. In writing this book, our aim was also to draw the attention of the applied mathematics community to interesting two-dimensional models arising in solid mechanics, involving a single nonlinear partial differential equation that has the virtue of relative mathematical simplicity without loss of essential physical relevance.

Our book, divided into four parts with 11 chapters, is intended as a unified and readily accessible source for mathematicians, applied mathematicians, engineers, and scientists, as well as advanced graduate students. It is organized with two different aims, so that readers who are not interested in modeling and applications can skip Parts III and IV and will find an elementary introduction to the theory of variational inequalities in Part II of the book; alternatively, readers who are interested in modeling and applications will find in Parts III and IV the mechanical models that lead to the various classes of variational inequalities presented in Part II of the book.

A brief description of the parts of the book follows.

*Part I* is devoted to the basic notation and results that are fundamental to the developments later in this book. We review the background on functional analysis and function spaces that we need in the study of variational inequalities. The material presented is standard and can be found in many textbooks and monographs. For this reason, we present only very few details of the proofs.

*Part II* represents one of the main parts of the book and includes original results. We present various classes of variational inequalities for which we prove existence results and, for some of them, we prove uniqueness, regularity, and convergence results. To this end we use convexity, monotonicity, compactness, time discretization, regularization, and fixed point arguments. Most of the concepts and results presented in this part can be extended to more general variational inequalities involving nonlinear operators on reflexive Banach spaces or to hemivariational inequalities; however, since our aim is to provide an accessible presentation of the theory of variational inequalities with emphasis in the study of antiplane frictional contact problems, we restrict ourselves to the framework of Hilbert spaces, linear operators, and convex analysis, as is sufficient for later development.

The terminology we use in this part of book is the following: if the time derivative of the unknown function $u$ appears in the formulation of a variational inequality (and, therefore, an initial condition for $u$ is needed), we refer to it as an *evolutionary variational inequality*. Otherwise, we refer to it as an *elliptic variational inequality*. If the nondifferentiable convex functional $j$ depends explicitly on $u$ or on its time derivative $\dot{u}$, we refer to the corresponding variational inequality as a *quasivariational inequality*. If both the data and the solution of a variational inequality depend on the time variable that plays the role of a parameter, the corresponding variational inequality is called a *time-dependent variational inequality.* Finally, if an integral term containing the solution or its derivative appears in the formulation of a variational inequality, we refer to it as a *history-dependent variational inequality.* This classification is not strict and is intended to distinguish among the types of variational inequalities used in the mathematical theory of contact mechanics, as it is illustrated in Part IV.

*Part III* presents preliminary material of contact mechanics that is needed in the rest of the book. We summarize basic notions and equations of mechanics of continua, then we introduce the frictional contact conditions as well as the constitutive laws that are used in the rest of the book. We then specialize the equations and conditions in the context of the antiplane shear and, as an example, we study a displacement-traction problem involving linearly elastic materials. The material presented in this part provides the background for the modeling of the antiplane frictional contact problems studied in Part IV of the book.

*Part IV* represents the other main part of the book and is partially based on our original research. It deals with the study of static and quasistatic frictional antiplane contact problems. We model the material behavior with isotropic linearly elastic and viscoelastic constitutive laws and, in the case of viscoelastic materials, we consider both short and long memory. Friction is modeled with versions of Coulomb's law in which the friction bound is either a function that does not depend on the process variables or depends on the slip or slip rate. Particular attention is paid to history-dependent frictional problems in which the friction bound depends on the total slip or the total slip rate. For each one of the problems, we provide a variational formulation then we use the abstract results in Part II in order to establish existence and sometimes uniqueness, regularity, and convergence results.

Each of the four parts of the book is divided into several chapters. All the chapters are numbered consecutively. Mathematical relations (equalities, inequalities, and inclusions) are numbered by chapter and their order of occurrence. For example, (4.3) is the third numbered mathematical relation in Chapter 4. Definitions, problems, theorems, propositions, lemmas, and corollaries are numbered consecutively within each chapter. For example, in Chapter 9, Problem 9.5 is followed by Theorem 9.6.

Each part ends with a section in which we present bibliographical comments. We provide references for the principal results presented, as well as information on important topics related to but not included in the body of the text. The list of the references at the end of the book includes only papers or books that are closely related to the subjects treated in this monograph.

This book is a result of cooperation between the authors during the past several years and was partially supported by the Integrated Action France-Romania *Brâncuşi* No. 06080RF/03. Part of the material is based on the Ph.D. thesis of the second author as well as on our joint work with several collaborators to whom we express our thanks. We especially thank Weimin Han, Constantin Niculescu, Vicenţiu Rădulescu, Meir Shillor, and Juan M. Viaño for our beneficial cooperation and for their constant support. We extend our gratitude to David Y. Gao for inviting us to make the contribution in the Springer book series on *Advances in Mechanics and Mathematics* (AMMA). Finally, we thank the unknown referees for their valuable suggestions, which improved the final form of the book.

Perpignan, France *Mircea Sofonea*
Craiova, Romania *Andaluzia Matei*

July 2008

# Contents

# *List of Symbols*

## Sets

$\mathbb{N}$: the set of positive integers;

$\mathbb{Z}_+$: the set of non-negative integers;

$\mathbb{R}$: the real line;

$\mathbb{R}_+$: the set of non-negative real numbers;

$\mathbb{R}^d$: the $d$-dimensional Euclidean space;

$\mathbb{S}^d$: the space of second-order symmetric tensors on $\mathbb{R}^d$;

$\Omega$: an open, bounded, connected set in $\mathbb{R}^d$ with a Lipschitz boundary $\Gamma$;

$\Gamma$: the boundary of the domain $\Omega$, which is decomposed as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ having mutually disjoint interiors;

$\Gamma_1$: the part of the boundary where displacement condition is specified; $\operatorname{meas}(\Gamma_1) > 0$ is assumed throughout the book;

$\Gamma_2$: the part of the boundary where traction condition is specified;

$\Gamma_3$: the part of the boundary where contact takes place;

$[0, T]$: time interval of interest, $T > 0$.

## Operators

$\nabla$: the gradient operator (pages 14, 29, 149);

Div: the divergence operator (page 130);

div: the divergence operator (page 149);

$\gamma$: the trace operator (page 28);

$\partial_\nu$: the normal derivative operator (page 151);

$\mathcal{P}_K$: the projection operator onto a set $K$ (page 11);

$\boldsymbol{I}$: the identity operator on $\mathbb{R}^3$ (page 137).

**Function spaces**

$C^m(\overline{\Omega})$: the space of functions whose derivatives up to and including order $m$ are continuous up to the boundary $\Gamma$ (page 22);

$C_0^\infty(\Omega)$: the space of infinitely differentiable functions with compact support in $\Omega$ (page 22);

$L^p(\Omega)$: the Lebesgue space of $p$-integrable functions, with the usual modification if $p = \infty$ (page 23);

$W^{k,p}(\Omega)$: the Sobolev space of functions whose weak derivatives of orders less than or equal to $k$ are $p$-integrable on $\Omega$ (page 25);

$H^k(\Omega) \equiv W^{k,2}(\Omega)$ (page 25);

$W_0^{k,p}(\Omega)$: the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ (page 26);

$H_0^k(\Omega) \equiv W_0^{k,2}(\Omega)$ (page 26);

$V = \{ v \in H^1(\Omega) : v = 0$ a.e. on $\Gamma_1 \}$, with inner product $(u, v)_V = (\nabla u, \nabla v)_{L^2(\Omega)^2}$ (page 152);

$X$: a Hilbert space with inner product $(\cdot, \cdot)_X$, or a Banach space with norm $\| \cdot \|_X$;

$\mathcal{L}(X, Y)$: the space of linear continuous operators from $X$ to a normed space $Y$ (page 6);

$\mathcal{L}(X) \equiv \mathcal{L}(X, X)$ (page 6);

$X \times Y$: the product of the Hilbert spaces $X$ and $Y$, with inner product $(\cdot, \cdot)_{X \times Y}$ (page 12);

$0_X$: the zero element of $X$;

$C^m([0, T]; X) = \{ v \in C([0, T]; X) : v^{(j)} \in C([0, T]; X),\ j = 1, \ldots, m \}$ (page 33);

$L^p(0, T; X) = \{ v : (0, T) \to X$ measurable: $\|v\|_{L^p(0,T;X)} < \infty \}$ (page 33);

$W^{k,p}(0, T; X) = \{ v \in L^p(0, T; X) : \|v^{(j)}\|_{L^p(0,T;X)} < \infty\ \forall j \le k \}$ (page 35);

$H^k(0, T; X) \equiv W^{k,2}(0, T; X)$ (page 35).

## Other symbols

$c$: a generic positive constant;

$r_+ = \max\{0, r\}$: positive part of $r$;

$\forall$: for all;

$\exists$: there exist(s);

$\Longrightarrow$: implies;

$\overline{A}$: the closure of the set $A$;

$\partial A$: the boundary of the set $A$;

$\delta_{ij}$: the Kronecker delta;

a.e.: almost everywhere;

iff: if and only if;

l.s.c.: lower semicontinuous (page 12);

$\psi_K$: the indicator function of the set $K$ (page 13);

$\partial\varphi$: the subdifferential of the function $\varphi$ (page 15);

$\dot{f}$: the time derivative of the function $f$.

# Part I
# Background on Functional Analysis

# Chapter 1
# Preliminaries

This chapter presents preliminary material from functional analysis that will be used in subsequent chapters. Most of the results are stated without proofs, as they are standard and can be found in many references. We start with a review of definitions and properties of linear normed spaces and Banach spaces, including results on duality and weak convergence. We then recall some properties of the Hilbert spaces. Finally, we present miscellaneous results that will be applied repeatedly in this book; they include elements of convex analysis, fixed point theorems, and well-known inequalities. All the linear spaces considered in this book including abstract normed spaces, Banach spaces, Hilbert spaces, and various function spaces are assumed to be real spaces. We assume that the reader has some knowledge of linear algebra and general topology.

## 1.1 Linear Operators on Normed Spaces

The notion of a norm in a general linear space is an extension of the ordinary length of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ and is provided by the following definition.

**Definition 1.1.** Given a linear space $X$, a *norm* $\|\cdot\|_X$ is a function from $X$ to $\mathbb{R}$ with the following properties.

1. $\|u\|_X \geq 0 \ \forall\, u \in X$, and $\|u\|_X = 0$ iff $u = 0_X$.
2. $\|\alpha\, u\|_X = |\alpha|\, \|u\|_X \ \forall\, u \in X,\ \forall\, \alpha \in \mathbb{R}$.
3. $\|u + v\|_X \leq \|u\|_X + \|v\|_X \ \forall\, u, v \in X$.

The pair $(X, \|\cdot\|_X)$ is called a *normed space*.

Here and everywhere in this book, $0_X$ will denote the zero element of $X$. Also, we will simply say $X$ is a normed space when the definition of the norm is understood from the context.

On a linear space, various norms can be defined. Sometimes, it is desirable to know if two norms are related and, for this reason, we introduce the following definition.

**Definition 1.2.** Let $\|\cdot\|^{(1)}$ and $\|\cdot\|^{(2)}$ be two norms over a linear space $X$. The two norms are said to be *equivalent* if there exist two constants $c_1$, $c_2 > 0$ such that

$$c_1 \|u\|^{(1)} \le \|u\|^{(2)} \le c_2 \|u\|^{(1)} \quad \forall\, u \in X. \tag{1.1}$$

The notion of a seminorm is useful in the study of various nonlinear boundary problems and in error estimates of some numerical approximations.

**Definition 1.3.** Given a linear space $X$, a *seminorm* $|\cdot|_X$ is a function from $X$ to $\mathbb{R}$ satisfying the following properties.

1. $|u|_X \ge 0 \;\forall\, u \in X$.
2. $|\alpha\, u|_X = |\alpha|\, |u|_X \;\forall\, u \in X,\ \forall\, \alpha \in \mathbb{R}$.
3. $|u + v|_X \le |u|_X + |v|_X \;\forall\, u, v \in X$.

It follows from above that a seminorm satisfies the properties of a norm except that $|u|_X = 0$ does not necessarily imply $u = 0_X$.

With a norm at our disposal, we use the quantity $\|u - v\|_X$ to measure the distance between $u$ and $v$. Consequently, the norm is used to define the bounded sets and the convergence of sequences in the space $X$.

**Definition 1.4.** Let $(X, \|\cdot\|_X)$ be a normed space. A subset $A \subset X$ is *bounded* if there exists $M > 0$ such that $\|u\|_X \le M$ for all $u \in A$. A sequence $\{u_n\} \subset X$ is *bounded* if there exists $M > 0$ such that $\|u_n\|_X \le M$ for all $n \in \mathbb{N}$ or, equivalently, if $\sup_n \|u_n\|_X < \infty$.

**Definition 1.5.** Let $X$ be a normed space. A sequence $\{u_n\} \subset X$ is said to converge (strongly) to $u \in X$ if

$$\|u_n - u\|_X \to 0 \quad \text{as } n \to \infty.$$

In this case, $u$ is called the (strong) *limit* of the sequence $\{u_n\}$ and we write

$$u = \lim_{n\to\infty} u_n \quad \text{or} \quad u_n \to u \quad \text{in } X.$$

It is easy to verify that a limit of a sequence, if it exists, is unique. The adjective "strong" is introduced in the previous definition to distinguish this convergence from other types of convergence that will be introduced in the next section. Using (1.1) it is easy to see that, for two equivalent norms, convergence in one norm implies the convergence in the other norm.

The convergence of sequences is used to introduce closed sets and dense sets in a normed space.

**Definition 1.6.** Let $A$ be a subset of a normed space $X$. The *closure* $\overline{A}$ of $A$ is the union of $A$ and the set of the limits of all the convergent sequences from $A$. The set $A$ is said to be *closed* if $\overline{A} = A$ and *dense* if $\overline{A} = X$.

To test the convergence of a sequence without knowing the limiting element, it is usually convenient to refer to the notion of a Cauchy sequence.

**Definition 1.7.** Let $X$ be a normed space. A sequence $\{u_n\} \subset X$ is called a *Cauchy sequence* if $\|u_m - u_n\|_X \to 0$ as $m, n \to \infty$.

Obviously, a convergent sequence is a Cauchy sequence but in a general infinite dimensional space, a Cauchy sequence may fail to converge. This justifies the following definition.

**Definition 1.8.** A normed space is said to be *complete* if every Cauchy sequence from the space converges to an element in the space. A complete normed space is called a *Banach space*.

Given two linear spaces $X$ and $Y$, an *operator* $T : X \to Y$ is a rule that assigns to each element in $X$ a unique element in $Y$. A real-valued operator defined on a linear space $X$ is called a *functional*. If both $X$ and $Y$ are normed spaces, we can consider the continuity and Lipschitz continuity of the operators.

**Definition 1.9.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. An operator $T : X \to Y$ is said to be

1. *continuous* at $u \in X$ if

$$u_n \to u \text{ in } X \Longrightarrow T(u_n) \to T(u) \text{ in } Y;$$

2. *continuous* if it is continuous at each element of the space $X$;
3. *Lipschitz continuous* if there exists $L_T > 0$ such that

$$\|T(u) - T(v)\|_Y \le L_T \|u - v\|_X \quad \forall\, u,\, v \in X.$$

Clearly, if $T$ is Lipschitz continuous, then it is a continuous operator, but the converse is not true in general.

We now consider a particular, yet important, type of operators called linear operators.

**Definition 1.10.** Let $X$ and $Y$ be two linear spaces. An operator $L : X \to Y$ is called *linear* if

$$L(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 L(u_1) + \alpha_2 L(u_2) \quad \forall\, u_1, u_2 \in X,\ \alpha_1, \alpha_2 \in \mathbb{R}.$$

For a linear operator $L$, we usually write $L(v)$ as $Lv$. For the sake of simplicity, we sometimes write $Lv$ even when $L$ is not linear. A well-known important property of a linear operator is the following.

**Theorem 1.11.** *Let $X$ and $Y$ be normed spaces and let $L : X \to Y$ be a linear operator. Then $L$ is continuous on $X$ iff there exists $M > 0$ such that*

$$\|Lu\|_Y \leq M\|u\|_X \quad \forall\, u \in X.$$

From Theorem 1.11, we conclude that for a linear operator, continuity and Lipschitz continuity are equivalent.

We will use the notation $\mathcal{L}(X, Y)$ for the space of linear continuous operators from a normed space $X$ to another normed space $Y$. In the special case $Y = X$, we use $\mathcal{L}(X)$ to replace $\mathcal{L}(X, X)$. For $L \in \mathcal{L}(X, Y)$, the quantity

$$\|L\|_{\mathcal{L}(X,Y)} = \sup_{0_X \neq u \in X} \frac{\|Lu\|_Y}{\|u\|_X} \tag{1.2}$$

is called the *operator norm* of $L$ and $L \mapsto \|L\|_{\mathcal{L}(X,Y)}$ defines a norm on the space $\mathcal{L}(X, Y)$. The norm (1.2) enjoys the following compatibility property

$$\|Lu\|_Y \leq \|L\|_{\mathcal{L}(X,Y)} \|u\|_X \quad \forall\, u \in X.$$

Moreover, the following result holds.

**Theorem 1.12.** *Let $X$ be a normed space and let $Y$ be a Banach space. Then $\mathcal{L}(X, Y)$ is a Banach space.*

Later in the book, we will need the concept of compact operators.

**Definition 1.13.** Let $X$ and $Y$ be two normed spaces and $L : X \to Y$ be a linear operator. The operator $L$ is said to be *compact* if for every bounded sequence $\{u_n\} \subset X$, the sequence $\{Lu_n\} \subset Y$ has a subsequence converging in $Y$.

The previous definition shows, in other words, that a linear operator $L : X \to Y$ is compact if for each sequence $\{u_n\} \subset X$ that satisfies the inequality $\sup_n \|u_n\|_X < \infty$, we can find a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and an element $y \in Y$ such that $Lu_{n_k} \to y$ in $Y$. Compact operators are also called *completely continuous* operators.

We now consider an important type of real valued mappings defined on a product of linear spaces.

**Definition 1.14.** Let $X$ and $Y$ be linear spaces. A mapping $a : X \times Y \to \mathbb{R}$ is called *bilinear form* if it is linear in each variable, that is, for every $u_1$, $u_2$, $u \in X$, $v_1$, $v_2$, $v \in Y$, and $\alpha_1$, $\alpha_2 \in \mathbb{R}$,

$$\begin{aligned} a(\alpha_1 u_1 + \alpha_2 u_2, v) &= \alpha_1\, a(u_1, v) + \alpha_2\, a(u_2, v), \\ a(u, \alpha_1 v_1 + \alpha_2 v_2) &= \alpha_1\, a(u, v_1) + \alpha_2\, a(u, v_2). \end{aligned}$$

In the case $X = Y$, we say that a bilinear form is *symmetric* if

$$a(u, v) = a(v, u) \quad \forall\, u, v \in X.$$

If both $X$ and $Y$ are normed spaces, we can consider the continuity of the bilinear forms.

**Definition 1.15.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. A bilinear form $a : X \times Y \to \mathbb{R}$ is said to be *continuous* if there exists a constant $M > 0$ such that

$$|a(u,v)| \le M\,\|u\|_X \|v\|_Y \quad \forall\, u \in X,\ \forall\, v \in Y.$$

In the case $X = Y$, we say that a bilinear form is $X$*-elliptic* if there exists a constant $m > 0$ such that

$$a(u,u) \ge m\,\|u\|_X^2 \quad \forall\, u \in X.$$

Bilinear symmetric continuous and $X$-elliptic forms defined on a Hilbert space $X$ will be used in Part II of this book in the study of variational and quasivariational inequalities.

## 1.2 Duality and Weak Convergence

For a normed space $X$, the space $\mathcal{L}(X, \mathbb{R})$ is called the *dual space* of $X$ and is denoted by $X'$. The elements of $X'$ are *linear continuous functionals* on $X$. The duality pairing between $X'$ and $X$ is usually denoted by $\ell(u)$ or $\langle u', u\rangle$ for $\ell, u' \in X'$ and $u \in X$. As it follows from (1.2), the norm on $X'$ is given by

$$\|\ell\|_{X'} = \sup_{0_X \ne u \in X} \frac{|\ell(u)|}{\|u\|_X}.$$

Also, by Theorem 1.12 we know that $(X', \|\cdot\|_{X'})$ is a Banach space.

We can now introduce another kind of convergence in a normed space.

**Definition 1.16.** Let $X$ be a normed space. A sequence $\{u_n\} \subset X$ is said to *converge weakly* to $u \in X$ if for every $\ell \in X'$,

$$\ell(u_n) \to \ell(u) \quad \text{as } n \to \infty.$$

In this case, $u$ is called the *weak limit* of $\{u_n\}$ and we write $u_n \rightharpoonup u$ in $X$.

It follows from the Hahn-Banach theorem that the weak limit of a sequence, if it exists, is unique. Also, it is easy to see that the strong convergence implies the weak convergence, i.e., if $u_n \to u$ in $X$, then $u_n \rightharpoonup u$ in $X$. The converse of this property is not true in general.

The weak convergence of sequences is used to define weakly closed sets in a normed space.

**Definition 1.17.** Let $X$ be a normed space. A subset $A \subset X$ is said to be *weakly closed* if it contains the limits of all weakly convergent sequences $\{u_n\} \subset A$.

Clearly, every weakly closed subset of $X$ is closed, but the converse of this property is not true, in general. An important exception is provided by the class of convex sets that is introduced below.

**Definition 1.18.** Let $X$ be a linear space. A subset $K \subset X$ is said to be *convex* if it has the property

$$u, v \in K \Rightarrow (1-t)\,u + t\,v \in K \quad \forall t \in [0,1].$$

For $t \in [0,1]$, the expression $(1-t)\,u+t\,v$ is said to be a *convex combination* of $u$ and $v$. The set $\{\,(1-t)\,u+t\,v : t \in [0,1]\,\}$ consists of all the points on the line segment connecting $u$ and $v$. We see that if $K$ is convex and $u, v \in K$, then the line segment connecting $u$ and $v$ is contained in $K$.

**Theorem 1.19.** *A convex subset of a normed space $X$ is closed if and only if it is weakly closed.*

We now introduce the concept of reflexive spaces. To this end, consider a normed space $X$ and denote by $X'' = (X')'$ the dual of the Banach space $X'$, which will be called the *bidual* of $X$. The bidual $X''$ is a Banach space. Each element $u \in X$ induces a linear continuous functional $\ell_u \in X''$ by the relation $\ell_u(u') = \langle u', u\rangle$ for every $u' \in X'$. The mapping $u \mapsto \ell_u$ from $X$ into $X''$ is linear and isometric, i.e., $\|\ell_u\|_{X''} = \|u\|_X$ for all $u \in X$. Therefore, the normed space $X$ may be viewed as a linear subspace of the Banach space $X''$ by the embedding $u \mapsto \ell_u = \chi(u)$. We introduce the following definition.

**Definition 1.20.** A normed space $X$ is said to be *reflexive* if $X$ may be identified with $X''$ by the canonical embedding $\chi$ (i.e., if $\chi(X) = X''$).

A reflexive space must be complete and is hence a Banach space. We have the following important property of a reflexive space.

**Theorem 1.21.** (Eberlein-Smulyan) *If $X$ is a reflexive Banach space, then each bounded sequence in $X$ has a weakly convergent subsequence.*

It follows that if $X$ is a reflexive Banach space and the sequence $\{u_n\} \subset X$ is bounded (i.e., $\sup_n \|u_n\|_X < \infty$), then we can find a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and an element $u \in X$ such that $u_{n_k} \rightharpoonup u$ in $X$. Furthermore, it can be proved that if the limit $u$ is independent of the subsequence extracted, then the *whole* sequence $\{u_n\}$ converges weakly to $u$.

On the dual of a normed space, besides the weak convergence, we can introduce the notion of weak * convergence.