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Homogenization of Partial Differential Equations

Translated from the original Russian by M. Goncharenko and D. Shepelsky
This book is devoted to homogenization problems for partial differential equations describing various physical phenomena in microinhomogeneous media. This direction in the theory of partial differential equations has been intensively developed for the last forty years; it finds numerous applications in radiophysics, filtration theory, rheology, elasticity theory, and many other areas of physics, mechanics, and engineering sciences.

A medium is called microinhomogeneous if its local parameters can be described by functions rapidly varying with respect to the space variables. We will always assume that the length scale of oscillations is much less than the linear sizes of the domain in which a physical process is considered but much greater than the sizes of molecules, so that the process can be described using the differential equations of the mechanics of solids. These differential equations either have rapidly oscillating coefficients (with respect to the space variables) or are considered in domains with complex microstructure, such as domains with fine-grained boundary [112] (called later by the better-known term strongly perforated domains). The microstructure is understood as the local structure of a domain or the coefficients of equations in the scale of microinhomogeneities.

Obviously, it is practically impossible to solve the corresponding boundary (initial boundary) value problems by either analytical or numerical methods. However, if the microscale is much less than the characteristic scale of the process under investigation (e.g., the wavelength), then it is possible to give a macroscopic description of the process. If it is the case, the medium usually has stable characteristics (heat conductivity, dielectric permeability, etc.), which, in general, may differ substantially from the local characteristics. Such stable characteristics are referred to as homogenized, or effective, characteristics, because they are usually determined by methods of the homogenization theory for differential equations or the relevant mean field methods, effective medium methods, etc.

The term homogenization is associated, first of all, with methods of nonlinear mechanics and ordinary differential equations developed by Poincaré, Krylov, Bogolyubov, and Mitropolskii (see, e.g., [21, 123]). For partial differential equations, homogenization problems have been studied by physicists from Maxwell's times,
but they remained for a long time outside the interests of mathematicians. However, since the mid 1960s, homogenization theory for partial differential equations began to be intensively developed by mathematicians as well, which was motivated not only by numerous applications (first of all, in the theory of composite media [142]) but also by the emergence of new deep ideas and concepts important for mathematics itself. Currently, there is a great number of publications devoted to mathematical aspects of homogenization such as asymptotic analysis, two-scale convergence, $G$-convergence, and $\Gamma$-convergence. Making no claim to cite all of the available monographs on the subject, we would like to mention the books by Allaire [3], Bakhvalov and Panasenko [9], Bensoussan, Lions, and Papanicolaou [13], Braides and Defranceschi [26], Cioranescu and Donato [42], Cioranescu and Saint Jean Paulin [45], Dal-Maso [46], Marchenko and Khruslov [113], Oleinik, Iossifiyan, and Shamaev [131], Pankov [133], Sanches-Palencia [148], Skrypnik [161], Zhikov, Kozlov, and Oleinik [181].

In the mathematical description of a physical phenomenon in microinhomogeneous media, the local characteristics depends on a small parameter $\varepsilon$, which is the characteristic scale of the microstructure. It is the asymptotic analysis, as $\varepsilon \to 0$, of the problem that leads to the homogenized model of the process. It turns out that the limits of solutions of the original problem can be described by certain new differential equations with coefficients smoothly varying in simple domains. These equations constitute a mathematical model of the physical process in a microinhomogeneous medium, their coefficients being effective characteristics of the medium. For example, in the simplest case, the local characteristics of a microinhomogeneous medium are described by periodic functions of the form $a\left(\frac{x}{\varepsilon}\right)$, $x \in \mathbb{R}^n$. The corresponding effective characteristics appear to be independent of $x$; moreover, the homogenized equations have the same structure as the original ones. Therefore, in this case, the main problem of mathematical modeling is to determine the coefficients of the homogenized equations; these coefficients can then be viewed as the effective parameters of the medium. This situation is typical for various microinhomogeneous media encountered in nature.

However, there exist media with more complicated microstructure, the macroscopic description of which cannot be reduced to the determination of the effective characteristics only, since homogenization leads to equations substantially different from the original ones. Such a situation usually occurs when the microstructure is characterized by several small parameters, of different order of smallness; artificial composite materials as well as some natural media provide the relevant examples. The corresponding homogenized models differ substantially from the original, "microscopic," ones; depending on the microstructure, they appear to be either nonlocal models or multicomponent models or models with memory. This book is basically devoted to the study of structure of microinhomogeneous media leading to "nonstandard" models; therefore, it has almost no intersections with the monographs cited above, except [113]. We began to write this book (which was initially thought of as a revised edition of [113]) in the late 1980s; but since then, new results have been obtained, which now constitute the main contents of the book, the needed results from [113] being presented in more convenient fashion.
In the book, we restrict ourselves mainly to physical phenomena described by the Dirichlet and Neumann boundary value problems in strongly perforated domains and by linear elliptic and parabolic differential equations with rapidly oscillating coefficients; but the developed methods can be applied as well in the study of boundary value problems of elasticity theory, electrodynamics, Fourier boundary value problems, nonlinear problems, etc.

Acknowledgments

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Kharkov, March 2004

Vladimir Marchenko
Evgenii Khruslov
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Homogenization of
Partial Differential Equations
In contrast to the majority of available monographs on homogenization theory dealing with media of relatively simple microstructure (such as periodic, or close to periodic, structures depending on a single small parameter), in this book we study phenomena in media of arbitrary microstructure characterized by several small parameters (or even more complicated media). For such media, homogenized models of physical processes may have various forms differing substantially from an original model. In order to give some ideas about the possible types of models and the topology of microstructure of the corresponding media, in this introduction we consider typical examples of microstructures leading, in the limit, to particular homogenized models. To be specific, we consider a nonstationary heat conduction process (a nonstationary diffusion), described by the heat equation, in microinhomogeneous media of various types.

In the main part of the book (Chapters 2–8), we will consider problems in the general setting and present necessary and sufficient conditions for the convergence of solutions of the original problems to solutions of the corresponding homogenized equations. These condition are formulated in terms of local “mean” characteristics of the microstructure (“mesocharacteristics”), which are then used to express the coefficients of the limiting equations. These characteristics are introduced in cubes (“mesocubes”), which are small relative to the whole domain but at the same time are large relative to the microscale. Since we define the mesocharacteristics following the penalty method and therefore they may seem to be introduced somewhat artificially, we present in this introduction a certain motivation for our approach. Here we also discuss, without proofs, the basic notions needed for the characterization of general microstructures such as the notions of strongly connected domains and weakly connected domains.

1.1 The Simplest Homogenized Model

We begin with the study of a two-phase medium consisting of a bulk homogeneous material in which small grains (inclusions) of another homogeneous material are
embedded. More precisely, let \( \Omega \) and \( G \) be bounded domains in the \( n \)-dimensional space \( \mathbb{R}^n (n \geq 2) \) with smooth boundaries \( \partial \Omega \) and \( \partial G \), respectively, \( \overline{G} \) (the closure of \( G \)) lying in the parallelogram \( \Pi = \{ x \in \mathbb{R}^n : |x_i| < \frac{h_i}{2} \} \). Construct the disjoint domains

\[
G_{j\varepsilon} = \varepsilon G + \varepsilon \sum_{k=1}^{n} h_k m_{j\varepsilon} e^k, \quad j = 1, 2, \ldots, \tag{1.1}
\]

arranged periodically in \( \mathbb{R}^n \); here \( m_{j\varepsilon} \) are entire numbers, \( \{ e^k \}_{k=1}^{n} \) is an orthonormal basis in \( \mathbb{R}^n \), and \( \varepsilon > 0 \) is a small parameter.

Assume that the bulk material occupies the domain \( \Omega_{0\varepsilon} = \Omega \setminus \bigcup_j \overline{G}_{j\varepsilon} \), whereas the inclusions occupy the domain \( \Omega_{1\varepsilon} = \bigcup_j G_{j\varepsilon} \), where the union is taken over a finite number of domains \( G_{j\varepsilon} \) lying entirely in \( \Omega \), i.e., \( \overline{G}_{j\varepsilon} \subset \Omega \) for \( j = 1, 2, \ldots, \mathcal{N}(\varepsilon) < \infty \); see Figure 1.1.

Denote by \( a_k \) and \( b_k \) \( (k = 0, 1) \) the heat conductivity and heat capacity, respectively, of the bulk material \( (k = 0) \) and the inclusions \( (k = 1) \), i.e., the local characteristics of the phases.

A nonstationary heat conduction process in such a medium can be described by the temperature function \( u^\varepsilon(x, t) \) \( (x \in \Omega, t > 0) \), which, assuming that there are no
1.1 The Simplest Homogenized Model

internal heat sources in $\Omega_{\epsilon_k}$ ($k = 0, 1$), satisfies the heat equations

$$b_k \frac{\partial u^\epsilon}{\partial t} = a_k \Delta u^\epsilon, \quad x \in \Omega_{\epsilon_k}, \quad t > 0, \quad k = 0, 1, \quad (1.2)$$

and the conjugation conditions

$$\left( u^\epsilon \right)_0 = \left( u^\epsilon \right)_1, \quad x \in \partial G_{j\epsilon}, \quad t > 0, \quad (1.3)$$

$$a_0 \left( \frac{\partial u^\epsilon}{\partial \nu} \right)_0 = a_1 \left( \frac{\partial u^\epsilon}{\partial \nu} \right)_1, \quad x \in \partial G_{j\epsilon}, \quad t > 0, \quad (1.4)$$

on the interphase boundaries $\partial G_{j\epsilon}(j = 1, \ldots, N(\epsilon))$. Here $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $\frac{\partial}{\partial \nu}$ is the normal derivative with respect to $\partial G_{j\epsilon}$, and the subscripts 0 and 1 denote the limiting values of functions as their argument approaches a point $x \in \partial G_{j\epsilon}$ from $\Omega_0\epsilon$ and $\Omega_1\epsilon = \bigcup_{j=1}^{N(\epsilon)} G_{j\epsilon}$, respectively.

Condition (1.3) means that the temperature is continuous; condition (1.4) means that the heat fluxes are equal on the interphase boundaries.

It is well known (see, e.g., [105]) that a function $u^\epsilon(x, t)$ satisfying (1.2)–(1.4) is a generalized solution of the equation

$$b^\epsilon(x) \frac{\partial u^\epsilon}{\partial t} - \sum_{i,k=1}^n \frac{\partial}{\partial x_i} \left( a^\epsilon_{ik}(x) \frac{\partial u^\epsilon}{\partial x_k} \right) = 0, \quad x \in \Omega, \quad t > 0, \quad (1.5)$$

where $b^\epsilon(x)$ and $a^\epsilon_{ik}(x)$ ($i, k = 1, \ldots, n$) are piecewise constant periodic functions in $\mathbb{R}^n$ defined by

$$b^\epsilon(x) = \begin{cases} b_0, & x \in \Omega_{0\epsilon}, \\ b_1, & x \in \Omega_{1\epsilon}; \end{cases} \quad a^\epsilon_{ik}(x) = \begin{cases} a_0 \delta_{ik}, & x \in \Omega_{0\epsilon}, \\ a_1 \delta_{ik}, & x \in \Omega_{1\epsilon}; \end{cases}$$

here $\delta_{ik}$ is the Kronecker symbol.

Consider (1.5) together with the initial condition

$$u^\epsilon(x, 0) = U(x), \quad x \in \Omega, \quad (1.6)$$

where $U(x) \in W_2^1(\Omega)$, and the Dirichlet boundary condition

$$u^\epsilon(x, t) = 0, \quad x \in \partial \Omega, \quad (1.7)$$

on $\partial \Omega$.

Problem (1.5)–(1.7) has the unique generalized solution $u^\epsilon(x, t)$ in $W_2^{1,1}(\Omega \times (0, T))$ (see, e.g., [105]); it describes the cooling of the composite body $\Omega$. For small $\epsilon$, the functions $b^\epsilon(x)$ and $a^\epsilon_{ik}(x)$ (the local characteristics of the medium) oscillate rapidly in $\Omega$, which makes the direct solution of problem (1.5)–(1.7) (or, equivalently, problem (1.2)–(1.4), (1.6), (1.7)) extremely difficult. On the other hand, it is intuitively clear that for small $\epsilon$, a number of important quantities (for example, the damping factor for the mean body temperature) can be found with the help of the
homogenized characteristics of the medium such as the “effective” heat conductivity and heat capacity. The asymptotic analysis of \( u^\varepsilon(x, t) \) as \( \varepsilon \to 0 \) shows that the considered microinhomogeneous medium \( \Omega \) indeed possesses such effective characteristics, \( b \) and \( a_{ij} \). Namely, the following assertion holds: The solutions \( u^\varepsilon(x, t) \) of problem (1.5)–(1.7) converge in \( L^2(\Omega) \), as \( \varepsilon \to 0 \), to a solution \( u(x, t) \) of the following initial boundary value problem:

\[
\begin{align*}
    b \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} &= 0, & x \in \Omega, & t > 0, \\
    u(x, 0) &= U(x), & x \in \Omega, & (1.8) \\
    u(x, t) &= 0, & x \in \partial \Omega, & t > 0, \quad (1.9)
\end{align*}
\]

uniformly with respect to \( t \geq 0 \). Here the coefficients \( b \) and \( a_{ij} \) are constants defined by

\[
b = \frac{1}{|\Pi|} \int B(x) \, dx, \\
\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j = \inf_v \frac{1}{|\Pi|} \int \sum_{i,j=1}^{n} A_{ij}(x) \left( \frac{\partial v}{\partial x_i} + \xi_i \right) \left( \frac{\partial v}{\partial x_j} + \xi_j \right) \, dx,
\]

where \( \Pi = \left\{ x \in \mathbb{R}^n : |x_i| < \frac{h_i}{2}, i = 1, \ldots, n \right\} \) includes \( \overline{G} \), \( |\Pi| \) is the volume of \( \Pi \),

\[
B(x) = \begin{cases} 
    b_0, & x \in \Pi \setminus \overline{G} \\
    b_1, & x \in G,
\end{cases} \quad A_{ij}(x) = \begin{cases} 
    a_{0 \delta_{ij}}, & x \in \Pi \setminus \overline{G} \\
    a_{1 \delta_{ij}}, & x \in G,
\end{cases}
\]

and \( \xi = \{\xi_1, \ldots, \xi_n\} \in \mathbb{R}^n \) is an arbitrary vector; the infimum is taken over the class of smooth \( \Pi \)-periodic functions.

This statement remains valid (see, e.g., [9, 180]) even in the more general setting in which the coefficients of (1.5) have the form \( b^\varepsilon(x) = B \left( \frac{x}{\varepsilon} \right) \) and \( a_{ij}^\varepsilon(x) = A_{ij} \left( \frac{x}{\varepsilon} \right) \); here \( B(x) \) and \( A_{ij}(x) \) are arbitrary bounded \( \Pi \)-periodic functions satisfying the conditions

\[
B(x) > 0, \quad \sum_{i,j=1}^{n} A_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2 \quad (\alpha > 0).
\]

Similar results have been obtained also for \( B(x) \) and \( A_{ij}(x) \) being almost-periodic functions or realizations of certain random uniform fields in \( \mathbb{R}^n \) [100], [101], [178].

Now consider the general setting. We assume that the coefficients of (1.5) are measurable functions of \( x \in \Omega \) depending on a parameter \( \varepsilon > 0 \) and satisfying the estimates

\[
b \leq b^\varepsilon(x) \leq B \quad (1.11)
\]

and
1.2 Nonlocal Homogenized Model

\[ a|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^\varepsilon(x)|\xi_i|\xi_j x f \leq A|\xi|^2, \]  
\( i,j=1 \)

where 0 < b < B < \infty and 0 < a < A < \infty; \( \xi = \{\xi_1, \ldots, \xi_n\} \) is an arbitrary vector in \( \mathbb{R}^n \) (notice that for each case mentioned above, these estimates are obviously satisfied).

Using the Laplace transform and well-known results for stationary problems (see, e.g., [181] and [58]), it can be shown that under the conditions stated above, the set of solutions \( \{u^\varepsilon(x, t), 0 < \varepsilon < \varepsilon_0\} \) of the initial boundary value problem (1.5)–(1.7) is compact in \( L_2(\Omega \times (0, T)) \) \( (\forall T > 0) \) and that all limits of \( u^\varepsilon(x, t) \) as \( \varepsilon \to 0 \) (i.e., the limits of all convergent subsequences of \( \{u^\varepsilon\} \)) are solutions of the boundary value problem

\[ b(x) \frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right), \quad x \in \Omega, \ t > 0, \]  
\( i,j=1 \)

u(x, 0) = U(x), \quad x \in \Omega, \quad (1.14)

u(x, t) = 0, \quad x \in \partial \Omega, \ t > 0. \quad (1.15)

Here the coefficients \( b(x) \) and \( a_{ij}(x) \) satisfy the estimates (1.11) and (1.12) and, in general, depend on a chosen convergent subsequence.

Under some additional conditions on the local medium characteristics \( b^\varepsilon(x) \) and \( a_{ij}^\varepsilon(x) \), the coefficients \( b(x) \) and \( a_{ij}(x) \) are independent of the choice of the subsequence, so that the \( u^\varepsilon(x, t) \) converge, as \( \varepsilon \to 0 \), to a solution of problem (1.13)–(1.15). In this case, (1.13) is said to be the homogenized model of the heat conduction process (the nonstationary diffusion process) in a microinhomogeneous medium. We emphasize that due to the conditions of uniform ellipticity and boundedness, (1.11) and (1.12), the homogenized equation (1.13) has the same form as the original equation (1.5); this is the simplest (standard) homogenized model.

Now let us assume that the conditions of uniform boundedness and ellipticity are not satisfied, i.e., there exist subsets \( G_\varepsilon \subset \Omega \) in which the coefficients of (1.5) either grow infinitely or vanish, as \( \varepsilon \to 0 \). Then solutions of (1.5)–(1.7) do not, in general, converge to a solution of (1.13)–(1.15). Physically, this means that the homogenized heat conduction model is not described by (1.13), although the effective heat conductivity \( a_{ij}(x) \) and heat capacity \( b(x) \) can still be defined locally. In this case, the homogenized model appears to be more complicated: depending on the structure of the set \( G_\varepsilon \), it can be either a nonlocal model or a multiphase model or a model with memory.

In Sections 1.2–1.5, we will consider simple examples of microinhomogeneous media for which homogenization leads to such nonstandard models.

1.2 Nonlocal Homogenized Model

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain and let \( \tilde{G}_\varepsilon \subset \mathbb{R}^3 \) be a connected periodic set consisting of round cylinders of radius \( r_\varepsilon \ll \varepsilon \) with axes making an \( \varepsilon \)-periodic lattice in \( \mathbb{R}^3 \); a fragment of \( \tilde{G}_\varepsilon \) is shown in Figure 1.2.
Set \( G_\varepsilon = \Omega \cap \tilde{G}_\varepsilon \) and define the coefficients of (1.5) by

\[
b^\varepsilon(x) \equiv 1, \quad x \in \Omega
\]

and

\[
a^{\varepsilon}_{ij}(x) = \begin{cases} 
\delta_{ij}, & x \in \Omega \setminus G_\varepsilon, \\
\delta_{ij} \mes \Omega (\mes G_\varepsilon)^{-1}, & x \in G_\varepsilon,
\end{cases}
\]

where \( \delta_{ij} \) is the Kronecker symbol and \( \mes \) denotes the Lebesgue measure in \( \mathbb{R}^3 \).

Since \( r_\varepsilon = o(\varepsilon) \), we have

\[
\lim_{\varepsilon \to 0, r_\varepsilon > 0} \mes G_\varepsilon = \frac{3\pi r_\varepsilon^2}{\varepsilon^2} \mes \Omega \to 0 \quad \text{as} \quad \varepsilon \to 0,
\]

and therefore, \( a^{\varepsilon}_{ij} \to \infty \) in \( G_\varepsilon \). Thus, the first inequality in (1.12), i.e., the condition of uniform (with respect to \( \varepsilon \)) boundedness of \( a^{\varepsilon}_{ij}(x) \), is violated. On the other hand, since the other inequalities in (1.11) and (1.12) remain true, the set \( \{u^\varepsilon(x, t), \varepsilon > 0\} \) of solutions of problem (1.5)–(1.7) is compact in \( L_2(\Omega \times [0, T]) \) for any \( T > 0 \).

Suppose that the limit

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2 |\ln r_\varepsilon|} = q,
\]

finite or infinite, exists. If \( q < \infty \), then solutions \( u^\varepsilon(x, t) \) of (1.5)–(1.7) converge, as \( \varepsilon \to 0 \), to a solution \( u(x, t) \) of the following initial boundary value problem (see Section 7.4):
where \( c = 6\pi q \), \( R(x, y) = c^2 G(x, y) \), and \( G(x, y) \) is Green's function of the following problem:

\[
-\frac{1}{3} \Delta G + cG = \delta(x, y), \quad x, y \in \Omega,
\]

\[
G(x, y) = 0, \quad x \in \partial \Omega.
\]

In this case, the radius \( r_\varepsilon \) is exponentially small with respect to the square of the period \( \varepsilon \); for example, \( r_\varepsilon \) can be thought of as

\[
r_\varepsilon = a_1 \exp \left\{ -\frac{a_2}{\varepsilon^2} \right\}, \quad a_1, a_2 > 0.
\]

Thus, the grid \( G_\varepsilon \) on which the heat conductivity increases is very rapidly becoming thinner. Therefore, for \( 0 < q < \infty \), the homogenized model (1.18)-(1.20) appears to be nonlocal: equation (1.18) contains an integral (with respect to the space variables) term.

From the physical point of view, this can be explained as follows. The mean heat capacity of \( G_\varepsilon \) becomes, in view of (1.16), arbitrarily small; at the same time, the mean heat conductivity of the grid does not vanish, because the local heat conductivity of \( G_\varepsilon \) grows rapidly. Therefore, the grid can transfer a part of the heat flux in \( \Omega \) almost instantly, which leads to the nonlocal model (1.18)-(1.20).

It is essential that the parameter \( q \) be finite: then the connection between \( G_\varepsilon \) and \( \Omega \setminus G_\varepsilon \) is weak, so that the temperatures at a point of \( G_\varepsilon \) and a nearby point of \( \Omega \setminus G_\varepsilon \) can differ substantially; as a consequence, the dependence between the heat fluxes in \( G_\varepsilon \) and in \( \Omega \setminus G_\varepsilon \) is weak. If \( q = \infty \), then the homogenized model appears to be the standard local model of the form (1.13)-(1.15).

Thus, for the sets \( G_\varepsilon \) having the structure (grids) considered above, the nonlocality effect is determined by the parameter \( q \) (1.17). For the sets \( G_\varepsilon \subset \Omega \) of more general type, conditions providing nonlocality of the homogenized model can be formulated in terms of the strong connectivity and weak connectedness of families of domains; these terms will be defined below, see Chapter 4. Roughly speaking, a homogenized model is a nonlocal model of the type (1.18)-(1.20) if and only if the domains \( G_\varepsilon \) and \( \Omega \setminus G_\varepsilon \) are strongly connected (considered separately) but weakly connected between each other.

### 1.3 Two-Component Homogenized Model

Now consider the case in which both inequalities in (1.11) and the second inequality in (1.12) are fulfilled but the first inequality in (1.12) is not: \( a_{ij}^\varepsilon(x) \) in (1.5) vanish,
as $\varepsilon \to 0$, in $G_\varepsilon$ constructed in the following way. Let $\tilde{G}_\varepsilon \in \mathbb{R}^3$ be a periodic set consisting of round tubes with axes making an $\varepsilon$-periodic lattice in $\mathbb{R}^3$. Let the tube radius be $r_\varepsilon = r\varepsilon$ and let the thickness of the tube walls be $\delta_\varepsilon = \delta\varepsilon^{1+x}$, where $r < \frac{1}{2}$, and $\delta$ and $x$ are fixed positive numbers. A fragment (cell) of $\tilde{G}_\varepsilon$ is shown in Figure 1.3.

![Figure 1.3](image-url)

Set $G_\varepsilon = \Omega \cap \tilde{G}_\varepsilon$ and denote by $\Omega_{1\varepsilon}$ and $\Omega_{2\varepsilon}$ the subdomains of $\Omega$ lying in $\Omega$ interior to $G_\varepsilon$ and exterior to $G_\varepsilon$, respectively, so that $\Omega = \Omega_{1\varepsilon} \cup \Omega_{2\varepsilon} \cup G_\varepsilon$.

Define the coefficients of (1.5) by

$$b^\varepsilon(x) = \begin{cases} 1, & x \in \Omega, \\ x \in \Omega_{1\varepsilon} \cup \Omega_{2\varepsilon}, \\ x \in G_\varepsilon, \end{cases}$$

and

$$a_{ij}^\varepsilon = \begin{cases} \delta_{ij}, & x \in \Omega_{1\varepsilon} \cup \Omega_{2\varepsilon}, \\ \alpha \varepsilon^{2+x} \delta_{ij}, & x \in G_\varepsilon, \end{cases}$$

where $\alpha$ is a fixed positive number.

Since the condition of uniform ellipticity is violated in this case, the set of solutions $\{u^\varepsilon(x, t), \varepsilon > 0\}$ of (1.5)–(1.7) is not compact in $L_2(\Omega \times [0, T])$. However, for the configuration of $G_\varepsilon$ described above, it is natural to introduce the notion of convergence of $u^\varepsilon(x, t)$ to a two-component function $u(x, t) = (u_1(x, t), u_2(x, t)) \in L_2(\Omega \times [0, \infty))$: $u^\varepsilon$ is said to converge to $u = (u_1, u_2)$ if

$$\lim_{\varepsilon \to 0} \sum_{k=1}^2 \| \chi_k^\varepsilon u^\varepsilon - \chi_k^\varepsilon u_k \|_{L_2(\Omega \times [0, T])} = 0, \quad \forall T > 0,$$

where $\chi_k^\varepsilon(x)$ is the characteristic functions of $\Omega_{k\varepsilon} \subset \Omega$ ($k = 1, 2$).
Then one can show that $u^e(x,t)$ converges, in the sense of (1.21), to a solution $(u_1(x,t), u_2(x,t))$ of the following initial boundary value problem (see Section 7.2):

\begin{align}
   b_1 \frac{\partial u_1}{\partial t} - a_1 \Delta u_1 + c(u_1 - u_2) &= 0, \quad x \in \Omega, \ t > 0, \\
   b_2 \frac{\partial u_2}{\partial t} - a_2 \Delta u_2 + c(u_2 - u_1) &= 0, \quad x \in \Omega, \ t > 0,
\end{align}

\begin{align}
   u_1 &= 0, \quad u_2 = 0, \quad x \in \partial \Omega, \quad t > 0, \\
   u_1(x,0) &= U(x), \quad u_2(x,0) = U(x), \quad x \in \Omega.
\end{align}

Here $b_k, a_k \ (k = 1, 2),$ and $c$ are positive constants defined as follows. Let $K = \{ x \in \mathbb{R}^3 : 0 \leq x < \frac{1}{2}, \ i = 1, 2, 3 \}$ and let $G \subset K$ consist of three round cylinders of radius $r < \frac{1}{2}$ and length 1, with axes parallel to the coordinate axes in $\mathbb{R}^3$; see Figure 1.4.

Introduce the following notation: $G_1 = G, \ G_2 = K \setminus G, \ S = \partial G_1 \cap \partial G_2$ (lateral area of $\partial G$), and $S_k^\pm = \partial G_k \cap \{ x_3 = \pm \frac{1}{2} \}$ ("horizontal" pieces of the surfaces $\partial G_k$). Denote by $|S|$ the area of a surface $S$. Let $W_k$ be the class of functions in $W_2^1(G_k)$ equal to $\pm \frac{1}{2}$ on $S_k^{\pm}$. Then the coefficients of (1.22) and (1.23) are determined by

\begin{align}
   b_k &= \text{mes} \ G_k, \quad a_k = \min_{v \in W_k} \int_{G_k} |\nabla v|^2 \ dx, \quad k = 1, 2, \quad c = \frac{\alpha}{\delta} |S|.
\end{align}
Therefore, the homogenized heat conduction model (1.22)-(1.25) is a two-component model. Physically, it can be explained as follows: a weakly conducting layer \( C_\varepsilon \) divides the body \( \Omega \) into two intertwining, weakly interacting parts (phases), \( \Omega_{1\varepsilon} \) and \( \Omega_{2\varepsilon} \), which conduct heat fluxes almost independently, so that the temperature values \( u_1(x, t) \) and \( u_2(x, t) \) at nearby points of \( \Omega_{1\varepsilon} \) and \( \Omega_{2\varepsilon} \), respectively, can differ substantially. Mathematically, the (qualitative) reason for this effect is again the strong connectivity of \( \Omega_{1\varepsilon} \) and \( \Omega_{2\varepsilon} \) and their (mutual) weak connectedness.

**1.4 Homogenized Model with Memory**

Assume again that the first inequality in (1.12) is not satisfied but, in contrast to the previous case, the set \( G_\varepsilon \subseteq \Omega \), in which the coefficients \( a_i^\varepsilon(x) \) vanish as \( \varepsilon \to 0 \), has a different topological structure. Namely, let \( G_\varepsilon = \bigcup_i G_{i\varepsilon} \) be a union of spherical shells lying in a bounded domain \( \Omega \subseteq \mathbb{R}^n \) \((n \geq 2)\) and centered at the nodes of an \( \varepsilon \)-periodic lattice in \( \mathbb{R}^n \). Let the external radius of the shells be given by \( r_\varepsilon = r \varepsilon \) and let the thickness of their walls be \( \delta_\varepsilon = \delta \varepsilon^{1+\kappa} \), where \( 0 < r < \frac{1}{2}, \kappa > 0, \) and \( \delta > 0 \). Denote by \( D_{i\varepsilon} \) the spheres interior to \( G_{i\varepsilon} \). Then \( \Omega_{1\varepsilon} = \Omega \setminus (D_{i\varepsilon} \cup G_{i\varepsilon}) \) is the exterior domain with respect to the shells; the two-dimensional case is illustrated in Figure 1.5.

\[ \text{Fig. 1.5.} \]

Define the coefficients of (1.5) as follows:
In this case, solutions $u_\varepsilon(x, t)$ of (1.5)–(1.7) converge, as $\varepsilon \to 0$, to a solution $u(x, t)$ of the following initial boundary value problem (see Section 7.2):

$$
\begin{align*}
 b \frac{\partial u}{\partial t} - a \Delta u + \frac{\partial}{\partial t} \int_0^t q e^{-p(t-\tau)} u(x, \tau) d\tau &= q e^{-pt} U(x), & x \in \Omega, & t > 0, \\
 u(x, t) &= 0, & x \in \partial\Omega, & t > 0, \\
 u(x, 0) &= U(x), & x \in \Omega,
\end{align*}
$$

where $b$, $a$, $p$, and $q$ are positive constants. The convergence is understood in the norm of $L_2(\Omega_{1\varepsilon} \times [0, T])$ for any $T$, i.e.,

$$
\lim_{\varepsilon \to 0} \|u_\varepsilon - u\|_{L_2(\Omega_{1\varepsilon} \times [0, T])} = 0,
$$

and the constants $a$, $b$, $p$, and $q$ are defined by

$$
a = \min_{v \in W} \int_{K \setminus G} |\nabla v|^2 dx,
$$

$$
b = 1 - |G|, \quad p = \frac{a n}{\delta r}, \quad q = p|G|.
$$

Here $K = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| < \frac{1}{2}, i = 1, \ldots, n\}$ is the unit cube centered at 0, $G = \{x \in K, |x| < r\}$ is the ball of radius $r < \frac{1}{2}$ centered also at 0, $|G|$ is the volume of $G$, and $W$ is the class of functions in $W_2^1(K \setminus G)$ equal to $\pm \frac{1}{2}$ on the cube faces $x_n = \pm \frac{1}{2}$, respectively.

Since (1.26) contains an integral (with respect to time) term, the homogenized model (1.26)–(1.28) can be viewed as a model with memory (delayed model). From the physical point of view, this can be easily understood by considering the diffusion in the microinhomogeneous medium shown in Figure 1.5. Indeed, some of particles moving through $\Omega_{1\varepsilon}$ fall into the traps $D_{1\varepsilon}$. Since the transmitting capacity of the trap walls is low, it is difficult for particles to go out (as well as to go into) the traps, which causes a delay. Notice that in this case, the carrier $\Omega_{1\varepsilon}$ is a strongly connected domain in $\Omega$; but the set of traps $\bigcup_i D_{1\varepsilon}$ becomes, on the one hand, “asymptotically” disconnected almost everywhere (i.e., in each subdomain of $\Omega$) and, on the other hand, weakly connected with the carrier.
1.5 Homogenized Model with Memory: The Case of Violated Uniform Boundedness of \( b^\varepsilon(x) \)

In all the examples given above, the both inequalities in (1.11) were satisfied. Violation of the first inequality in (1.11) does not cause, by itself, a qualitative changing of the homogenized model. On the other hand, if the second inequality is not valid, the homogenized model can be a model with memory.

Indeed, let \( G_\varepsilon = \bigcup_i G_i \) be a union of balls in \( \Omega \subset \mathbb{R}^3 \) of radius \( r_\varepsilon = \varepsilon \bar{r} \) centered at the nodes of an \( \varepsilon \)-periodic lattice in \( \mathbb{R}^3 \). Denote by \( \Omega_\varepsilon \) the complement (in \( \Omega \)) to the balls: \( \Omega_\varepsilon = \Omega \setminus G_\varepsilon \). Define the coefficients of (1.5) by

\[
a^{\varepsilon}_{ij}(x) \equiv \delta_{ij}, \quad x \in \Omega;
\]

\[
b^\varepsilon(x) = \begin{cases} 1, & x \in \Omega_\varepsilon, \\ b_\varepsilon = b_\varepsilon \kappa, & x \in G_\varepsilon, \end{cases}
\]

where \( r, b, \) and \( \kappa \) are arbitrary positive numbers. In this case, solutions of problems (1.5)–(1.7) converge, as \( \varepsilon \to 0 \), in \( L_2(\Omega \times [0, T]) \) to a solution \( u(x) \) of a homogenized problem, the particular form of which depends on the parameter \( \kappa \). The most interesting case corresponds to the critical value \( \kappa = 6 \). Indeed, for \( \kappa = 6 \) the limiting function \( u(x) \) is a solution of the following initial boundary value problem (see Section 7.5):

\[
\frac{\partial u}{\partial t} - \Delta u + \frac{\partial}{\partial t} \int_0^t B(t - \tau)u(x, \tau)d\tau = B(t)U(x), \quad x \in \Omega, \quad t > 0,
\]

\[
u(x, t) = 0, \quad x \in \partial \Omega, \quad t > 0, \tag{1.31}
\]

\[
u(x, 0) = U(x), \quad x \in \Omega, \tag{1.32}
\]

where

\[
B(t) = 4\pi r - \frac{8r}{\pi} \int_0^{mt} \Theta(\tau)d\tau.
\]

and the number \( m \) and the function \( \Theta(\tau) \) are defined by

\[
m = \frac{\pi}{br^2}, \quad \Theta(\tau) = \sum_{k=0}^{\infty} e^{(-k+\frac{1}{2})^2\tau}.
\]

If \( \kappa \neq 6 \), then \( u(x, t) \) solves the problem

\[
\frac{\partial u}{\partial t} - \Delta u + cu = cU(x), \quad x \in \Omega, \quad t > 0, \tag{1.33}
\]

\[
u(x, t) = 0, \quad x \in \partial \Omega, \quad t > 0, \tag{1.34}
\]

\[
u(x, 0) = U(x), \quad x \in \Omega, \tag{1.35}
\]

where \( c = 0 \) for \( \kappa < 6 \) and \( c = 4\pi r \) for \( \kappa > 6 \).
1.6 Homogenization of Boundary Value Problems in Strongly Perforated Domains

In the considered examples, the notion of weak connectedness between the heat carriers or between the main heat carrier and accumulators (traps) played an essential role. The connectedness was weak due to either low transmitting capacity of the layers $G_{\varepsilon}$ (Sections 1.3 and 1.4) or smallness of the contact area $\partial G_{\varepsilon}$ (Sections 1.2 and 1.5).

Apparently, the notion of weak connectedness appeared for the first time in the theory of superconductivity [157], where, in addition to the two above-mentioned types of weak connections, thin passages ("bridges") and microcontacts have been considered. Media with bulk weak connections can be seen in the theory of high-temperature superconductivity; they are also called Josephson's media [12]. The influence of bulk weak connections such as thin passages and microcontacts can be demonstrated using the homogenization of the Neumann problem for the heat equation in strongly perforated domains $\Omega_{\varepsilon} = \Omega \setminus F_{\varepsilon}$.

The Neumann Problem

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a fixed bounded domain and let $F_{\varepsilon} \in \Omega$ be a closed set, with the piecewise smooth boundary $\partial F_{\varepsilon}$, depending on a parameter $\varepsilon$. Consider in $\Omega_{\varepsilon} = \Omega \setminus F_{\varepsilon}$ the following initial boundary value problem:

$$
\frac{\partial u^{\varepsilon}}{\partial t} - \Delta u^{\varepsilon} = 0, \quad x \in \Omega_{\varepsilon}, \quad t > 0,
$$

(1.36)

$$
\frac{\partial u^{\varepsilon}}{\partial v} = 0, \quad x \in \partial F_{\varepsilon},
$$

(1.37)

$$
u^{\varepsilon} = 0, \quad x \in \partial \Omega, \quad t > 0,
$$

$$
u^{\varepsilon}(x, 0) = U(x), \quad x \in \Omega_{\varepsilon},
$$

(1.38)

where $U(x) \in W^{1}_{2}(\Omega)$ is the given initial data and $\frac{\partial}{\partial v}$ is the normal derivative with respect to $\partial F_{\varepsilon}$.

This problem describes the process of diffusion in a medium with impenetrable (reflecting) inclusions $F_{\varepsilon} \subset \Omega$, the initial particle density being $U(x)$. 

Therefore, if the parameter $x$ takes its critical value, $x = 6$, then the homogenized model appears to be a model with memory; but the physical nature of this effect is different from the one seen in the preceding example: now this is basically due to the large heat inertia of inclusions $G_{j\varepsilon}$ and low degree of their connectedness with the main heat carrier (the latter is because their sizes $r_{\varepsilon} = r\varepsilon^{3}$ are relatively small). The homogenized models (1.33)-(1.35) (for $c = 0$ or $c = 4\pi r$) correspond to the cases in which the rate of growth, as $\varepsilon \to 0$, of the heat capacity of inclusions $G_{j\varepsilon}$ is either too small or, conversely, too large relative to the rate of vanishing of their sizes.
There exists a unique solution $u^\varepsilon(x, t)$ of problem (1.36)-(1.38). The study of the asymptotic behavior of $u^\varepsilon(x, t)$ as $\varepsilon \to 0$ leads to the three basic types of homogenized models of diffusion determined by the structure of the family of sets $\{F_\varepsilon\}$: the simplest (standard) model of diffusion, a multicomponent model, and a model with memory.

Example (i). First, consider the simplest case. Let $F_\varepsilon = \overline{G_\varepsilon}$, where $G_\varepsilon = \bigcup_j G_{j\varepsilon}$ is the union of subdomains $G_{j\varepsilon} = \varepsilon G + x_{j\varepsilon}$ defined by (1.1) and periodically arranged in $\Omega \subset \mathbb{R}^n$. Then $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$ is a strongly perforated domain in $\mathbb{R}^n$; see Figure 1.6.

![Fig. 1.6.](image)

In this case, solutions $u^\varepsilon$ of (1.36)-(1.38) converge, as $\varepsilon \to 0$, to a solution $u(x, t)$ of the boundary value problem (1.8)-(1.10) formulated in the whole domain $\Omega$. The convergence is understood in the norm of $L^2(\Omega_\varepsilon)$ for any fixed $t$; the coefficients $b$ and $a_{ij}$ ($i, j = 1, \ldots, n$) of (1.8) are defined by (see Section 5.2 and Chapter 6)

$$b = \frac{|\Pi \setminus G|}{|\Pi|}$$

and

$$\sum_{i,j=1}^n a_{ij} \xi_i \xi_j = \inf_v \frac{1}{|\Pi|} \int_{\Pi \setminus G} |\nabla v + \xi|^2 \, dx,$$

where $\xi \in \mathbb{R}^n$ and the infimum is taken over the class of functions $v \in W^1_2(\Pi \setminus G)$ satisfying, with their derivatives, the periodicity conditions on opposite faces of $\Pi$ (all the notation corresponds to that of Section 1.1).

Therefore, for the configuration $F_\varepsilon = \bigcup_j \overline{G_{j\varepsilon}}$ considered above, the homogenization of the Neumann boundary value problem (1.36)-(1.38) leads to the simplest
model of diffusion, equation (1.8). The main reason for this is the fact that the family of domains \( \{ \Omega_\varepsilon = \Omega \setminus F_\varepsilon, \varepsilon > 0 \} \) satisfies the strong connectivity condition.

Example (ii). Next, consider impenetrable (reflecting) inclusions \( G_{j,\varepsilon} \subset \Omega \) of another form; namely, let \( G_{j,\varepsilon} \subset \mathbb{R}^n \) \((n > 2) \) be spherical shells penetrated by thin channels (punctured shells). The shells are centered at the nodes \( x_{j,\varepsilon} \) of a cubic \( \varepsilon \)-periodic lattice in \( \mathbb{R}^n \); the external radius of the shells is \( r_\varepsilon = r \varepsilon \); the wall thickness is \( \delta_\varepsilon = \delta \varepsilon^\gamma \) \((r > \delta > 0, 1 \leq \gamma < \frac{n}{n-2}) \). The channels in the shells have the form of round cylinders of radius \( \varrho_\varepsilon = \varrho_\varepsilon^{(n+\gamma)/(n-1)} \), their axes being directed radially. For the sake of simplicity, let us assume that each shell is penetrated by a single channel \( P_{j,\varepsilon} \); see Figure 1.7.

Let \( Q_{j,\varepsilon} \) and \( Q'_{j,\varepsilon} \) be the internal and external, with respect to \( G_{j,\varepsilon} \), balls of radius \( r_\varepsilon - \delta_\varepsilon \) and \( r_\varepsilon \), respectively. Introduce the notation \( \Omega_{1,\varepsilon} = \Omega \setminus \bigcup_j G_{j,\varepsilon} \) and \( \Omega_\varepsilon = \Omega \setminus \bigcup_j G_{j,\varepsilon} = \bigcup_j (Q_{j,\varepsilon} \cup P_{j,\varepsilon}) \) and consider in \( \Omega_\varepsilon \) the initial boundary value problem (1.36)–(1.38). It turns out (see Section 5.5) that as \( \varepsilon \to 0 \), solutions \( u_\varepsilon(x, t) \) of this problem converge, for each \( t \), in the norm of \( L_2(\Omega_{1,\varepsilon}) \), to a solution of problem (1.26)–(1.28) considered in \( \Omega \). In (1.26), the coefficients \( p \) and \( q \) are defined by

\[
p = \frac{n \omega_{n-1} Q^{n-1}}{(n-1) \omega_n [r - \delta \chi (\gamma - 1)]^n \delta}, \quad q = \frac{\omega_{n-1} Q^{n-1}}{(n-1) \delta},
\]

where \( \omega_n \) is the area of the unit sphere in \( \mathbb{R}^n \) and \( \chi(t) \) is the Heaviside step function; the coefficients \( a \) and \( b \) are defined as in Section 1.4.
Therefore, the homogenized model of diffusion in this case is a model with memory. From the physical point of view, a particle diffused on the carrier $\Omega_{1\varepsilon}$ passes, sometimes, thin bridges and falls into the traps $Q_{j\varepsilon}$, where it can stay for some time and then return to $\Omega_{1\varepsilon}$; this explains the time delay (memory) in the system. Mathematically, the explanation is again based on the fact that the main carrier $\Omega_{1\varepsilon}$ is strongly connected, whereas it is weakly connected to the traps $Q_{j\varepsilon}$.

Example (iii). Finally, consider a “reflecting” set $F_\varepsilon$ of the following structure. Let $\tilde{F}_\varepsilon \in \mathbb{R}^3$ be a periodic set consisting of round “punctured” tubes with the axes making a coordinate $\varepsilon$-periodic lattice. Let the internal radius of the tubes be $r_\varepsilon = r$, the thickness of their walls $\delta_\varepsilon = \delta_\varepsilon^\varepsilon$ ($0 < r, \delta < \frac{1}{4}, 1 \leq \gamma < 3$), and the radius of the round holes in the walls $\varrho_\varepsilon = \varrho_\varepsilon^{(3+\gamma)/(2)}$. An $\varepsilon$-fragment of $F_\varepsilon$ is shown in Figure 1.8.

![Fig. 1.8.](image)

Set $F_\varepsilon = \Omega \cap \tilde{F}_\varepsilon$ and denote by $\Omega_{1\varepsilon}$ and $\Omega_{2\varepsilon}$ the subdomains of $\Omega$ lying interior to $F_\varepsilon$ and exterior to $F_\varepsilon$, respectively, so that

$$
\Omega_\varepsilon = \Omega \setminus F_\varepsilon = \Omega_{1\varepsilon} \cup \Omega_{2\varepsilon} \cup \bigcup_j P_{j\varepsilon},
$$

where $\bigcup_j P_{j\varepsilon}$ is the system of channels in the tube walls.

Consider in $\Omega_\varepsilon$ the initial boundary value problem (1.36)--(1.38). It can be shown that solutions $u^\varepsilon(x, t)$ of this problem converge, in the sense of (1.21), to a solution of problem (1.22)--(1.25), where the connection coefficient is $c = \pi \varrho^2 m \delta^{-1}$ (here $m$ is the number of channels $P_{j\varepsilon}$ per period) and the other coefficients are defined as in Section 1.4 (see Chapters 5 and 6).
1.6 Homogenization in Strongly Perforated Domains

Hence, the corresponding homogenized model is a two-component model. This is basically due to the fact that \( \Omega_\varepsilon \) consists of two strongly connected carriers, \( \Omega_{1\varepsilon} \) and \( \Omega_{2\varepsilon} \), which are weakly connected between themselves by thin bridges \( P_{j\varepsilon} \).

A similar result takes place in the case that the tubes are thought of as having walls of zero thickness, so that the wall hole radius is \( Q_\varepsilon = q\varepsilon^3 \) and the connection coefficient is \( c = \frac{\pi m Q}{\varepsilon} \). In this case, the holes are relatively small; one can say that the carriers \( \Omega_{1\varepsilon} \) and \( \Omega_{2\varepsilon} \) are weakly connected to each other by the body-distributed “microcontacts.”

### The Dirichlet Problem

The examples considered above show the diversity of homogenized models corresponding to the Neumann boundary conditions on the surface of inclusions \( F_\varepsilon \subset \Omega \) (reflecting inclusions); a particular type of the model is determined by the topology of the microstructure of \( \Omega_\varepsilon = \Omega \setminus F_\varepsilon \).

A different picture is seen in the case of Dirichlet boundary conditions on \( \partial F_\varepsilon \) (absorbing inclusions). In this case, a homogenized equation always has the same form regardless of the topology of the microstructure of \( \Omega_\varepsilon \).

Indeed, let \( F_\varepsilon \) be an arbitrary closed set in a bounded domain \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) depending on a parameter \( \varepsilon > 0 \). As above, we assume that, as \( \varepsilon \to 0 \), \( F_\varepsilon \) is body-distributed and dense in some subdomain \( \Omega' \subset \Omega \). Consider in \( \Omega_\varepsilon = \Omega \setminus F_\varepsilon \) the initial boundary value problem

\[
\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = 0, \quad x \in \Omega_\varepsilon, \quad t > 0,
\]

\[
u_\varepsilon = 0, \quad x \in \partial\Omega_\varepsilon, \quad t > 0,
\]

\[
u_\varepsilon(x, 0) = U(x), \quad x \in \Omega_\varepsilon.
\]

We extend its solutions \( u_\varepsilon(x, t) \) into the sets \( F_\varepsilon \times [0, T] \) by zero. Then the set of the extended functions \( \{u_\varepsilon(x, t), \varepsilon > 0\} \) is compact in \( L_2(\Omega \times [0, T]) \) (\( \forall T > 0 \)) and all the limits \( u(x, t) \), by subsequences \( \{\varepsilon_k \to 0\} \), are described by the following initial boundary value problem:

\[
\frac{\partial u}{\partial t} - \Delta u + c u = 0, \quad x \in \Omega, \quad t > 0,
\]

\[
u = 0, \quad x \in \partial\Omega, \quad t > 0,
\]

\[
u(x, 0) = U(x), \quad x \in \Omega,
\]

where \( c = c(x) \) is a nonnegative distribution that depends on a chosen subsequence \( \{\varepsilon_k\} \) that can be expressed in terms of the capacities of parts of \( F_{\varepsilon_k} \) (see [77] and [78]).

Under some additional conditions on \( F_\varepsilon, u_\varepsilon(x, t) \) converge to a solution of problem (1.42)–(1.44), where \( c(x) \) is a (classical) function in \( L_p(\Omega) \). For example, if \( F_\varepsilon \) is a union of balls of radius \( r_\varepsilon = r\varepsilon^{(n)/(n-2)} \) located \( \varepsilon \)-periodically in \( \Omega \subset \mathbb{R}^n \) with \( n \geq 3 \), then \( c(x) \equiv const = (n - 2)\omega_n r^{n-2} \). Thus, the homogenization of
the Dirichlet problem (1.39)-(1.41) with zero boundary conditions always leads to (1.42), where the “potential” $c(x)$ is, in general, a distribution (measure).

**Remark 1.1** If the boundary condition on $\partial F_\varepsilon$ in (1.39)-(1.41) is inhomogeneous, e.g., $u(x) = U(x)$ for $x \in \partial F_\varepsilon$, then the homogenized equation has the form (1.33). This is quite natural, because this boundary condition can be obtained by passing to the limit $b_\varepsilon \to \infty$ in $F_\varepsilon$, the inclusions in (1.5)-(1.7) being fixed. Similarly, if, starting from the same original problem with fixed inclusions $F_\varepsilon$, we pass to the limit $a_\varepsilon \to 0$ in $F_\varepsilon$, then we arrive at the Neumann problem (1.36)-(1.38). This is in agreement with the fact that the basic homogenized models discussed in Sections 1.1, 1.3, 1.4, and 1.6 are all of the same form.

As follows from the considered examples, the simplest homogenized model (1.13) having the same form as the original one, (1.5), corresponds to the cases in which the medium microstructure depends on a single small parameter $\varepsilon$ (see Sections 1.1 and 1.6, example (i)). More complicated homogenized models correspond to microstructures depending on several parameters of different orders of smallness: $\varepsilon$ and $r_\varepsilon \ll \varepsilon$ in Section 1.2; $\varepsilon$ and $\delta_\varepsilon \ll \varepsilon$ in Section 1.3; $\varepsilon, \delta_\varepsilon \ll \varepsilon$, and $\varrho_\varepsilon \ll \delta_\varepsilon$ in Section 1.6 (examples (ii) and (iii)).

In Chapters 2–8, homogenization problems will be considered in the general setting, i.e., for sets $F_\varepsilon$ of arbitrary form. We will not point out explicitly the dependence of the microstructure on small parameters; instead we will consider sequences of problems, i.e., we will assume that the coefficients of original equations or original domains depend on a single natural parameter $s$. In the particular cases in which the dependence of the microstructure on the small parameters $\{\varepsilon_1, \ldots, \varepsilon_k\}$ is given explicitly, it is natural to set, e.g., $s = \min_i \left\lceil \frac{1}{\varepsilon_i} \right\rceil$.

### 1.7 Strongly Connected and Weakly Connected Domains: Definitions and Quantitative Characteristics

We have already mentioned in the examples considered above that in order to distinguish between different qualitative types of homogenized models, the medium microstructure should be described in terms of strongly connected and weakly connected domains. In Chapter 4, we will give rigorous definitions (of various degrees of generality) of these terms; here we discuss them, as well as quantitative characteristics of the corresponding domains, in qualitative terms only.

#### 1.7.1 Strongly Connected and Weakly Connected Domains

First of all, we emphasize that the notions of strongly connected and weakly connected domains refer to *sequences of domains* and not to a fixed domain.

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain and let $\{\Omega^{(s)} \subset \Omega, s = 1, 2, \ldots\}$ be a sequence of its subdomains. We assume that the subdomains $\Omega^{(s)}$ as well as their complements $\Omega \setminus \Omega^{(s)}$ are asymptotically dense in $\Omega$, i.e., for any ball $B \subset \Omega$, the